Globally bounded local edge colourings of hypergraphs

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Abstract

We consider edge colourings of $K_n^{(r)}$ – the complete r-uniform hypergraph on n vertices. Our main question is: how 'colourful' can such a colouring be if we restrict the number of colours locally?

The local restriction is formulated as follows: for a fixed hypergraph H and an integer k we call a colouring (H,k)-local, if every copy of H in the complete hypergraph $K_n^{(r)}$ picks up at most k different colours. We will investigate the threshold of k which guarantees that every (H,k)-local colouring must have a bounded global number of colours as n tends to infinity.

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1 Introduction and results

We consider edge colourings of hypergraphs. Our central question is: How many different colours can we allow 'locally' while keeping the 'global' number of colours bounded?

Let $r \geq 2$ and denote by $E(K_n^{(r)})$ the edge set of the r-uniform complete hypergraph on n vertices. Fix an r-uniform hypergraph H and a positive integer k. An (H,k)-local colouring is a mapping $\gamma : E(K_n^{(r)}) \to \mathbb{Z}$ that guarantees that (the edges of) every copy of H in $K_n^{(r)}$ are coloured with *at most* k different colours. Let us denote the set of all such local colourings by $\mathcal{L}_n^{(r)}(H,k)$. Local colourings of this kind were introduced by Truszczyński [6]. We are interested in the maximum total number of colours that a local colouring of $K_n^{(r)}$ can achieve, which we denote by

$$t(H,k,n) := \max\left\{ |\operatorname{im}(\gamma)| \colon \gamma \in \mathcal{L}_n^{(r)}(H,k) \right\}.$$

For given H and k, how does $t(H,k,n)$ behave as a function in n? To warm up, consider the following example for graphs. Let $r = 2$ and $H = K_5$. We have that

$$t(K_5,1,n) = 1 \quad \text{and} \quad t(K_5,2,n) = 2.$$

Indeed, the first is trivial and the latter is immediately verified as follows. Suppose for a contradiction that a colouring $\gamma \in \mathcal{L}_n^{(2)}(K_5,2)$ uses colours 1, 2, and 3 on the edges $\{x_1,y_1\}$, $\{x_2,y_2\}$, and $\{x_3,y_3\}$. If these six vertices were not pairwise distinct, they would be contained in a copy of a K_5 picking up 3 colours, which is forbidden. Also, the edge $\{x_1,x_2\}$ cannot have colour 3, so w.l.o.g. it has colour 1. But then the vertices x_1,x_2,y_2,x_3,y_3 span a K_5 with 3 colours. Continuing with our example, we claim next that

$$t(K_5,3,n) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

This can be verified by considering a colouring γ_{match}, which assigns pairwise different colours to the edges of a fixed matching of size $\lfloor \frac{n}{2} \rfloor$, and colours all the other edges with an extra colour 0. It is clear that $\gamma_{\text{match}} \in \mathcal{L}_n^{(2)}(K_5,3)$, because any copy of a K_5 can contain at most 2 matching edges. In other words, when we move from $t(K_5,2,n)$ to $t(K_5,3,n)$, the function suddenly changes from bounded to unbounded.

For a given H, we would like to determine the maximal k, for which

$t(H,k,n)$ is bounded. More precisely we are interested in

$$\mathrm{Fin}(H) := \max_{k\in\mathbb{N}} \big\{k\colon\ \exists t_0\,\forall n\ t(H,k,n) \le t_0\big\}.$$

The above example shows that $\mathrm{Fin}(K_5) = 2$. Clapsadle and Schelp [3] gave a nice description of $\mathrm{Fin}(H)$ for an arbitrary *graph* H.

Theorem 1.1 (Clapsadle & Schelp [3]) *Let H be a graph with at least two edges and let $\nu(H)$ be the cardinality of a maximum matching in H and $\Delta(H)$ the maximum degree of a vertex in H. Then* $\mathrm{Fin}(H) = \min\{\nu(H),\ \Delta(H)\}$.

Clapsadle and Schelp consider in particular the case where $t(H,k,n) = k$ and observe that then H must contain every graph on k edges as a subgraph. They conjecture that the converse is also true.

The central aim of our paper is to generalise Theorem 1.1 to r-uniform hypergraphs. For this we introduce the following definitions. A *sunflower* (often also called a Δ-*system*) with *core* L is an r-uniform hypergraph with set of edges $\{e_1,\dots,e_s\}$ such that $e_i \cap e_j = L$ for all $i \ne j$. The sets $p_i := e_i \setminus L$ are called the *petals*, the cardinality of the core $|L|$ is denoted as the *type*, and the number of edges (or petals) is called the *size* of the sunflower. If $\ell = |L|$ denotes the type and s the size of the sunflower, we will speak of an (ℓ, s)-sunflower and denote it by $S = (L, p_1, \dots, p_s)$.

