Fachbereich Mathematik
Prof. Dr. J. Behrens

WiSe 2022/23

Klausur Differentialgleichungen II

??. März 2023

Bitte kennzeichnen Sie jedes Blatt
mit Ihrem Namen und Ihrer Matrikelnummer.

Tragen Sie bitte zunächst Ihren Namen, Ihren Vornamen und Ihre Matrikelnummer in **DRUCKSCHRIFT** in die folgenden jeweils dafür vorgesehenen Felder ein. Diese Eintragungen werden auf Datenträger gespeichert.

Name:																
Vorname:																
Matr.-Nr.:																

Studiengang:	AIW	CI	ET	GES	IIW	MB	MTB	SB	

Ich bin darüber belehrt worden, dass die von mir zu erbringende Prüfungsleistung nur dann bewertet wird, wenn die Nachprüfung durch das Zentrale Prüfungsamt der TUHH meine offizielle Zulassung vor Beginn der Prüfung ergibt.

Unterschrift:

Aufg.	Punkte	Korrekteur
1		
2		
3		
4		

$$\sum =$$

Exercise 1: [4 points]

Compute the solution to the following initial value problem for $u(x,t)$:

$$u_t - 4t\,u_x = 5\,, \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+,$$
$$u(x,0) = \cos(x) \qquad x \in \mathbb{R}\,.$$

Solution:

With the characteristics method one computes:

$\frac{dx}{dt} = -4t \implies dx = -4tdt \implies x = 2t^2 + C_1$ **(1 point)**

$\frac{du}{dt} = 5 \implies du = 5dt \implies u = 5t + C_2\,.$ **(1 point)**

With $C_1 = x - 2t^2$ and $C_2 = u - 5t$ we make an ansatz

$C_2 = f(C_1)$

and obtain

$u - 5t = f(x - 2t^2)$

and thus we have the general solution: $u(x,t) = 5t + f(x - 2t^2)$.

The initial condition requires:

$u(x,0) = 5 \cdot 0 + f(x - 2 \cdot 0^2) = f(x) \overset{!}{=} \cos(x)\,.$ **(1 point)**

Hence $u(x,t) = 5t + \cos(x - 2t^2)$. **(2 points)**

Exercise 2: [6= 2+4 points]

Given a differential equation

$$u_t + (f(u))_x = 0$$

with the flow function $f(u) = \frac{(2u+1)^2}{3}$.

a) Is the solution constant along the characteristics?
Are the characteristics straight lines?
Justify your answers.

b) Determine the entropy solution for the differential equation for the initial values

$$u(x,0) = \begin{cases} 1 & x \leq 0, \\ -1 & 0 < x. \end{cases}$$

Note: Only solutions for the given initial values are required. You don't need to compute solutions for general initial values!

Solution:

a) With the usual designations we have $f(u) = \frac{(2u+1)^2}{3}$.
On the characteristic curves it holds
$\dot{x}(t) = f'(u) = \frac{4(2u+1)}{3}$ and $\dot{u}(t) = 0$.
So u is constant along the characteristics and the slope of the characteristics depends only on u. Thus the characteristics are straight lines with a constant slope $\frac{8u(x(0),0)+4}{3}$.
(2 points)

b) For the data from part b), an ambiguity of the solution would arise immediately (that is, already at $t = 0$) if we were to use the characteristics method. A shock front $s(t)$ must be introduced with $u_l = 1$ and $u_r = -1$. **(1 point)**
With $f(u_l) = \frac{(2+1)^2}{3} = 3, \quad f(u_r) = \frac{(-2+1)^2}{3} = \frac{1}{3}$ **(1 point)**
we obtain

$$\dot{s}(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{3 - \frac{1}{3}}{1 - (-1)} = \frac{4}{3}$$ **(1 point)**

and hence

$$u(x,t) = \begin{cases} u_l = 1 & x \leq s(t) = \frac{4t}{3} \\ u_r = -1 & \frac{4t}{3} < x. \end{cases}$$ **(1 point)**

Exercise 3: [6 points]

Determine the solution to the initial boundary value problem

$$\begin{aligned} u_t - 9u_{xx} &= 5\sin(3x) && 0 < x < 2\pi\,,\ 0 < t\,, \\ u(x,0) &= 0 && 0 \le x \le 2\pi, \\ u(0,t) &= 0 && 0 \le t, \\ u(\pi,t) &= 0 && 0 \le t. \end{aligned}$$

