$$\int \frac{1}{x}\,dx = \ln|x| + C:$$

$$\ln|x| = \begin{cases} \ln x \,; & x > 0 \\ \ln(-x)\,; & x < 0 \end{cases}$$

$$\underline{x>0}: (\ln x)' = \frac{1}{(e^y)'\big|_{y=\ln x}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

$$\Rightarrow (\ln|x|)' = \begin{cases} \frac{1}{x}\,; & x > 0 \\ \frac{1}{-x}\cdot(-1) = \frac{1}{x}\,; & x < 0 \end{cases}$$

$$\int \frac{1}{1-x^2}\,dx \overset{\text{PBZ}}{=} \int \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx$$

$$= \frac{1}{2}\left\{ \int \frac{1}{1+x}\,dx + \int \frac{1}{1-x}\,dx \right\}$$

$$= \frac{1}{2}\left(\ln|1+x| - \ln|1-x|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$

Zur Linearität:

$$F'(x) = f(x) \; ; \quad G'(x) = g(x)$$

$$\Rightarrow \alpha f(x) + \beta g(x) = \alpha F'(x) + \beta G'(x)$$
$$= (\alpha F(x) + \beta G(x))'$$

$$\Rightarrow \int \alpha f(x) + \beta g(x)\, dx = \int (\alpha F(x) + \beta G(x))'\, dx$$

$\downarrow$ Hauptsatz

$$= \alpha F(x) + \beta G(x)$$

$$= \alpha \int f(x)\, dx + \beta \int g(x)\, dx$$

Zur partiellen Integration:

$$u(x) \cdot v(x) = \int (u(x) \cdot v(x))'\, dx$$

$\downarrow$ Produktregel

$$= \int u'(x) \cdot v(x) + u(x) \cdot v'(x)\, dx$$

$$= \int u'(x) \cdot v(x)\, dx + \int u(x) \cdot v'(x)\, dx$$

$$\Rightarrow \int u(x) \cdot v'(x)\, dx = u(x) \cdot v(x) - \int u'(x) \cdot v(x)\, dx$$

zur Substitutionsregel:

$$\frac{dF(h)}{dh} = f(h) \ ; \quad \frac{dh}{dt} = h'(t)$$

$$\int f(h)\,\underline{dh} \overset{\text{Hauptsatz}}{=} F(h) = F(h(t)) \overset{\text{Hauptsatz}}{=} \int \frac{d}{dt}\Big(F(h(t))\Big)\,dt$$

$$\overset{\text{Kettenregel}}{=} \int \frac{dF}{dh}(h(t)) \cdot \frac{dh(t)}{dt}\,dt$$

$$= \int f(h(t)) \cdot \underline{h'(t)\,dt}$$

<u>Merkregel</u>: Substituiere $h = h(t)$
und setze $dh = h'(t)\,dt$

<u>bestimmtes Integral</u>: setze $h_a = h(a)$; $h_b = h(b)$

$$\int_{h_a}^{h_b} f(h)\,dh \overset{\text{Hauptsatz}}{=} F(h_b) - F(h_a)$$

$$= F(h(b)) - F(h(a))$$

$$\overset{\text{Hauptsatz}}{=} \int_a^b \frac{d}{dt}\Big(F(h(t))\Big)\,dt$$

$$\overset{\text{Kettenregel}}{=} \int_a^b f(h(t)) \cdot h'(t)\,dt$$

Beispiel Linearität:

$$\int 28x^3 + 12x^2 - 2x + 3\,dx$$

$$= 28\int x^3\,dx + 12\int x^2\,dx - 2\int x\,dx + 3\int dx$$

$$= \frac{28}{4}x^4 + \frac{12}{3}x^3 - 2\,\frac{x^2}{2} + 3x + C$$

Beispiel: partielle Integration

$$v' = \frac{1}{x}$$

$$u = x$$

$$\int \ln x\,dx = \int \underset{u'}{1} \cdot \underset{v}{\ln x}\,dx$$

$$= \underbrace{x \ln x}_{u \cdot v} - \int \underbrace{x \cdot \frac{1}{x}}_{u \cdot v'}\,dx$$

$$= x \ln x - x + C$$

Beispiel: Substitutionsregel (bestimmtes Integral)

