Differential Equations II

19.04.2022 / J. Behrens

Week 02

① Example:

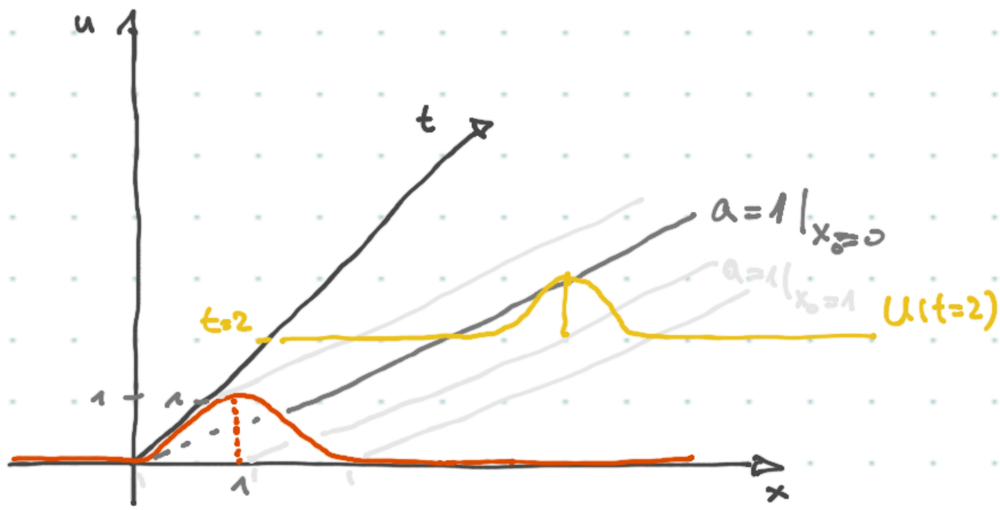
- Consider: $x u_x + y u_y + (x^2+y^2) u_z = 0$
- Characteristic System:
$$\dot{x} = x$$
$$\dot{y} = y$$
$$\dot{z} = (x^2+y^2)$$
- General Solution:
$$x(t) = c_1 e^t$$
$$y(t) = c_2 e^t$$
$$z(t) = \frac{1}{2}(c_1^2+c_2^2)e^{2t} + c_3$$
- With this: $u(x(t), y(t), z(t)) = u(c_1 e^t, c_2 e^t, \frac{1}{2}(c_1^2+c_2^2)e^{2t}+c_3) = \text{const.}$
- We know: $e^t = \frac{x(t)}{c_1} = \frac{y(t)}{c_2} \Rightarrow \frac{y(t)}{x(t)} = \frac{c_2}{c_1} = c \in \mathbb{R}$

  and: $z(t) = \frac{1}{2}(x^2+y^2)+c_3 \Rightarrow z(t) - \frac{1}{2}(x(t)^2+y(t)^2) = d \in \mathbb{R}$

  So: only c and d determine the value of u along charact. curves.
- Representation of solution:
$$u(x,y,z) = \phi\left(\frac{y}{x}, z - \frac{1}{2}(x^2+y^2)\right)$$
with arbitrary C^1-function $\phi: \mathbb{R}^2 \to \mathbb{R}$

② Interpretation of the transport eq.

- Transport eq in $\mathbb{R}\times]0,\infty[$: $u_t + a\cdot u_x = 0$ with $u=u_0$ on $\mathbb{R}\times\{t=0\}$
- Solution : $u(x,t) = u_0(x-at)$



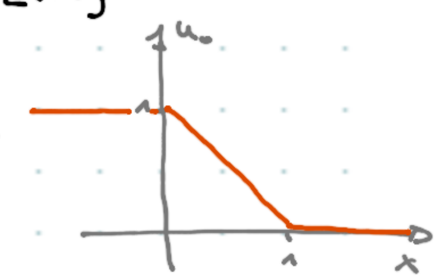
Week 03

① Burgers' eq.

- Consider Cauchy problem with flux function $f(u) = \frac{u^2}{2}$

$$\begin{aligned} u_t + u\,u_x &= 0 && \text{in } \mathbb{R}\times]0,\infty[\\ u &= u_0 && \text{on } \mathbb{R}\times\{t=0\} \end{aligned}$$

- Initial conditions $u_0(x) = \begin{cases} 1 & , x \le 0 \\ 1-x & , 0 < x < 1 \\ 0 & , 1 \le x \end{cases}$