Solution:

With $L = 2\pi$, $\omega = \frac{\pi}{2\pi} = \frac{1}{2}$ and $c = 9$ we make an ansatz

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(n\omega x).$$

Inserting the ansatz into the differential equation yields:

$$\sum_{n=1}^{\infty} \left(\dot{a}_n(t) + cn^2\omega^2 a_n(t)\right) \sin\left(\frac{n}{2}x\right) \stackrel{!}{=} 5\sin(3x)$$

Inserting into the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} a_n(0) \sin\left(\frac{n}{2}x\right) \stackrel{!}{=} 0 \implies a_n(0) = 0,\ \forall n \in \mathbb{N}.\ \textbf{(1 point)}$$

Together we have: $\dot{a}_n(t) + cn^2\omega^2 a_n(t) = 0$, $a_n(0) = 0$, $\forall n \neq 6$,

and hence $a_n(t) \equiv 0$, $\forall n \neq 6$. **(1 point)**

For $n = 6$ we obtain:

$\dot{a}_6(t) + 9 \cdot 6^2 (\frac{1}{2})^2 a_6(t) = 5,\ a_6(0) = 0$

The associated homogeneous differential equation

$\dot{a}_{6h}(t) + 81 a_{6h}(t) = 0 \implies a_{6h}(t) = k \cdot e^{-81t}$.

Ansatz for a particular solution of the inhomogeneous problem: $a_{6p}(t) = \alpha$

Plugging into the differential equation : $9 \cdot 6^2 (\frac{1}{2})^2 a_6(t) = 5 \implies a_{6p}(t) = \frac{5}{81}$.

Alternatively: variation of the constants with the ansatz $a_{6p}(t) = k(t)e^{-81t}$.

$a_6(t) = \gamma e^{-81t} + \frac{5}{81}$ and $a_6(0) = \gamma e^0 + \frac{5}{81} = 0 \implies \gamma = -\frac{5}{81}$

$\implies \boxed{a_6(t) = \frac{5}{81}(1 - e^{-81t})}$ **(3 points)**

$$u(x,t) = \sum_{k=1}^{\infty} a_n(t) \sin(n\omega x) = a_6(t) \sin(\frac{6}{2}x) = \frac{5}{81}(1 - e^{-81t}) \sin(3x) \qquad \textbf{(1 point)}$$

Exercise 4: [4 points]

Given the following initial value problem for the wave equation with continuos functions $f:\mathbb{R}\to\mathbb{R}$ und $g:\mathbb{R}\to\mathbb{R}$

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \qquad x\in\mathbb{R},\quad t>0 \quad c>0,\\ u(x,0) = u_0(x) = f(x), &\quad u_t(x,0) = v_0(x) = g(x) \qquad x\in\mathbb{R}. \end{aligned}$$

Show that this initial value problem is well posed for finite $t\in[0,T]$.

Solution:

Using d'Alembert formula

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz\,.$$

we obtain that there is a unique solution. **(1 point)**

Let $\tilde{u}$ be the solution

$$\begin{aligned} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} &= 0 \qquad x\in\mathbb{R},\quad t>0 \quad c>0,\\ \tilde{u}(x,0) = \tilde{u}_0(x) = \tilde{f}(x), &\quad \tilde{u}_t(x,0) = \tilde{v}_0(x) = \tilde{g}(x) \qquad x\in\mathbb{R}. \end{aligned}$$

Then it holds

$$\begin{aligned} |u(x,t)-\tilde{u}(x,t)| = &\left|\frac{1}{2}\left[f(x+ct) + f(x-ct)\right] - \frac{1}{2}\left[\tilde{f}(x+ct) + \tilde{f}(x-ct)\right]\right.\\ &\left. + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)-\tilde{g}(z)\,dz\right|\\ &\le \frac{1}{2}|f(x+ct) - \tilde{f}(x+ct)| + \frac{1}{2}|f(x-ct) - \tilde{f}(x-ct)| + \frac{2ct}{2c}\|g-\tilde{g}\|_\infty\\ &\le \|f-\tilde{f}\|_\infty + T\|g-\tilde{g}\|_\infty \end{aligned}$$

So

$\|u-\tilde{u}\|_\infty \le \|f-\tilde{f}\|_\infty + T\|g-\tilde{g}\|_\infty$. **(3 points)**