Differential Equations II

Wave Equation

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Preliminary Remarks:

- **Aim:** Investigation of the wave equation

$$u_{tt} - \Delta u = 0$$

- Inhomogeneous wave equation

$$u_{tt} - \Delta u = f$$

- Let suitable initial/boundary conditions be given.
- Let $t > 0$ denote the time variable and $\mathbf{x} \in \Omega$, $\Omega \subset \mathbb{R}^n$ open, the spatial variable.
- **Search:** Function $u : \overline{\Omega} \times [0, \infty[\to \mathbb{R}$, $u = u(\mathbf{x}, t)$, where the Laplace operator acts on the spatial variable $\mathbf{x} = (x_1, \ldots, x_n)^\top$.
- For the inhomogeneous equation assume $f : \Omega \times [0, \infty[\to \mathbb{R}$ a given function.

Remark: (Formula of d'Alembert)
Consider the one-dimensional initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times [0, \infty[, \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

with g, h given initial conditions.
Goal: direct solution method. 1

Proposition: (Formula of d'Alembert)
A solution of the one-dimensional initial value problem

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with g, h initial conditions, is given by the formula of d'Alembert:

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy.$$

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Remark: In order that $u(x,t)$ is a *differentiable* solution of the wave equation, we require:

$$u \in C^2(\mathbb{R}) \quad \text{and} \quad h \in C^1(\mathbb{R}).$$

2

Reflection Method

Preliminary Remarks:

- Consider the initial value problem on the half space $\mathbb{R}_+$:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times]0,\infty[\\ u = g,\ u_t = h & \text{on } \mathbb{R}_+ \times \{t=0\} \\ u = 0 & \text{on} \{x=0\} \times]0,\infty[\end{cases}$$

  with given functions g and h where $g(0) = h(0) = 0$.

- Problem: not simply solvable by formula of d'Alembert, since it requires a Cauchy-Problem.

- Idee: Extend the *half space problem* to a *whole space problem* and then use formula of d'Alembert.

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Conclusion: (Reflection of half space $\mathbb{R}_+$)
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Example:
The solution of the initial boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times]0,\infty[\\ u = 0, u_t = \sin x & \text{on } \mathbb{R}_+ \times \{t=0\} \\ u = 0 & \text{on } \{x=0\} \times]0,\infty[\end{cases}$$

is given by

$$u(x,t) = \frac{1}{2}[\cos(x-t) - \cos(x+t)].$$

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Spherical Averaging

Preliminary Remarks:

- Consider the higher dimensional case ($n \geq 2$) of the initial value problem:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times]0, \infty[\\ u = g, \; u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

- Idea: Derive by suitable spherical averages a simplified PDE, which yields an explicit solution formula.

Remark: (Mean/Average over Sphere) For $x \in \mathbb{R}^n$, $t > 0$ and $r > 0$ define the Average of $u(x,t)$ over the Sphere $\partial B_r(x)$ (or $\partial B(x,r)$)

$$U(x; r, t) := \fint_{\partial B_r(x)} u(y,t) \, dS(y).$$

Furthermore, let

$$\begin{aligned} G(x; r) &:= \fint_{\partial B_r(x)} g(y) \, dS(y) \\ H(x, r) &:= \fint_{\partial B_r(x)} h(y) \, dS(y) \end{aligned}$$

Proposition: (Euler-Poisson-Darboux Equation)
let $x \in \mathbb{R}^n$ be fixed and u a solution of the wave equation

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Then $U(x; r, t)$ solves the Euler-Poisson-Darboux equation

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } \mathbb{R}_+ \times]0, \infty[\\ U = G, \; U_t = H & \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases}$$

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2

Kirchhoff's Formula

Remark: (Kirchhoff's Formula for $n = 3$)
The solution to the initial value problem of the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times]0, \infty[\\ u = g,\ u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

for $x \in \mathbb{R}^3$, $t > 0$ is given by (Kirchhoff's Formula):

$$u(x,t) = ⨍_{\partial B_t(x)} (th(y) + g(y) + Dg(y) \cdot (y - x))\, dS(y)$$

Derivation: Via Euler-Poisson-Darboux equation:
Define

$$\begin{aligned} \tilde{U} &:= rU, \\ \tilde{G} &:= rG, \\ \tilde{H} &:= rH. \end{aligned}$$

Then

$$\tilde{U}_{tt} = rU_{tt} = r\left(U_{rr} + \frac{2}{r}U_r\right) = rU_{rr} + 2U_r = (U + rU_r)_r = \tilde{U}_{rr}.$$

Replacing this in our previous equation, we obtain

$$\begin{aligned} u(x,t) = & ⨍_{\partial B(x,t)} tDg(y) \cdot \left(\frac{y-x}{t}\right)\, dS(y) + ⨍_{\partial B(x,t)} g(y)\, dS(y) \\ & + ⨍_{\partial B(x,t)} th(y)\, dS(y) \end{aligned}$$

This is – after reorganization – **Kirchhoff's formula**.

