The distributivity spectrum

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joint work with Vera Fischer and Marlene Koelbing

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Definition

$\mathbb{P}$ is λ-distributive if it does not add a function $f : \lambda \to Ord$ with $f \notin V$.

$\mathfrak{h}(\mathbb{P}) :=$ least λ such that $\mathbb{P}$ is not λ-distributive (the distributivity of $\mathbb{P}$).

For maximal antichains A and B,

$$B \text{ refines } A :\Longleftrightarrow \forall q \in B\ \exists p \in A\ (q \leq p).$$

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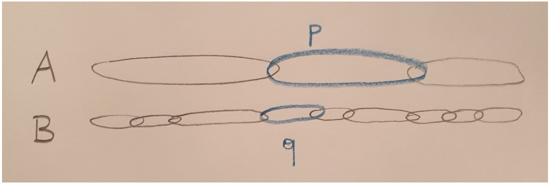

Proposition

$\mathbb{P}$ is λ-distributive if and only if for each family $\mathcal{A} = \{A_\xi : \xi < \lambda\}$ of maximal antichains in $\mathbb{P}$, there exists a common refinement (i.e., a maximal antichain B such that B refines A_ξ for each $\xi < \lambda$).

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$\mathfrak{h}(\mathbb{P}) :=$ least λ such that $\mathbb{P}$ is not λ-distributive (the distributivity of $\mathbb{P}$).

- $\omega \leq \mathfrak{h}(\mathbb{P}) \leq |\mathbb{P}|$
- $\mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\text{fin})$ ("the distributivity number")
  - $\omega_1 \leq \mathfrak{h} \leq \mathfrak{c}$ (since $\mathcal{P}(\omega)/\text{fin}$ is σ-closed and hence ω-distributive)
- Is there a generalization of $\mathfrak{h}$ to regular uncountable κ?
  - ...what about $\mathfrak{h}_\kappa := \mathfrak{h}(\mathcal{P}(\kappa)/{<}\kappa)$??
  - note that $\mathcal{P}(\kappa)/{<}\kappa$ is NOT σ-closed
  - in fact, $\mathcal{P}(\kappa)/{<}\kappa$ is NOT even ω-distributive; in other words: $\mathfrak{h}_\kappa = \omega$
  - so this definition is not interesting :-(
- The tower number $\mathfrak{t}$ has been generalized to κ:
  - $\mathcal{P}(\kappa)/{<}\kappa$ is not σ-closed, so straightforward generalization of $\mathfrak{t}$ yields ω
  - $\mathfrak{t}_\kappa :=$ shortest (regular) length above κ of a tower in $\mathcal{P}(\kappa)/{<}\kappa$
  - $\kappa^+ \leq \mathfrak{t}_\kappa \leq 2^\kappa$
- So let us look at the (distributivity) spectrum instead!

Definition (Distributivity spectrum (with respect to fresh functions))

We say that $\lambda \in \mathit{FRESH}(\mathbb{P})$ if in some extension of V by $\mathbb{P}$,

there exists a fresh function on λ,

i.e., a function $f : \lambda \to \mathit{Ord}$ with

1. $f \notin V$, but
2. $f \restriction \gamma \in V$ for every $\gamma < \lambda$.

Note: $\lambda \in \mathit{FRESH}(\mathbb{P}) \iff \mathrm{cf}(\lambda) \in \mathit{FRESH}(\mathbb{P})$

So from now on, we only talk about regular cardinals λ.

