Fachbereich Mathematik der Universität Hamburg **SoSe 2021**
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Analysis II für Studierende der Ingenieurwissenschaften

Lösungen zu Blatt 2

Aufgabe 5:

Man berechne alle Stammfunktionen zu

a) $f_1(x) = -3x^4 + 10\cosh x$, b) $f_2(x) = 6\sin x - 8e^x$, c) $f_3(x) = \dfrac{2}{1+x^2} + \dfrac{3}{x}$,

d) $f_4(x) = \dfrac{3x - 9x^2\cos x}{x^2}$, e) $f_5(x) = \dfrac{6x^4 + 8x^2 - 10}{\sqrt[3]{x}}$.

Lösung:

a) $$\int -3x^4 + 10\cosh x\, dx \;=\; -\frac{3x^5}{5} + 10\sinh x + C\,,$$

b) $$\int 6\sin x - 8e^x\, dx \;=\; -6\cos x - 8e^x + C\,,$$

c) $$\int \frac{2}{1+x^2} + \frac{3}{x}\, dx \;=\; 2\arctan x + 3\ln|x| + C$$

d) $$\int \frac{3x - 9x^2\cos x}{x^2}\, dx \;=\; \int \frac{3}{x} - 9\cos x\, dx \;=\; 3\ln|x| - 9\sin x + C\,,$$

e) $$\int \frac{6x^4 + 8x^2 - 10}{\sqrt[3]{x}}\, dx \;=\; \int 6x^{11/3} + 8x^{5/3} - 10x^{-1/3}\, dx$$
$$= \frac{9}{7}x^{14/3} + 3x^{8/3} - 15x^{2/3} + C$$

Aufgabe 6:

Mit Hilfe der partiellen Integrationsregel berechne man:

a) $\int (2x+5)\cos x\,dx$, b) $\int (x^2-x+2)\sinh x\,dx$, c) $\int 9t^2 \ln t\,dt$,

d) $\int e^x \sin x\,dx$, e) $\int 105t^2\sqrt{t+2}\,dt$, f) $\int \cot x\,dx$.

Lösung:

a) partielle Integration: $u = 2x+5\,,\ v' = \cos x$

$$\int (2x+5)\cos x dx = (2x+5)\sin x - \int 2\sin x dx + C = (2x+5)\sin x + 2\cos x + C$$

b) partielle Integration: $u = x^2-x+2\,,\ v' = \sinh x$

$$\int (x^2-x+2)\sinh x\,dx = (x^2-x+2)\cosh x - \int (2x-1)\cosh x\,dx + C$$

weitere partielle Integration: $u = 2x-1\,,\ v' = \cosh x$

$$= (x^2-x+2)\cosh x - ((2x-1)\sinh x - \int 2\sinh x\,dx) + C$$

$$= (x^2-x+4)\cosh x - (2x-1)\sinh x + C$$

c) partielle Integration: $u = \ln t\,,\ v' = 9t^2$

$$\int 9t^2 \ln t\,dt = 3t^3\ln t - \int \frac{3t^3}{t}\,dt = 3t^3\ln t - t^3 + C\,,$$

d) partielle Integration: $u = e^x\,,\ v' = \sin x$

$$\int e^x \sin x\,dx = -e^x\cos x + \int e^x\cos x\,dx$$

weitere partielle Integration: $u = e^x\,,\ v' = \cos x$

$$= -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx + \tilde{C}$$

$$\Rightarrow \quad \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

e) partielle Integration: $u = 105t^2$, $v' = \sqrt{t+2}$

$$\begin{aligned}\int 105t^2\sqrt{t+2}\,dt &= \frac{2\cdot 105t^2}{3}(t+2)^{3/2} - \int 210t\frac{2}{3}(t+2)^{3/2}\,dt + C\\ &= 70t^2(t+2)^{3/2} - \int 140t(t+2)^{3/2}\,dt + C\end{aligned}$$

