Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Work sheet 7

Exercise 1:

Given the following initial boundary value problem for $u = u(x,t)$:

$$\begin{aligned}
&u_{tt} - 4u_{xx} = e^{-t}\left(1 - \frac{x}{3}\right) && x \in (0,\,3),\, t > 0,\\
&u(x,0) = 1 + 2\sin(\pi x) && x \in [0,3],\\
&u_t(x,0) = \frac{x}{3} && x \in (0,3),\\
&u(0,t) = e^{-t} && t \geq 0,\\
&u(3,t) = 1 && t \geq 0.
\end{aligned} \tag{1}$$

a) Show that the homogenization of the boundary data according to

$$v = u - e^{-t} - \frac{x}{3}(1 - e^{-t})$$

leads to the following initial boundary value problem for v:

$$\begin{aligned}
&v_{tt} - 4v_{xx} = 0 && x \in (0,\,3),\, t > 0,\\
&v(x,0) = 2\sin(\pi x) && x \in [0,3],\\
&v_t(x,0) = 1 && x \in (0,3),\\
&v(0,t) = 0 && t \geq 0,\\
&v(3,t) = 0 && t \geq 0.
\end{aligned} \tag{2}$$

b) Solve the initial boundary value problem (2) from part a) and compute the solution to the initial boundary value problem (1).

Exercise 2:

For the numerical solution of a differential equation for $u(x,t)$, $x \in]0,(n+1)\Delta x[, t > 0$ with given initial data at $t = 0$ and boundary data at $x = 0$ and $x = (n+1)\Delta x$ the following grid is defined

$$x_j = j \cdot \Delta x, \qquad j = 0, 1, \ldots, n+1, \qquad\qquad t_m = m \cdot \Delta t, \qquad m = 0, 1, 2, \ldots$$

u_j^m is an approximation of $u(x_j, t_m)$.

Which PDEs are approximated by the following difference equations with the corresponding initial data ($m = 0$) and boundary data ($j = 0$ or $j = n+1$)?

For $j = 0, \ldots, N$ and $m = 1, 2, 3 \ldots$:

a) $$\frac{u_j^{m+1} - u_j^m}{\Delta t} + c\,\frac{u_j^m - u_{j-1}^m}{\Delta x} = 0,$$

b) $$\frac{u_j^{m+1} - u_j^m}{\Delta t} + c\,\frac{u_j^{m+1} - u_{j-1}^{m+1}}{\Delta x} = 0,$$

c) $$\frac{u_j^{m+1} - 2u_j^m + u_j^{m-1}}{\Delta t^2} = \frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{\Delta x^2},$$

d) $$\frac{u_j^{m+1} - u_j^m}{\Delta t} + c\,\frac{u_{j+1}^m - u_{j-1}^m}{2\Delta x} = \frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{\Delta x^2}$$

e) $$\frac{u_j^{m+1} - u_j^m}{\Delta t} + c\,\frac{u_j^m - u_{j-1}^m}{\Delta x} = \frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{\Delta x^2}$$

For which difference equations can the data at time point $m+1$ be calculated directly if the data at time m is known? So which method is an explicit method?

Discussion: 11.07. – 15.07.2022