Fachbereich Mathematik der Universität Hamburg **SoSe 2021**
Dr. K. Rothe

Analysis II

für Studierende der Ingenieurwissenschaften

Hörsaalübung mit Beispielaufgaben zu Blatt 2

Definition

Gegeben sei die Funktion
$$f : [a, b] \to \mathbb{R}$$
und die differenzierbare Funktion
$$F : [a, b] \to \mathbb{R}.$$

Man nennt man F **Stammfunktion** von f,
wenn gilt
$$F' = f \ .$$

Ist F eine Stammfunktion von f, dann heißt
$$\int f(x)\, dx := F + C \ , \quad C \in \mathbb{R}, \quad C = \text{konst.}$$

unbestimmtes Integral von f.

Die Funktion f heißt **Integrand** und C **Integrationskonstante**.

Tabelle einiger Stammfunktionen:

$f(x)$	$F(x) = \int f(x)\,dx + C\,,$	$C \in \mathbb{R}$
a	$ax + C\,,$	$a \in \mathbb{R}$
x	$\dfrac{x^2}{2} + C$	
x^α	$\dfrac{x^{\alpha+1}}{\alpha+1} + C\,,$	$\alpha \in \mathbb{R}\backslash\{-1\}$
e^x	$e^x + C$	
$\dfrac{1}{x}$	$\ln\|x\| + C$	
$\sin x$	$-\cos x + C$	
$\cos x$	$\sin x + C$	
$\tan x$	$-\ln\|\cos x\| + C$	
$\dfrac{1}{1+x^2}$	$\arctan x + C$	
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$	
$\sinh x$	$\cosh x + C$	
$\cosh x$	$\sinh x + C$	

Integrationsregeln

Satz: (Linearität des unbestimmten Integrals)

Für $f, g : [a, b] \to \mathbb{R}$ seien Stammfunktionen bekannt.

Mit den Konstanten $\alpha, \beta \in \mathbb{R}$ gilt dann

$$\int \alpha f(x) + \beta g(x)\,dx \;=\; \alpha \int f(x)dx + \beta \int g(x)\,dx + C\,, \quad \alpha, \beta \in \mathbb{R}\,.$$

Satz: (Substitutionsregel)

Sei f stetig auf dem Intervall $[a,b]$

und g stetig differenzierbar auf dem Intervall $[c,d]$

und es existiere die Umkehrfunktion g^{-1} zu g,

dann gilt mit der Substitution $x = g(t)$ und der

Merkregel: $\dfrac{dx}{dt} = g'(t) \leftrightarrow dx = g'(t)dt$

a)
$$\begin{aligned}\int f(g(t))g'(t)\,dt &= \int f(x)\,dx\,, && \text{Substitution}\\ &= F(x)+C\,, && \text{Stammfunktion}\\ &= F(g(t))+C\,, && \text{Rücksubstitution}\end{aligned}$$

b)
$$\begin{aligned}\int f(x)\,dx &= \int f(g(t))g'(t)\,dt\,, && \text{Substitution}\\ &=: \int \tilde{f}(t)\,dt\\ &= \tilde{F}(t)+C\,, && \text{Stammfunktion}\\ &= \tilde{F}(g^{-1}(x))+C\,, && \text{Rücksubstitution}\\ &=: F(x)+C\end{aligned}$$

Satz: (partielle Integrationsregel)

Für stetig differenzierbare Funktionen $u, v : [a,b] \to \mathbb{R}$ gilt

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx + C\,.$$

Aufgabe 5:

Man berechne alle Stammfunktionen zu

a) $f_1(x) = 2x^5 - 5\sin x\,,$ b) $f_2(x) = 4\cos x - 7\sinh x$

c) $f_3(x) = \dfrac{2 + xe^x}{x}\,,$ d) $f_4(x) = \dfrac{3x^5 - 5x^3 + 4x}{\sqrt{x}}\,.$

Lösung:

a) $$\int 2x^5 - 5\sin x\,dx = \frac{x^6}{3} + 5\cos x + C\,,$$

b) $$\int 4\cos x - 7\sinh x\,dx = 4\sin x - 7\cosh x + C\,,$$

c) $$\int \frac{2 + xe^x}{x}\,dx = \int \frac{2}{x} + e^x\,dx = 2\ln|x| + e^x + C\,,$$

d) $$\int \frac{3x^5 - 5x^3 + 4x}{\sqrt{x}}dx = \int 3x^{9/2} - 5x^{5/2} + 4x^{1/2}dx = \frac{6}{11}x^{11/2} - \frac{10}{7}x^{7/2} + \frac{8}{3}x^{3/2} + C$$

Aufgabe 6:

Mit Hilfe der partiellen Integrationsregel berechne man

a) $\displaystyle\int (3x-1)\cosh x\,dx$, b) $\displaystyle\int x\ln x\,dx$, c) $\displaystyle\int x^2\cos x\,dx$,

d) $\displaystyle\int \cos t\sinh t\,dt$, e) $\displaystyle\int 15x\sqrt{x-1}\,dx$.