weitere partielle Integration: $u = 140t$, $v' = (t+2)^{3/2}$

$$\begin{aligned}\int 105t^2\sqrt{t+2}\,dt &= 70t^2(t+2)^{3/2} - \frac{2\cdot 140t}{5}(t+2)^{5/2} + \int 140\frac{2}{5}(t+2)^{5/2}\,dt + C\\ &= 70t^2(t+2)^{3/2} - 56t(t+2)^{5/2} + \frac{2\cdot 56}{7}(t+2)^{7/2} + C\\ &= 70t^2(t+2)^{3/2} - 56t(t+2)^{5/2} + 16(t+2)^{7/2} + C\\ &= 2(t+2)^{3/2}(35t^2 - 28t(t+2) + 8(t+2)^2) + C\\ &= 2(t+2)^{3/2}(35t^2 - 28t^2 - 56t + 8t^2 + 32t + 32) + C\\ &= 2(t+2)^{3/2}(15t^2 - 24t + 32) + C\end{aligned}$$

f) $\int \cot x\,dx = \int \cos x\cdot(\sin x)^{-1}\,dx$

partielle Integration: $u = (\sin x)^{-1}$, $v' = \cos x$

$$\begin{aligned}\int \cot x\,dx &= \int \cos x\cdot(\sin x)^{-1}\,dx = \sin x(\sin x)^{-1} + \int \sin x(\sin x)^{-2}\cos x\,dx + C\\ &= 1 + \int \cot x\,dx + C\end{aligned}$$

$$\Rightarrow\quad 0 = 1 + C \quad\Rightarrow\quad C = -1$$

Mit partieller Integration kann keine Stammfunktion gefunden werden.

Aufgabe 7:

Mit Hilfe der Substitutionsregel berechne man:

a) $\int \cos x \sin^3 x \, dx$, b) $\int 6x^2\sqrt{8+x^3}\, dx$, c) $\int 8xe^{x^2}\, dx$,

d) $\int \frac{24x \ln^3(x^2+1)}{x^2+1}\, dx$, e) $\int \frac{e^{2x}-e^x}{e^{2x}-1}\, dx$, f) $\int \cot x \, dx$.

Lösung:

a) Substitution: $s = \sin x \;\to\; ds = \cos x \, dx$

$$\int \cos x \sin^3 x \, dx = \int s^3 \, ds = \frac{s^4}{4} + C = \frac{\sin^4 x}{4} + C\,.$$

b) Substitution: $t = 8 + x^3 \;\to\; dt = 3x^2 dx$ erhält man

$$\int 6x^2\sqrt{8+x^3}\, dx = \int 2\sqrt{t}\, dt = \frac{4}{3}t^{3/2} + C = \frac{4}{3}(8+x^3)^{3/2} + C\,,$$

c) Substitution: $t = x^2 \;\to\; dt = 2xdx$ erhält man

$$\int 8xe^{x^2}\, dx = \int 4e^t\, dt = 4e^t + C = 4e^{x^2} + C\,,$$

d) Substitution: $t = x^2 + 1 \;\to\; dt = 2x\, dx$

$$\int \frac{24x \ln^3(x^2+1)}{x^2+1}\, dx \;=\; \int \frac{12\ln^3(t)}{t}\, dt$$

weitere Substitution: $u = \ln t \;\to\; du = \frac{1}{t}dt$

$$= \int 12u^3\, du = 3u^4 + C = 3\ln^4 t + C = 3\ln^4(x^2+1) + C$$

e) Substitution: $t = e^x \;\rightarrow\; \frac{dt}{dx} = e^x \;\rightarrow\; dx = \frac{dt}{t}$

$$\int \frac{e^{2x} - e^x}{e^{2x} - 1}\, dx = \int \frac{t^2 - t}{t^2 - 1}\, \frac{dt}{t} = \int \frac{t - 1}{t^2 - 1}\, dt = \int \frac{1}{t + 1}\, dt$$

weitere Substitution: $u = t + 1 \;\rightarrow\; du = dt$

$$= \int \frac{1}{u}\, du = \ln|u| + C = \ln|t + 1| + C = \ln|e^x + 1| + C$$

f) $\int \cot x \, dx = \int \frac{\cos x}{\sin x}\, dx$

Substitution: $u = \sin x \;\rightarrow\; du = \cos x \, dx$

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln|\sin x| + C$$

Aufgabe 8:

Man berechne die unbestimmten Integrale

a) $\int (x+1)e^{1-x}\,dx$, b) $\int x\sqrt{2x+1}\,dx$, c) $\int \cos^2 t\,dt$,

d) $\int \sqrt{4-x^2}\,dx$, e) $\int \frac{\sin^3 t}{\cos^2 t}\,dt$, f) $\int \arctan x\,dx$.