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx$$

Substitution
$x = \cos t$
$\Rightarrow dx = -\sin t\,dt$

$$= \int_{\pi}^{0} \underbrace{\sqrt{1-\cos^2 t}}_{=\sin t} \cdot (-\sin t)\,dt$$

$1 = \cos t_2 \Rightarrow t_2 = 0$
$-1 = \cos t_1 \Rightarrow t_1 = \pi$

$$= \int_{0}^{\pi} \sin^2 t\,dt$$

Beispiel: (unbestimmtes Integral)

Substitution:
$x = \cos t$
$t = \arccos x$

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\cos^2 t}\,(-\sin t)\,dt$$

$$= -\int \sin^2 t\,dt$$

$$= -\frac{1}{2}\left(t - \sin t \cos t\right) + C'$$

Rücksubstitution

$$= -\frac{1}{2}\left(\arccos x - x\sqrt{1-x^2}\right) + C'$$

1. Fall $C_1 = 0 = \int_a^b p(x)\,dx$ Ergänzung zum Beweis Mittelwertsatz

$$\Rightarrow \int_a^b f(x)\,p(x)\,dx = 0 = f(\xi)\cdot \underbrace{C_1}_{\int_a^b p(x)\,d} \checkmark$$

2. Fall: $C \neq 0$

$$\Rightarrow \min_{x\in[a,b]} f(x) \leq \underbrace{\frac{1}{C}\cdot\int_a^b f(x)\,p(x)\,dx}_{= f(\xi) \text{ Zwischenwertsatz}} \leq \max_{x\in[a,b]} f(x)$$

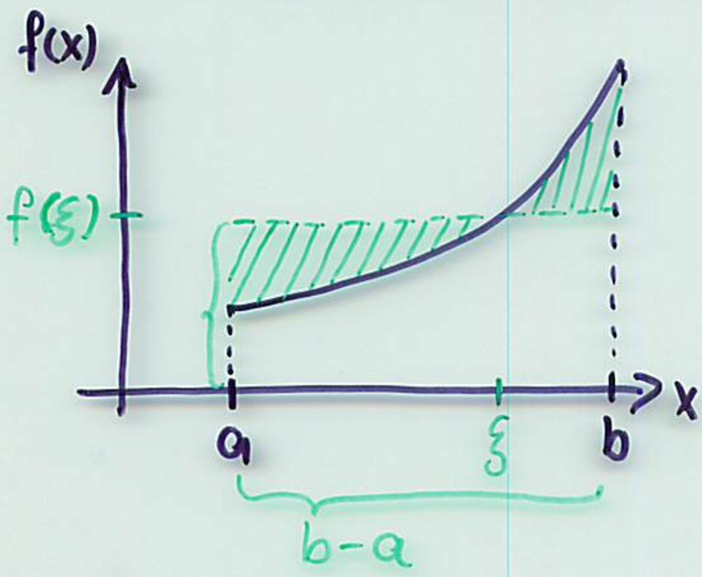
$$\Rightarrow \int_a^b f(x)\,p(x) = f(\xi)\cdot C = f(\xi)\cdot\int_a^b p(x)\,dx$$

Spezialfall:

$$p(x)=1 \Rightarrow \int_a^b p(x)\,dx = \int_a^b 1\,dx = b-a$$

Mittelwertsatz:

$$\int_a^b f(x)\,dx = f(\xi)(b-a) \quad ; \quad a \le \xi \le b$$



Mittelpunktsregel (näherungsweise Integralberechnung)

$$\xi \approx \frac{b-a}{2} \Rightarrow \int_a^b f(x)\,dx \approx f\left(\frac{b-a}{2}\right)\cdot(b-a)$$

Nachtrag zum Hauptsatz:

mit $F(x)=\int_{x_0}^{x} f(t)\,dt$

$$F'(a) = \lim_{b\to a} \frac{F(b)-F(a)}{b-a} = \lim_{b\to a} \frac{1}{b-a}\cdot\int_a^b f(t)\,dt$$

$$\underset{\uparrow}{=} \lim_{b\to a} \frac{1}{b-a}\cdot f(\xi)\cdot(b-a) = \lim_{b\to a} f(\xi) = f(a)$$

Mittelwertsatz $a \le \xi \le b$