Some basic facts:

- $\min(\mathit{FRESH}(\mathbb{P})) = \mathfrak{h}(\mathbb{P})$
- $\mathit{FRESH}(\mathbb{P}) \subseteq [\mathfrak{h}(\mathbb{P}), |\mathbb{P}|]$

Let $\mathbb{C}_\mu$ be the forcing for adding μ many Cohen reals (μ arbitrarily large).

$$\mathit{FRESH}(\mathbb{C}_\mu) = ?$$

Let $\mathbb{C}_\mu$ be the forcing for adding μ many Cohen reals (μ arbitrarily large).

$$\mathit{FRESH}(\mathbb{C}_\mu) = \{\omega\}$$

Theorem

If $\mathbb{P}$ satisfies $\mathbb{P} \times \mathbb{P}$ is δ-c.c. and $\lambda \geq \delta$, then $\lambda \notin \mathit{FRESH}(\mathbb{P})$.

Is $\mathbb{P}$ being δ-c.c. sufficient? No: consider a Suslin tree T (on ω_1)

- T is c.c.c. (i.e., ω_1-c.c.)
- BUT: $\omega_1 \in \mathit{FRESH}(T)$
- $\omega_2, \omega_3, \ldots \notin \mathit{FRESH}(T)$

Theorem

If $\mathbb{P}$ is δ-c.c. and $\lambda > \delta$, then $\lambda \notin \mathit{FRESH}(\mathbb{P})$.

Recall: $\mathfrak{h}(\mathbb{P}) = \min(\mathit{FRESH}(\mathbb{P}))$

Lemma

If $\mathbb{P}$ collapses λ to $\mathfrak{h}(\mathbb{P})$, then $\lambda \in \mathit{FRESH}(\mathbb{P})$.

Recall: $\mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\mathrm{fin})$

Theorem (Balcar-Pelant-Simon (Base Matrix Theorem))

$\mathcal{P}(\omega)/\mathrm{fin}$ collapses $\mathfrak{c}$ to $\mathfrak{h}$.

Corollary

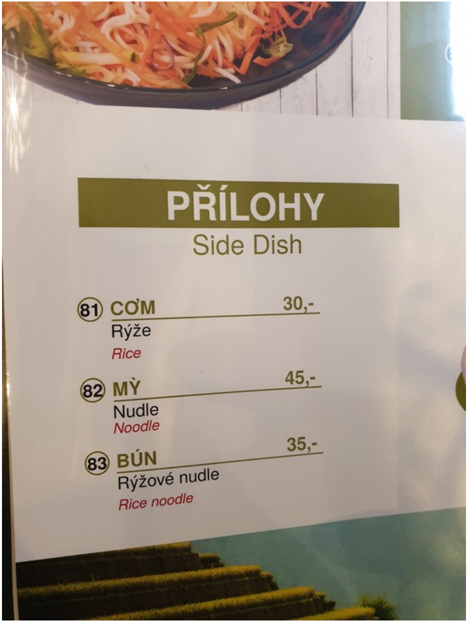
$\mathit{FRESH}(\mathcal{P}(\omega)/\mathrm{fin}) = [\mathfrak{h}, \mathfrak{c}]$.

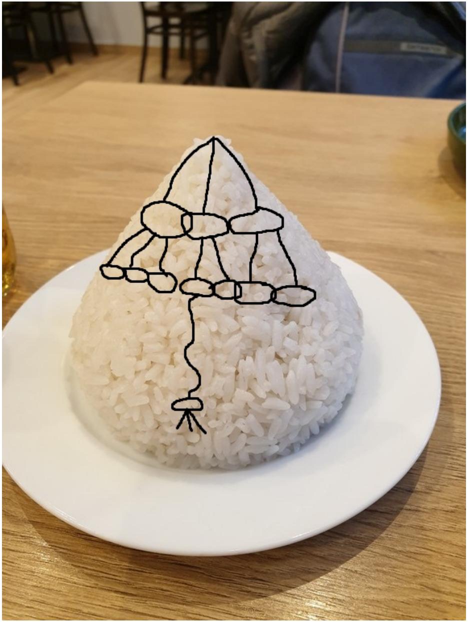
Theorem (Balcar-Simon; Shelah)

$\mathcal{P}(\kappa)/{<}\kappa$ collapses 2^κ to ω (assuming $2^{<\kappa} = \kappa$).