Lösung:

a) partielle Integration: $u = 3x-1$, $v' = \cosh x$

$$\begin{aligned}\int (3x-1)\cosh x\,dx &= (3x-1)\sinh x - \int 3\sinh x\,dx + C\\ &= (3x-1)\sinh x - 3\cosh x + C\end{aligned}$$

b) partielle Integration: $u' = x$, $v = \ln x$

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

c) partielle Integration: $u = x^2$, $v' = \cos x$

$$\int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx + C$$

weitere partielle Integration: $u = 2x$, $v' = \sin x$

$$\begin{aligned}&= x^2\sin x + 2x\cos x - \int 2\cos x\,dx + C\\ &= x^2\sin x + 2x\cos x - 2\sin x + C\end{aligned}$$

d) partielle Integration: $u = \sinh t$, $v' = \cos t$

$$\int \cos t\sinh t\,dt = \sin t\sinh t - \int \sin t\cosh t\,dt + \tilde{C}$$

weitere partielle Integration: $u = \cosh t$, $v' = \sin t$

$$= \sin t\sinh t - (-\cos t\cosh t - \int -\cos t\sinh t\,dt) + \tilde{C}$$

$$= \sin t\sinh t + \cos t\cosh t - \int \cos t\sinh t\,dt + \tilde{C}$$

$$\Rightarrow \quad \int \cos t\sinh t\,dt = \frac{\sin t\sinh t + \cos t\cosh t}{2} + C,$$

e) partielle Integration: $u = 15x\ ,\ v' = \sqrt{x-1}$

$$\begin{aligned}
\int 15x\sqrt{x-1}\,dx &= \frac{2\cdot 15x}{3}(x-1)^{3/2} - \int \frac{2\cdot 15}{3}(x-1)^{3/2}\,dx + C \\
&= 10x(x-1)^{3/2} - \int 10(x-1)^{3/2}\,dx + C \\
&= 10x(x-1)^{3/2} - \frac{20}{5}(x-1)^{5/2} + C \\
&= (x-1)^{3/2}(10x - 4(x-1)) + C \\
&= 2(x-1)^{3/2}(3x+2) + C
\end{aligned}$$

Aufgabe 7:

Mit Hilfe der Substitutionsregel berechne man

a) $\displaystyle\int \sin(x)\cos^2(x)\,dx$, b) $\displaystyle\int 2x\sqrt{x^2+1}\,dx$, c) $\displaystyle\int x^2 e^{x^3}\,dx$,

d) $\displaystyle\int \frac{(\ln(2t+3))^4}{6t+9}\,dt$, e) $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$, f) $\displaystyle\int \tan x\,dx$.

Lösung:

a) Substitution: $u = \cos x \;\rightarrow\; du = -\sin(x)\,dx$

$$\int \sin(x)\cos^2(x)\,dx \;=\; -\int u^2\,du \;=\; -\frac{u^3}{3} + C \;=\; -\frac{\cos^3 x}{3} + C$$

b) Substitution: $s = x^2 + 1 \rightarrow ds = 2x\,dx$

$$\int 2x\sqrt{x^2+1}\,dx = \int \sqrt{s}\,ds = \frac{2s^{3/2}}{3} + C = \frac{2(x^2+1)^{3/2}}{3} + C\,.$$

c) Substitution: $t = x^3 \;\rightarrow\; dt = 3x^2\,dx \;\rightarrow\; dx = \dfrac{dt}{3x^2}$

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^t\,dt = \frac{1}{3}e^t + C = \frac{1}{3}e^{x^3} + C\,.$$

d) Substitution: $x = 2t + 3 \;\rightarrow\; dx = 2\,dt \;\rightarrow\; dt = \dfrac{dx}{2}$

$$\int \frac{(\ln(2t+3))^4}{6t+9}\,dt = \frac{1}{2}\int \frac{(\ln(x))^4}{3x}\,dx = \frac{1}{6}\int \frac{1}{x}\cdot(\ln(x))^4\,dx$$

weitere Substitution: $u = \ln x \;\rightarrow\; du = \dfrac{1}{x}\,dx$

$$= \frac{1}{6}\int u^4\,du = \frac{u^5}{30} + C = \frac{(\ln x)^5}{30} + C = \frac{(\ln(2t+3))^5}{30} + C$$

e) Substitution: $t = e^x \;\rightarrow\; \dfrac{dt}{dx} = e^x \;\rightarrow\; dx = \dfrac{dt}{t}$

$$\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{t}{t^2+1}\,\frac{dt}{t} = \int \frac{1}{t^2+1}\,dt = \arctan t + C = \arctan e^x + C$$

f) $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Substitution: $t = \cos x \;\rightarrow\; dt = -\sin x\,dx$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{t}\,dt = -\ln|t| + C = -\ln|\cos x| + C$$

