Mathematics Department
Prof. Dr. I. Gasser

SoSe 2022

Mathematik III Exam
(Modul: Analysis III)

06. September 2022

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Task no.	Points	Evaluater
1		
2		
3		

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Exercise 1: (3+1 points)

A local minimum of the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) := x^2 + 4y^2 - 6x + 24y + 6 .$$

subject to the constraint

$$g(x,y) := \cos(\frac{x-3}{2}) + \sin(y+1) - 1 = 0$$

is sought.

$P_0 = (3, -1)^T$ is an admissible point for which the regularity condition is satisfied. This information may be used without proof.

a) Show that P_0 is a stationary point of the corresponding Lagrangian function for a suitable multiplier.

b) Show that $P_0 = (3, -1)^T$ is a local minimum of the function f subject to the constraint $g = 0$ by investigating the sufficient condition of second order.

Solution:

a) **(3 points)** $F := f + \lambda g$.

grad $F(x,y) = (F_x(x,y),\ F_y(x,y))$.

$$F_x(x,y) = 2x - 6 + \lambda(-\sin(\tfrac{x-3}{2})\tfrac{1}{2}),$$
$$F_y(x,y) = 8y + 24 + \lambda\cos(y+1).$$

$$F_x(3,-1) = 6 - 6 - \lambda \cdot 0 = 0,$$

$$F_y(3,-1) = -8 + 24 + \lambda \cdot 1 = 0 \iff \lambda = -16.$$

Hence, P_0 is a stationary point of the function $F := f - 16g$.

b) The Hessian matrix of $F := f + \lambda g$ is

$$HF(x,y;\lambda) = \begin{pmatrix} 2 - \lambda(\cos(\frac{x-3}{2})\frac{1}{4}) & 0 \\ 0 & 8 - \lambda\sin(y+1) \end{pmatrix}.$$

For $\lambda = -16$ we obtain:

$$HF(3,-1) = \begin{pmatrix} 2+4 & 0 \\ 0 & 8 \end{pmatrix}.$$

This matrix has two positive eigenvalues. Therefore, P_0 is a (local) minimum.

Exercise 2: (3+3 points)

Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) := y\cos(x) + x\sin(y) + 2.$$

a) Determine the second-degree Taylor polynomial T_2 of f at the point $(x_0, y_0) = (0, 0)$.

b) Show that

$$|f(x, y) - T_2(x, y)| \le \frac{4}{100}$$

for all $(x, y) \in D := [-0.3,\, 0.3] \times [-0.3,\, 0.3]$.

Solution 2:

a) **(3 points)**

$$\begin{aligned} f(x,y) &= y\cos(x) + x\sin(y) + 2 & f(0,0) &= 2\\ f_x(x,y) &= -y\sin(x) + \sin(y) & f_x(0,0) &= 0\\ f_y(x,y) &= \cos(x) + x\cos(y) & f_y(0,0) &= 1\\ f_{xx}(x,y) &= -y\cos(x) & f_{xx}(0,0) &= 0\\ f_{xy}(x,y) &= -\sin(x) + \cos(y) & f_{xy}(0,0) &= 1\\ f_{yy}(x,y) &= -x\sin(y) & f_{yy}(0,0) &= 0 \end{aligned}$$

$$T_2(x, y) = 2 + y + \frac{1}{2}\,(2xy) = 2 + y + xy.$$

b) **(3 points)**

For the estimated approximation error, we compute an upper bound for the absolute value of all partial derivatives of order three that holds true for all $(x, y) \in D$.

$$\begin{aligned} |\, f_{xxx}(x,y)\,| &= |y\sin(x)\,| \le |y| \cdot |\sin(x)| \le \frac{3}{10}\\ |\, f_{xxy}(x,y)\,| &= |-\cos(x)| \le 1\\ |\, f_{xyy}(x,y)\,| &= |-\sin(y)| \le 1\\ |\, f_{yyy}(x,y)\,| &= |-x\cos(y)\,| \le \frac{3}{10}. \end{aligned}$$

The absolute values of all partial derivatives of order three of f are therefore bounded from above by 1 in all points in D.

The approximation error $|f(x, y) - T_2(x, y)|$ can be estimated as follows:

$$|f(x,y) - T_2(x,y)| \le \frac{2^3}{3!} \cdot \|(x,y)\|_\infty^3 \cdot C \le \frac{8}{6} \cdot \frac{3^3}{10^3} \cdot 1 = \frac{4 \cdot 9}{10^3} < \frac{40}{1000} = \frac{4}{100}.$$

Exercise 3: (5+1+3+1 points)

Consider the half ball $K := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \, : \, x^2 + y^2 + z^2 \le 4 \, , \; z \le 0 \right\}$

and the vector field $\boldsymbol{f}\,(x, y, z) = \begin{pmatrix} xz + x \\ yz + y \\ x^2 + y^2 \end{pmatrix}$.

a) Compute the integral $\displaystyle\int_K \text{div } \boldsymbol{f}\,(x, y, z)\, d(x, y, z)$.