Corollary

$\mathit{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, 2^\kappa]$ (assuming $2^{<\kappa} = \kappa$).


PŘÍLOHY
Side Dish
81 CƠM 30,-
Rýže
Rice
82 MỲ 45,-
Nudle
Noodle
83 BÚN 35,-
Rýžové nudle
Rice noodle



Definition

We say $\mathcal{A} = \{A_\xi : \xi < \lambda\}$ is a distributivity matrix of height λ (for $\mathbb{P}$) if

- A_ξ is a maximal antichain in $\mathbb{P}$ (for each $\xi < \lambda$),
- A_η refines A_ξ (for each $\xi < \eta < \lambda$),
  - A_η refines $A_\xi :\Longleftrightarrow \forall q \in A_\eta \, \exists p \in A_\xi \, (q \leq p)$
- the set $\{q \in \mathbb{P} : q$ intersects $\mathcal{A}\}$ is not dense in $\mathbb{P}$.
  - q intersects $\mathcal{A} :\Longleftrightarrow \forall \xi < \lambda \, \exists p \in A_\xi \, (q \leq p)$

Let $COM(\mathbb{P})$ denote the combinatorial distributivity spectrum of $\mathbb{P}$:

$$\lambda \in COM(\mathbb{P}) :\Longleftrightarrow \text{there exists a distributivity matrix of height } \lambda \text{ for } \mathbb{P}.$$

Proposition

$\min(COM(\mathbb{P})) = \min(FRESH(\mathbb{P})) = \mathfrak{h}(\mathbb{P})$

Is $COM(\mathbb{P}) = FRESH(\mathbb{P})$?

Proposition

$COM(\mathbb{P}) \subseteq FRESH(\mathbb{P})$

Proposition

$COM(\mathbb{P}) = FRESH(\mathbb{P})$ in case $\mathbb{P}$ is a complete Boolean Algebra

Recall:

- $FRESH(\mathcal{P}(\omega)/\mathrm{fin}) = [\mathfrak{h}, \mathfrak{c}]$
- $FRESH(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, 2^\kappa]$ (assuming $2^{<\kappa} = \kappa$)

But note:

The Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ is NOT complete!!

The same is true in the κ-case: $\mathcal{P}(\kappa)/{<}\kappa$ is NOT complete.

$$\{\mathfrak{h}\} \subseteq COM(\mathcal{P}(\omega)/\mathrm{fin}) \subseteq [\mathfrak{h}, \mathfrak{c}]$$

Observe that $\mathfrak{h} = \mathfrak{c}$ implies that

$$\{\mathfrak{h}\} = FRESH(\mathcal{P}(\omega)/\text{fin}) = COM(\mathcal{P}(\omega)/\text{fin}).$$

Theorem

It is consistent that $\mathfrak{h} < \mathfrak{c} = \omega_2$, and

$$[\mathfrak{h}, \mathfrak{c}] = FRESH(\mathcal{P}(\omega)/\text{fin}) = COM(\mathcal{P}(\omega)/\text{fin}) = \{\omega_1, \omega_2\}.$$

To get a model in which both ω_1 and ω_2 are in $COM(\mathcal{P}(\omega)/\text{fin})$,

- we use a forcing (iteration) which adds a distributivity matrix of height ω_2, and
- we show that $\mathfrak{h} = \omega_1$ in the final model.