Aufgabe 8:

Man berechne die unbestimmten Integrale

$$\text{a)} \int x e^{5x-2}\,dx\ , \qquad \text{b)} \int \frac{x}{\sqrt{x+4}}\,dx\ , \qquad \text{c)} \int \cosh^2 t\,dt\ ,$$

$$\text{d)} \int \sqrt{1+x^2}\,dx\ , \qquad \text{e)} \int \sin^3 t \cos^3 t\,dt\ , \qquad \text{f)} \int \arcsin x\,dx\ .$$

Lösung:

a) partielle Integration: $u = x,\ v' = e^{5x-2}$

$$\int x e^{5x-2}\,dx = \frac{xe^{5x-2}}{5} - \int \frac{e^{5x-2}}{5}\,dx + C = \frac{xe^{5x-2}}{5} - \frac{e^{5x-2}}{25} + C\ ,$$

b) partielle Integration: $u = x$ und $v' = \dfrac{1}{\sqrt{x+4}}$

$$\begin{aligned}\int \frac{x}{\sqrt{x+4}}\,dx &= x \cdot 2\sqrt{x+4} - 2\int \sqrt{x+4}\,dx + C \\ &= 2x\sqrt{x+4} - \frac{4}{3}(x+4)^{3/2} + C \ = \ \frac{2}{3}(x-8)\sqrt{x+4} + C\end{aligned}$$

Alternative, Substitution:

$u = \sqrt{x+4} \ \Rightarrow\ x = u^2 - 4, \quad dx/du = 2u \to dx = 2u\,du$

$$\begin{aligned}\int \frac{x}{\sqrt{x+4}}\,dx &= \int \frac{u^2-4}{u} \cdot 2u\,du \ = \ 2\int u^2 - 4\,du \\ &= 2\left(\frac{u^3}{3} - 4u\right) + C = \frac{2u}{3}\left(u^2 - 12\right) + C = \frac{2}{3}(x-8)\sqrt{x+4} + C\end{aligned}$$

c) Additionstheorem: $\cosh^2 t - \sinh^2 t = 1$ und

partielle Integration: $u = \cosh t,\ v' = \cosh t$

$$\begin{aligned}\int \cosh^2 t\,dt &= \int \cosh t \cosh t\,dt \ = \cosh t \sinh t - \int \sinh t \sinh t\,dt + \tilde{C} \\ &= \cosh t \sinh t - \int \cosh^2 t - 1\,dt + \tilde{C} \qquad \Rightarrow\end{aligned}$$

$$2\int \cosh^2 t\, dt = t + \cosh t \sinh t + \tilde{C} \quad \Rightarrow$$

$$\int \cosh^2 t\, dt = \frac{1}{2}\left(t + \cosh t \sinh t\right) + C$$

d) Substitution: $x = \sinh t \;\Rightarrow\; dx = \cosh t\, dt$ und $t = \text{arsinh} x$

$$\int \sqrt{1+x^2}\, dx = \int \sqrt{1+\sinh^2 t}\, \cosh t\, dt = \int \cosh^2 t\, dt$$

$$= \frac{1}{2}\left(t + \cosh t \sinh t\right) + C = \frac{1}{2}\left(\text{arsinh} x + x\sqrt{1+x^2}\right) + C$$

e) Additionstheorem: $\cos^2 t + \sin^2 t = 1$ und

Substitution: $x = \sin t \;\Rightarrow\; dx = \cos t\, dt$

$$\int \sin^3 t \cos^3 t\, dt = \int \sin^3 t (1 - \sin^2 t) \cos t\, dt = \int x^3 (1 - x^2)\, dx$$

$$= \frac{x^4}{4} - \frac{x^6}{6} + C = \frac{\sin^4 t}{4} - \frac{\sin^6 t}{6} + C$$

f) $\displaystyle\int \arcsin x\, dx = \int 1 \cdot \arcsin x\, dx$

partielle Integration: $u' = 1,\; v = \arcsin x \;\Leftrightarrow\; x = \sin v \quad \Rightarrow$

$$v' = (\arcsin x)' = \frac{1}{(\sin v)'} = \frac{1}{\cos v} = \frac{1}{\sqrt{1 - \sin^2 v}} = \frac{1}{\sqrt{1-x^2}}$$

$$\int \arcsin x\, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx + C$$

Substitution: $u = 1 - x^2 \;\rightarrow\; du = -2x\, dx \;\rightarrow\; -\dfrac{du}{2} = x\, dx$

$$\int \arcsin x\, dx = x \arcsin x + \int \frac{1}{2\sqrt{u}}\, du + C$$

$$= x \arcsin x + \sqrt{u} + C = x \arcsin x + \sqrt{1-x^2} + C$$