Hint: $2\sin(\alpha)\cos(\alpha) = \sin(2\alpha)$.

b) The solid K is bounded by a flat surface D and a non-flat surface M . State a parametrization of the flat surface D .

c) Compute the flux (flow) of $\boldsymbol{f}$ through the flat surface D .

d) According to a) and c), what is the flux (flow) of $\boldsymbol{f}$ through the non-flat surface M ?

Solution sketch

a) [**5 points**]

div $\boldsymbol{f}\,(x, y, z) = z + 1 + z + 1 + 0 = 2z + 2$. **(1 point)**

To compute the integral, we use spherical coordinates

$x = r\cos(\phi)\cos(\theta)\,, \quad y = r\sin(\phi)\cos(\theta)\,, \quad z = r\sin(\theta)\,,$

with

$0 \le r \le 2$, $0 \le \phi \le 2\pi$, $-\frac{\pi}{2} \le z \le 0$ **(1 point)**

and obtain

$$\int_K \text{div}\,(x, y, z)\, d(x, y, z) = \int_0^2 \int_{-\frac{\pi}{2}}^0 \int_0^{2\pi} (2r\sin(\theta) + 2) \cdot r^2 \cos(\theta)\, d\phi\, d\theta\, dr \qquad \textbf{(1 point)}$$

$$= \int_0^2 \int_{-\frac{\pi}{2}}^0 \left(2r^3 \sin(\theta)\cos(\theta) + 2r^2\cos(\theta)\right) [\phi]_0^{2\pi}\, d\theta\, dr$$

$$= 2\pi \int_0^2 \int_{-\frac{\pi}{2}}^0 \left(r^3 \sin(2\theta) + 2r^2\cos(\theta)\right) d\theta\, dr$$

$$= 2\pi \int_0^2 \left[-r^3 \frac{\cos(2\theta)}{2} + 2r^2 \sin(\theta)\right]_{-\frac{\pi}{2}}^0 dr$$

$$= 2\pi \int_0^2 -r^3 \frac{1 - (-1)}{2} + 2r^2(0 - (-1))\, dr$$

$$= 2\pi \int_0^2 -r^3 + 2r^2\, dr$$

$$= 2\pi \left[-\frac{r^4}{4} + \frac{2r^3}{3}\right]_0^2 = 2\pi\left(-4 + \frac{16}{3}\right) = \frac{8\pi}{3}$$

(Computation 2 points)

b) **[1 point]**

The solid is bounded by a flat surface D (as in Deckel, German for lid), which is parametrized by

$$p(r,\phi) := \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \\ 0 \end{pmatrix}, \qquad r \in [0,2], \quad \phi \in [0,2\pi],$$

as well as the lower half of the balls' surface M .

c) **[3 points]**

For the flux through D, one computes:

$$\frac{\partial p}{\partial r} = \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{pmatrix} \qquad \frac{\partial p}{\partial \phi} = \begin{pmatrix} -r\sin(\phi) \\ r\cos(\phi) \\ 0 \end{pmatrix}$$

$$\frac{\partial p}{\partial r} \times \frac{\partial p}{\partial \phi} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \qquad f(p(r,\phi)) = \begin{pmatrix} \text{irrelevant} \\ \text{irrelevant} \\ r^2 \end{pmatrix}$$

$$< f, \frac{\partial p}{\partial r} \times \frac{\partial p}{\partial \phi} >= r^3 .$$

$$\int_0^2 \int_0^{2\pi} < f, \frac{\partial p}{\partial r} \times \frac{\partial p}{\partial \phi} > d\phi dr$$

$$= \int_0^2 \int_0^{2\pi} r^3 \, d\phi dr = 2\pi \int_0^2 r^3 \, dr$$

$$2\pi \frac{2^4}{4} = 8\pi .$$

d) **[1 Punkt]**

According to Gauß' theorem, we have:

Total flux through boundary of K = flux through D + flux through M
$= \int_K \operatorname{div} \boldsymbol{f}\,(x,y,z)\, d(x,y,z)$

Therefore, the flux through the non-flat surface M is
$\frac{8\pi}{3} - 8\pi = -\frac{16\pi}{3}$.