Definition (The forcing for ω_2)

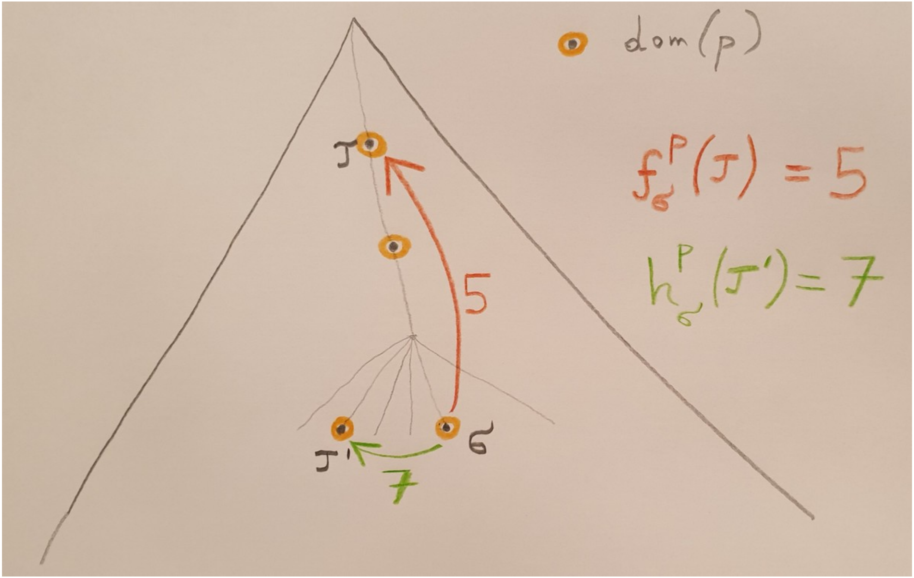
Let $T := \mathfrak{c}^{<\omega_2}$ and let $T^+ := \{\sigma \in T : |\sigma| \text{ is a successor}\}$. Define the forcing as follows: p is a condition if

- p is a finite function with $\mathrm{dom}(p) \subseteq T^+$,
- for each $\sigma \in \mathrm{dom}(p)$, $p(\sigma) = (s^p_\sigma, f^p_\sigma, h^p_\sigma)$, with $s^p_\sigma \in 2^{<\omega}$.

If G is a generic filter, let $a_\sigma := \bigcup_{p \in G} s^p_\sigma$, the matrix is $\{a_\sigma \mid \sigma \in T^+\}$.

- $f^p_\sigma : \{\tau \in \mathrm{dom}(p) : \tau \lhd \sigma\} \to \omega$ is a partial function,
- whenever $\tau \in \mathrm{dom}(f^p_\sigma)$ and $n = f^p_\sigma(\tau)$, we have $p \Vdash a_\sigma \setminus n \subseteq a_\tau$,
- $(\sigma = \rho^\frown\alpha)$ $h^p_\sigma : \{\rho^\frown\beta \in \mathrm{dom}(p) : \beta < \alpha\} \to \omega$ is a partial function,
- whenever $\tau \in \mathrm{dom}(h^p_\sigma)$ and $n = h^p_\sigma(\tau)$, we have $p \Vdash a_\tau \cap a_\sigma \subseteq n$,
- for each $\tau \in \mathrm{dom}(p)$ with $\tau \lhd \sigma$, $|s^p_\tau| \geq |s^p_\sigma|$.

$q \leq p$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$,
and for each $\sigma \in \mathrm{dom}(p)$, we have $s^p_\sigma \trianglelefteq s^q_\sigma$, $f^p_\sigma \subseteq f^q_\sigma$ and $h^p_\sigma \subseteq h^q_\sigma$.


dom(p)
$f_\sigma^p(\tau) = 5$
$h_\sigma^p(\tau') = 7$
τ
5
τ'
σ
7

Problem

The rows of the generic matrix are not mad families (new reals are added).

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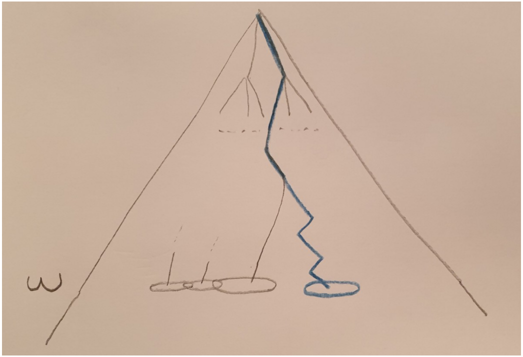

Solution

Use iterated forcing to make sure that the rows are maximal in the end.