Lösung:

a) partielle Integration: $u = x+1\,,\ \ v' = e^{1-x}$

$$\int (x+1)e^{1-x}\,dx = -(x+1)e^{1-x} - \int -e^{1-x}\,dx + C = -(x+1)e^{1-x} - e^{1-x} + C\,,$$

b) partielle Integration: $u = x$ und $v' = \sqrt{2x+1}$

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \int x(2x+1)^{1/2}\,dx \\ &= x\cdot\frac{2}{2\cdot 3}(2x+1)^{3/2} - \frac{2}{2\cdot 3}\int (2x+1)^{3/2}\,dx + C \\ &= \frac{x}{3}(2x+1)^{3/2} - \frac{1}{15}(2x+1)^{5/2} + C \\ &= (2x+1)^{3/2}\left(\frac{x}{3} - \frac{2x+1}{15}\right) + C = (2x+1)^{3/2}\cdot\frac{3x-1}{15} + C\end{aligned}$$

Alternative, Substitution: $u = \sqrt{2x+1} \Rightarrow x = \frac{u^2-1}{2} \to dx = u\,du$

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \int \frac{u^2-1}{2}\cdot u\cdot u\,du = \int \frac{u^4-u^2}{2}\,du = \frac{u^5}{10} - \frac{u^3}{6} + C \\ &= \frac{3u^5-5u^3}{30} + C = \frac{u^3}{30}(3u^2-5) + C = (2x+1)^{3/2}\cdot\frac{3x-1}{15} + C\end{aligned}$$

c) $\cos^2 t + \sin^2 t = 1$ und partielle Integration: $u = \cos t\,,\ \ v' = \cos t$

$$\begin{aligned}\int \cos^2 t\,dt &= \int \cos t\cos t\,dt = \cos t\sin t - \int -\sin t\sin t\,dt + \tilde{C} \\ &= \cos t\sin t + \int 1 - \cos^2 t\,dt + \tilde{C} \quad \Rightarrow\end{aligned}$$

$$2\int \cos^2 t\,dt \;=\; t+\cos t\sin t+\tilde{C} \;\Rightarrow\; \int \cos^2 t\,dt = \frac{1}{2}\,(t+\cos t\sin t)+C$$

d) Substitution: $x = 2\sin t \;\Rightarrow\; dx = 2\cos t\,dt$ und $t = \arcsin(x/2)$

$$\begin{aligned}
\int \sqrt{4-x^2}\,dx &= \int \sqrt{4-4\sin^2 t}\,2\cos t\,dt = 4\int \sqrt{1-\sin^2 t}\,\cos t\,dt \\
&= 4\int \cos^2 t\,dt = 2\,(t+\cos t\sin t)+C \\
&= 2\left(\arcsin\left(\frac{x}{2}\right)+\frac{x}{2}\sqrt{1-\left(\frac{x}{2}\right)^2}\right)+C \\
&= 2\arcsin\left(\frac{x}{2}\right)+\frac{x}{2}\sqrt{4-x^2}+C
\end{aligned}$$

e) $\cos^2 t+\sin^2 t = 1$ und Substitution: $x=\cos t \;\Rightarrow\; dx = -\sin t\,dt$

$$\begin{aligned}
\int \frac{\sin^3 t}{\cos^2 t}\,dt &= \int \frac{(1-\cos^2 t)}{\cos^2 t}\sin t\,dt = -\int \frac{(1-x^2)}{x^2}\,dx \\
&= \frac{1}{x}+x+C = \frac{1}{\cos t}+\cos t+C
\end{aligned}$$

f) $\displaystyle\int \arctan x\,dx = \int 1\cdot\arctan x\,dx$

partielle Integration: $u' = 1\,,\;\; v = \arctan x \;\Rightarrow\; v' = \dfrac{1}{1+x^2}$

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx + C$$

Substitution: $u = 1+x^2 \;\rightarrow\; du = 2x\,dx \;\rightarrow\; \dfrac{du}{2} = x\,dx$

$$\begin{aligned}
\int \arctan x\,dx &= x\arctan x - \int \frac{1}{2u}\,du + C \\
&= x\arctan x - \frac{1}{2}\ln|u| + C = x\arctan x - \frac{1}{2}\ln|1+x^2| + C
\end{aligned}$$

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