Using the definitions of G and H one obtains

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial t}(tG(x;t)) + tH(x;t)) \\ &= \frac{\partial}{\partial t}\left(t ⨍_{\partial B(x,t)} g\, dS\right) + t ⨍_{\partial B(x,t)} h\, dS \end{aligned}$$

Therefore, $\tilde{U}$ solves the initial value problem

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{in } \mathbb{R}_+ \times]0, \infty[\\ \tilde{U} = \tilde{G}, \tilde{U}_t = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ \tilde{U} = 0 & \text{on } \{r = 0\} \times \{t = 0\} \end{cases}$$

With the solution formula for the half space problem, we derive for $0 \leq r \leq t$ the representation

$$\tilde{U}(x;r,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(r-t)\right] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\, dy.$$

Since $\tilde{U}(x;r,t)$ is derived from $u(x,t)$ by spherical averaging, we have

$$u(x,t) = \lim_{r \to 0} U(x;r,t)$$

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With the definition of $\tilde{U}(x;r,t)$ obtain

$$\begin{aligned} u(x,t) &= \lim_{r\to 0} \frac{\tilde{U}(x;r,t)}{r} \\ &= \lim_{r\to 0} \left(\frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{2r} + \frac{1}{2r} \int_{-r+t}^{r+t} \tilde{H}(y)\ dy \right) \\ &= \tilde{G}'(t) + \tilde{H}(t) \end{aligned}$$

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$$
\begin{aligned}
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&= \frac{\partial}{\partial t}\left(t \fint_{\partial B(x,t)} g \; dS\right) + t \fint_{\partial B(x,t)} h \; dS
\end{aligned}
$$

But since

$$
\fint_{\partial B(x,t)} g(y) \; dS(y) = \fint_{\partial B(0,1)} g(x+tz) \; dS(z)
$$

it holds

$$
\begin{aligned}
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for $x \in \mathbb{R}^2$, $t > 0$ is given by (Poisson's formula):

$$u(x,t) = \frac{1}{2} \fint_{\partial B_t(x)} \frac{tg(y) + t^2 h(y) + tDg(y) \cdot (y - x)}{(t^2 - |y - x|^2)^{\frac{1}{2}}}\, dy$$

Remarks:

- To derive this solution representation, one considers the three dimensional initial value problem and assumes that the solution is independent of the third spatial coordinate x_3.
- Using an analogous derivation principle as for Kirchhoff's formula (use Euler-Poisson-Darboux equation and define $\tilde{U}$ suitably), solution formulas for the initial value problem of the wave equation in $\mathbb{R}^n$ can be derived.

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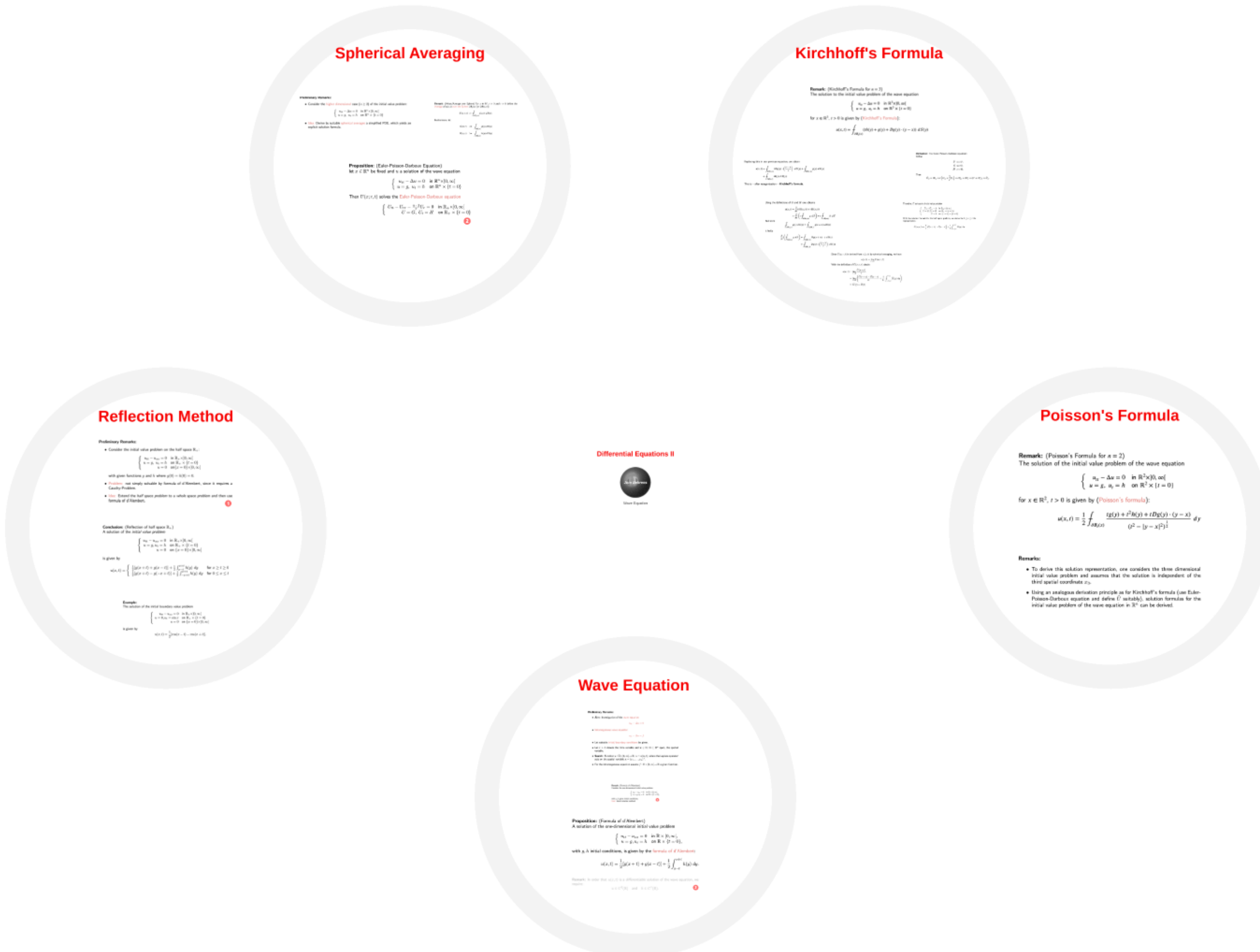
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