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Use iterated forcing to make sure that the rows are maximal in the end.

In each step of the iteration we force sets a_σ for all $\sigma \in \mathfrak{c}^{<\omega_2}$ for which they are not defined yet, and make sure that they are $\subseteq^*$ of the a_τ above (we get for free that they are almost disjoint to old sets in the same row).
The iterands are defined as the forcing above, with the following changes:

Definition

- $\mathrm{dom}(p) \subseteq \{\sigma \in \mathfrak{c}^{<\omega_2} \mid a_\sigma$ is not defined yet$\}$, i.e., $\mathrm{dom}(p)$ is a finite subset of the new nodes of $\mathfrak{c}^{<\omega_2}$.
- $\mathrm{dom}(f_\sigma^p) \subseteq \{\tau \lhd \sigma \mid \tau \in \mathrm{dom}(p)$ or a_τ is already defined$\}$ finite

We iterate for ω_2 many steps, hence all nodes of $\mathfrak{c}^{<\omega_2}$ appear at some intermediate stage of the iteration, thus a_σ is defined for all $\sigma \in \mathfrak{c}^{<\omega_2}$.

Lemma

The forcing has the c.c.c.

Proof.

This is an easy Δ-system argument. □

Lemma

In the final model, the following holds for the generic matrix:

1. along branches through $\mathfrak{c}^{<\omega_2}$ we have $\subseteq^*$-decreasing sequences,
2. rows are almost disjoint families.

Proof.

This follows directly from the definition of the forcing, because the f's and h's ensure it. □

Lemma

In the final model, the following holds for the generic matrix:

1. along branches through $\mathfrak{c}^{<\omega_2}$ we have towers, i.e., maximal $\subseteq^*$-decreasing sequences,
2. rows are mad families.

Sketch of the proof.

Let $b \subseteq \omega$ infinite in the final model. Show that b is not a pseudo-intersection of any branch, and that b has infinite intersection with one element of each row.

Tower Assume σ is a branch through $\mathfrak{c}^{<\omega_2}$ and b is a pseudointersection of the sets along this branch. Use that all the information which is needed to decide something about b is bounded in $\mathfrak{c}^{<\omega_2}$, thus there exists some $\gamma < \omega_2$ such that the information at $\sigma{\restriction}\gamma$ is not relevant for b. So it is possible to decide that $m \in b$ and that $m \notin a_{\sigma\restriction\gamma}$ for arbitrarily large m. Thus b is not $\subseteq^*$ of $a_{\sigma\restriction\gamma}$. □

Sketch of the proof continued.

Mad We show the following claim, which directly implies that the rows are mad families:

If $\sigma \in \mathfrak{c}^\alpha$ for some $\alpha < \omega_2$ and $b \cap a_{\sigma \restriction \beta}$ is infinite for each $\beta \leq \alpha$, then there exists some $i < \mathfrak{c}$ such that $b \cap a_{\sigma^\frown i}$ is infinite.

To show this claim, we use a similar argument as for the towers: this time, we use the node $\sigma^\frown\gamma$ (which is not relevant for b); it is possible to decide that $m \in b$ and that $m \in a_{\sigma^\frown\gamma}$ for arbitrarily large m. □

This finishes the generic construction of the distributivity matrix of height ω_2.

The matrix witnesses that (in the final model) ω_2 is in $COM(\mathcal{P}(\omega)/\text{fin})$.

