Mathematics Department
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WiSe 2021/2022

Mathematik III Exam
(Modul: Analysis III)
28. February 2022

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I was instructed about the fact that the required test performance will only be assessed if the TUHH examination office can assure my official admission before the exam's beginning.

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Task no.	Points	Evaluater
1		
2		
3		
4		

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Exercise 1: (5 points)

Let

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) := x^4 - 4xy^3 + 12y + 1.$$

a) Compute the gradient and the Hessian matrix of f.

b) Compute the stationary points of f and classify them.

Solution:

a) **(2 points)**

grad $f(x,y) = f_x(x,y),\ f_y(x,y))$.

$f_x(x,y) = 4x^3 - 4y^3$,

$f_y(x,y) = -12xy^2 + 12$.

Hessian matrix: $Hf(x,y) = \begin{pmatrix} 12x^2 & -12y^2 \\ -12y^2 & -24xy \end{pmatrix}$.

b) **(3 points)**

Stationary points: $f_x = f_y = 0$.

$f_x(x,y) = 4x^3 - 4y^3 = 0 \iff x = y$,

$f_y(x,y) = -12xy^2 + 12 = 0$ and $x = y \iff y^3 = 1$

So there is exactly one stationary point, namely $P := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For the Hessian matrix $Hf(P) = Hf(1,1)$ one obtains

det $Hf(1,1) = \begin{pmatrix} 12 & -12 \\ -12 & -24 \end{pmatrix} = -12 \cdot 24 - 12 \cdot 12 < 0.$

The matrix has one positive eigenvalue and one negative eigenvalue. So it is a saddle point.

Exercise 2: (4 points)

The equation

$$f(x,y) := x^2 - x^2 y + \frac{y^3}{3} - 1 = 0.$$

is an implicit definition of a curve in $\mathbb{R}^2$.
Show that the implicit function theorem gives us a function g, such that in the neighbourhood of $P_0 := \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ the following equivalence holds

$$f(x,y) = 0 \iff y = g(x), \qquad g(2) = 3\,.$$

Compute the Taylor polynomial of the first degree (the tangent) of g centered at the point $x_0 = 2$.

Solution 2:

$$f(2,3) := 2^2 - 2^2 \cdot 3 + \frac{3^3}{3} - 1 = 0$$

$$f_y(x,y) = -x^2 + y^2, \qquad f_y(2,3) = -2^2 + 3^2 \neq 0$$

By the implicit function theorem, with a suitable function g locally:

$$f(x,y) = 0 \iff y = g(x),\; g(2) = 3\,, \qquad g'(x) = -\frac{f_x}{f_y}\,.$$

$$f_x(x,y) = 2x - 2xy, \qquad f_x(2,3) = 4 - 12 = -8\,.$$

For the linearization one computes $\quad g'(2) = -\dfrac{4 - 4 \cdot 3}{-2^2 + 3^2} = \dfrac{8}{5}\,.$

So we obtain $\quad T_1(x) = g(2) + g'(2)(x-2) = 3 + \dfrac{8}{5}(x-2)\,.$

Exercise 3: (5+2 points)

a) Given

$$D := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,:\, 0 \leq x^2 + y^2 \leq 25,\, x \geq 0,\, y \geq 0 \right\}$$

and a vector field

$$\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2,\ \boldsymbol{f}(x, y) = \begin{pmatrix} -x^2 y + e^{\tan(x)} \\ xy^2 + \tan(e^y) \end{pmatrix},$$

compute curl $f(x, y)$ and the integral $\int_{\partial D} \boldsymbol{f}(x, y)\, d(x, y)$, where ∂D denotes positively oriented boundary of D.

b) Let f be a vector field

$$\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3,\ \boldsymbol{f}(x, y, z) = \begin{pmatrix} y^2 + z^2 + 2xz \\ x^2 + z^2 - 2yz \\ x^2 + y^2 - 2xy \end{pmatrix}.$$

Compute div $\boldsymbol{f}(x, y, z)$ and the flux (flow) of $\boldsymbol{f}$ through the surface of the sphere

$$K := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,:\, 0 \leq (x-1)^2 + (y-2)^2 + (z+3)^2 \leq 1 \right\}.$$

Solution sketch

a) **(5 points)**

$$\text{rot } \boldsymbol{f}(x, y) = (f_2)_x - (f_1)_y = y^2 + x^2. \qquad \textbf{(1 point)}$$

From Green's theorem we have:

$$\int_{\partial D} \boldsymbol{f}(x, y)\, d(x, y) = \int_D \text{rot} f(x, y)\, d(x, y) \qquad \textbf{(Ansatz: 1 point)}$$

and with $x = r\cos\phi,\ y = r\sin\phi,\ 0 \leq r \leq 5,\ 0 \leq \phi \leq \frac{\pi}{2}$ we obtain

$$\text{rot } \boldsymbol{f}(x, y) = (f_2)_x - (f_1)_y = y^2 + x^2 = r^2 \qquad \textbf{(2 points)}$$

$$\int_0^5 \int_0^{\frac{\pi}{2}} r^2 \cdot r\, d\phi\, dr = \frac{\pi}{2} \int_0^5 r^3\, dr = \frac{\pi}{2} \cdot \frac{5^4}{4} \qquad \textbf{(1 point)}$$

b) **(2 points)**

$$\text{div } \boldsymbol{f}(x, y, z) = 2z - 2z + 0 = 0$$

From Gauss' theorem it follows that the flux through the surface of the specified sphere is zero.

Exercise 4: (4 points)

Given a function

$$\boldsymbol{f} \,:\, \mathbb{R}^3 \to \mathbb{R}^3, \qquad \boldsymbol{f}\,(x,y,z) \;=\; (-xy\;,\; x^2\;,\; z)^T$$

und the curve

$$\boldsymbol{c} \;:\; [0\,,\,2\pi] \to \mathbb{R}^3, \qquad \boldsymbol{c}\,(t) \;=\; (2\cos(t),\, 2\sin(t),\, t)^T.$$

Compute the line integral

$$\int_{\boldsymbol{c}} \;\boldsymbol{f}\,(x,y,z)$$

Solution sketch:

(4 points)

$$\begin{aligned}
\int_c \boldsymbol{f}(x,y,z)d(x,y,z) \;&=\; \int_0^{2\pi} <\, \boldsymbol{f}(c(t)),\dot{c}(t)\,>\,dt \\
&= \int_0^{2\pi} < \begin{pmatrix} -4\sin(t)\,\cos(t) \\ 4\cos^2(t) \\ t \end{pmatrix} \,,\, \begin{pmatrix} -2\sin(t) \\ 2\cos(t) \\ 1 \end{pmatrix} > dt \qquad \textbf{(2 points)} \\
&= \int_0^{2\pi} 8\sin^2(t)\,\cos(t) \,+\, 8\cos^2(t)\cos(t) + t\,dt \\
&= \int_0^{2\pi} t \,+\, 8\cos(t)\,\big(\cos^2(t)\,+\sin^2(t)\big)\;dt \\
&= \int_0^{2\pi} t \,+\, 8\cos(t)\,dt \\
&= \left[\frac{t^2}{2} - 8\sin(t)\right]_0^{2\pi} \;=\; 2\pi^2\,.\textbf{(2 Punkte)}
\end{aligned}$$