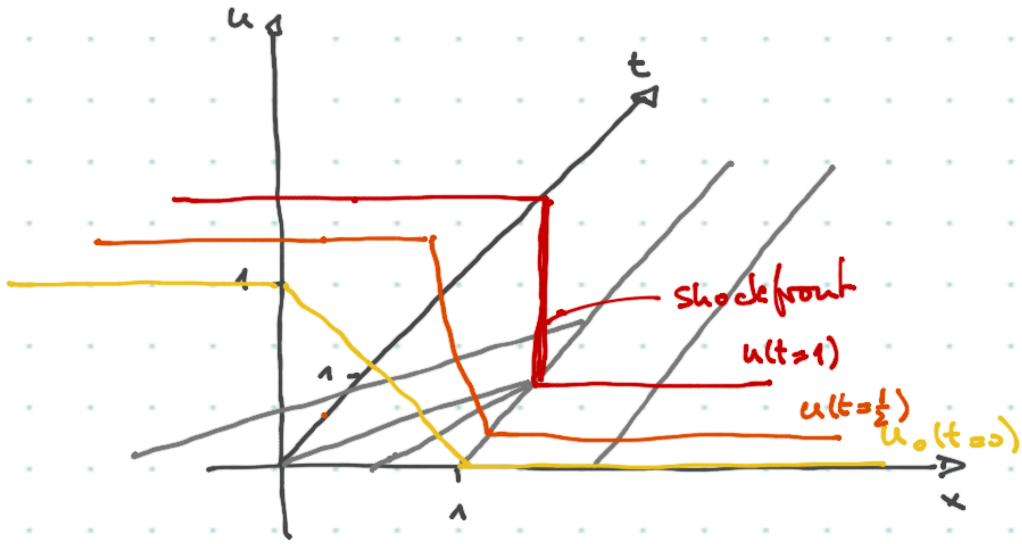


- Solve by method of char. $\dot{x} = u$, $x(0) = x_0$
- Solution is constant along characteristic curves $x(t)$, i.e.

$$\dot{x} = u_0(x_0) \Rightarrow x(t) = x_0 + t\, u_0(x_0)$$

- Fill this into the specific initial condition above:

$$x(t) = \begin{cases} t + x_0 & x_0 \le 0 \\ (1-x_0)t + x_0 & 0 < x_0 < 1 \\ x_0 & 1 \le x_0 \end{cases}$$



② Test functions + weak solution

- Let $v : \mathbb{R} \times [0, \infty[\to \mathbb{R}$ differentiable fct. with compact support.
- Let the Cauchy problem $u_t + f(u)_x = 0$ in $\mathbb{R} \times]0, \infty[$, $u = u_0$ on $\mathbb{R} \times \{t=0\}$
- Multiply an integrate:

$$0 = \int_0^\infty \int_{-\infty}^\infty (u_t + f(u)_x)\, v \, dx \, dt$$

$$= -\int_0^\infty \int_{-\infty}^{\infty} u\, v_t \, dx\, dt + \int_{-\infty}^{\infty} \cancel{u(x,\infty) \underbrace{v(x,\infty)}_{=0}}\, dx - \int_{-\infty}^{\infty} \underbrace{u(x,0)}_{=u_0(x)}\, v(x,0)\, dx$$

compact supp.

$$-\int_0^\infty \int_{-\infty}^{\infty} f(u)\, v_x \, dx\, dt + \int_0^\infty \cancel{u(\infty,t)\underbrace{v(\infty,t)}_{=0}}\, dt + \int_0^\infty \cancel{u(-\infty,t)\underbrace{v(-\infty,t)}_{=0}}\, dt$$

$$= -\int_0^\infty \int_{-\infty}^{\infty} u\, v_t \, dx\, dt + \int_{-\infty}^{\infty} u_0(x)\, v(x,0)\, dx - \int_0^\infty \int_{-\infty}^{\infty} f(u)\, v_x \, dx\, dt$$

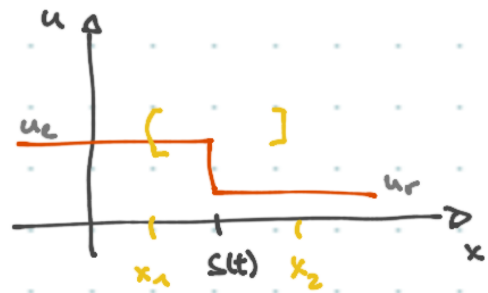
$$\Rightarrow \quad 0 = \int_0^\infty \int_{-\infty}^{\infty} \left(u\, v_t + f(u)\, v_x\right) dx\, dt - \int_{-\infty}^{\infty} u_0(x)\, v(x,0)\, dt$$

(4) Reasoning for Rankine-Hugoniot condition

- An integral solution fulfills:

$$\frac{d}{dt} \int_{x_1}^{x_2} u(\xi,t)\, d\xi = f(u(x_1,t)) - f(u(x_2,t))$$

- Chose $x_1 < s(t) < x_2$



$$\Rightarrow \frac{d}{dt}\left(\int_{x_1}^{s(t)} u(\xi,t)\, d\xi + \int_{s(t)}^{x_2} u(\xi,t)\, d\xi \right) = f(u(x_1,t)) - f(u(x_2,t))$$

- Now u is differentiable to the right and left of $s(t)$, resp., therefore

$$\Rightarrow \int_{x_1}^{s(t)} \frac{\partial u}{\partial t}\, d\xi + \dot{s}\, u(s(t)^-,t) + \int_{s(t)}^{x_2} \frac{\partial u}{\partial t}\, d\xi + \dot{s}\, u(s(t)^+,t) + f_2 - f_1 = 0$$

with $f_1 = f(u(x_1,t))$, $f_2 = f(u(x_2,t))$

- The limit process $x_1 \to s(t)^-$, $x_2 \to s(t)^+$ lead to vanishing integrals

$$\Rightarrow \dot{s}\, \underbrace{u(s(t)^-,t)}_{u_l} - \dot{s}\,(\underbrace{u(s(t)^+,t)}_{u_r}) = \underbrace{f(u(s(t)^-,t))}_{f_l} - \underbrace{f(u(s(t)^+,t))}_{f_r}$$

$[u]$ $\qquad\qquad$ $[f]$

$$\Rightarrow \dot{s} = \frac{[f]}{[u]}$$

$\boxtimes$