

Differential Equations II for Engineering Students

Homework sheet 3

Exercise 1: Determine the entropy solution to the Burgers' equation $u_t + uu_x = 0$ with the initial data

$$u(x, 0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

at the time $t = 2$. What new problem occurs at $t = 2$?

Alternatively: Determine the solution for $t > 2$.

Exercise 2:

Determine entropy solutions to the differential equation

$$u_t + (f(u))_x = 0$$

with the flow function $f(u) = \frac{(u-2)^4}{2}$ and initial conditions

$$\mathbf{a)} \quad u(x, 0) = \begin{cases} 2 & x \leq 0, \\ 1 & 0 < x, \end{cases} \quad \text{and} \quad \mathbf{b)} \quad u(x, 0) = \begin{cases} 1 & x \leq 0, \\ 2 & 0 < x. \end{cases}$$

Note: Only solutions for the given initial values are required. You don't need to give solutions for general initial values!

Exercise 3:

We discuss again the simple traffic flow model from Sheet 1 with the notation introduced there:

$u(x, t)$ = density of vehicles (vehicles/length) at point x at time t ,

$v(x, t)$ = velocity at point x at time t ,

$q(x, t)$ = flow = number of vehicles passing x at time t per time unit.

We improve our model from Sheet 1 by incorporating maximal density and a maximal velocity

u_{max} = maximal density of vehicles (bumper to bumper),

v_{max} = maximal velocity

This can be done, for example, as follows:

$$v(u(x, t)) = v_{max} \left(1 - \frac{u(x, t)}{u_{max}} \right)$$

- a) Set up the continuity equation ($u_t + q_x = 0$).
- b) Show again that the characteristics are straight lines and determine their slopes.
- c) Sketch the characteristics for

$$v_{max} = 1 \quad (\text{Here has been scaled appropriately!})$$
$$u(x, 0) = \begin{cases} u_l = u_{max}/2 & x < 0 \\ u_r = u_{max} & x > 0 \end{cases} \quad (\text{red traffic light/traffic jam etc.})$$

- d) For the Burgers' equation we allowed shock waves only in the case $u_l > u_r$. There must obviously be a different condition here. What could be the reason for that?

Note: This question can not be answered completely only with help of the lecture slides. You can only make a guess here!

Submission deadline: 09.05.-13.05.2022