To prove that (in the final model) both ω_1 and ω_2 are in $COM(\mathcal{P}(\omega)/\text{fin})$,

- it remains to show that $\omega_1 \in COM(\mathcal{P}(\omega)/\text{fin})$.
  - In other words, we have to show that $\mathfrak{h} = \omega_1$;
  - recall that $\mathfrak{h} \leq \mathfrak{b}$,
  - so it is enough to just show that $\mathfrak{b} = \omega_1$.
- In fact, we aim at showing that the ground model reals $\mathcal{B} := \omega^\omega \cap V$ are still unbounded in our final model:
- we represent our finite support iteration as a "finer" finite support iteration all of whose iterands are eqivalent to

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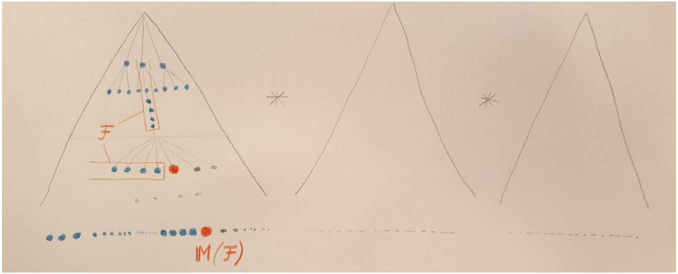

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- we will show that $\mathcal{B}$ remains unbounded at every successor step (i.e., the filtered Mathias forcings we use preserve the unboundedness of $\mathcal{B}$);
- due to a general theorem by Judah-Shelah for finite support iterations, the unboundedness of $\mathcal{B}$ is preserved at limit steps as well, finishing the argument that $\mathfrak{b}$ is small in the final model.

A filter $\mathcal{F}$ on ω is Canjar if $\mathbb{M}(\mathcal{F})$ does not add a dominating real.

Definition

A filter $\mathcal{F}$ on ω is $\mathcal{B}$-Canjar if $\mathbb{M}(\mathcal{F})$ preserves the unboundedness of $\mathcal{B}$.

Let X be a collection of finite subsets of ω. We say that

$$X \in (\mathcal{F}^{<\omega})^+ :\Longleftrightarrow \forall A \in \mathcal{F} \, \exists s \in X \, (s \subseteq A).$$

Theorem (Hrušák-Minami)

A filter $\mathcal{F}$ on ω is Canjar if and only if the following holds: whenever $X_n \in (\mathcal{F}^{<\omega})^+$ for each $n \in \omega$, there exists an $f \in \omega^\omega$ such that

$$\bigcup_{n \in \omega} X_n \cap \mathcal{P}(f(n)) \in (\mathcal{F}^{<\omega})^+.$$

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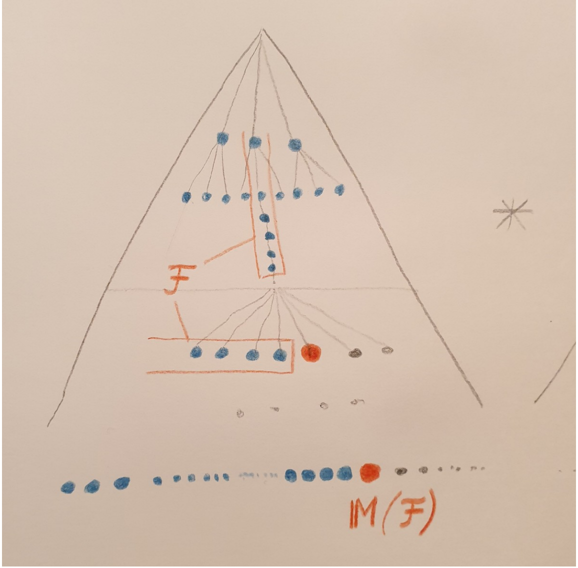
F
M(F)

Thank you for your attention and enjoy Hamburg. . . well. . .
. . . at least enjoy these pictures of Hamburg ;-)

View from my “subway” to university in Hamburg, near Kellinghusenstraße, June 2017

Thank you for your attention and enjoy Hamburg. . . well. . .
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Binnenalster in Hamburg, June 2017

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Planten un Blomen in Hamburg, May 2017