Denote by $\Delta_\ell(H)$ the maximum size of a sunflower of type ℓ in a hypergraph H. Obviously if H is a graph, then we have $\Delta_1(H) = \Delta(H)$ and $\Delta_0(H) = \nu(H)$. Motivated by Theorem 1.1, Bollobás, Kohayakawa, Taraz, and Rödl conjectured that $\mathrm{Fin}(H) = \min_{0\le\ell<r} \Delta_\ell(H)$ for every nontrivial r-uniform hypergraph H and they proved this conjecture for 3-uniform hypergraphs and for r-uniform hypergraphs H that satisfy $r \ge \min_{0\le\ell<r} \Delta_\ell(H)$. The main theorem of this note verifies the full conjecture.

Theorem 1.2 *For any r-uniform hypergraph H with at least two edges we have that* $\mathrm{Fin}(H) = \min_{0\le\ell<r} \Delta_\ell(H)$.

In the following section, we first prove that $\min_{0\le\ell<r} \Delta_\ell(H)$ is an upper bound on $\mathrm{Fin}(H)$. The proof that it is also a lower bound is more involved, and we will only sketch the most important ideas. The full proof of Theorem 1.2 and related results discussed in Section 3 will appear in a joint paper of Bollobás, Kohayakawa, Rödl, and the authors [2].

2 Proof of Theorem 1.2

Upper bound. To prove the upper bound in Theorem 1.2, we will show that

$$\mathrm{Fin}(H) < \min_{0 \le \ell < r} \Delta_\ell(H) + 1 =: k\,. \tag{1}$$

In order to verify (1) we give an example of a sequence of (H,k)-local colourings $\gamma_n\colon\ E(K_n^{(r)}) \to \mathbb{Z}$ such that $|\operatorname{im}(\gamma_n)|$ is unbounded.

By definition of k in (1), H contains no (ℓ_0,k)-sunflower for some $\ell_0 \in [0, r-1] := \{0,\dots,r-1\}$. Fix in $K_n^{(r)}$ an $(\ell_0,\bar{n})$-sunflower $S = (L, p_1, \dots, p_{\bar{n}})$, with $\bar{n} := \lfloor (n-\ell_0)/(r-\ell_0) \rfloor$. Consider the colourings $\gamma_n\colon\ E(K_n^{(r)}) \to \mathbb{Z}$, where edges of S are coloured with $1,\dots,\bar{n}$, and all other edges are coloured 0. As H contains no (ℓ_0,k)-sunflower, every copy of H in $K_n^{(r)}$ cannot pick up more than $k-1$ colours from those appearing in S, and thus at most k in total. Hence γ_n is (H,k)-local, but obviously $|\operatorname{im}(\gamma_n)| \to \infty$ as $n \to \infty$.

Lower bound (sketch). Now we outline the proof of the lower bound of Theorem 1.2: we have to show that for every r-uniform hypergraph H with at least two edges

$$\mathrm{Fin}(H) \ge \min_{0 \le \ell < r} \Delta_\ell(H) =: s_H\,. \tag{2}$$

That means we have to show that for every n, every (H, s_H)-local colouring $\gamma\colon\ E(K_n^{(r)}) \to \mathbb{Z}$ is t_0-bounded, i.e., $|\operatorname{im}(\gamma)| \le t_0$ for some constant $t_0 = t_0(H)$ independent of n. The special case $s_H = 1$ is rather uninteresting and from now on we assume that $s_H \ge 2$.

For a given colouring γ, an (ℓ,k)-sunflower in $K_n^{(r)}$ will be called *injective*, if all of its k edges receive different colours. A colouring γ that yields no injective (ℓ,k)-sunflower in $K_n^{(r)}$ *for all* $\ell \in [0, r-1]$ will be called *k-local*. The next proposition shows that it is sufficient to prove that every (H, s_H)-local colouring γ is k-local.

Proposition 2.1 *For all integers k, $r \ge 2$ there exists an integer $t_0 = t_0(k,r)$ such that for every n and every k-local colouring $\gamma : E(K_n^{(r)}) \to \mathbb{Z}$ we have $|\operatorname{im}(\gamma)| \le t_0$.*

We easily deduce Proposition 2.1 from the following Theorem of Erdős and Rado.

Theorem 2.2 (Erdős & Rado [4]) *If an r-uniform hypergraph contains more than $r!(k-1)^r$ edges, then it contains an (ℓ,k)-sunflower for some $\ell \in [0, r-1]$.*

In fact for $k = 3$ Erdős offered \$1000 for the proof that $r!$ can be replaced by c^r for some constant c independent of r. Currently the best bound for that case is given by Kostochka [5].

Proof of Proposition 2.1 Let integers k, $r \geq 2$ be given. Set $t_0 = r!(k-1)^r$ and suppose that $\gamma : E(K_n^{(r)}) \to \mathbb{Z}$ is a k-local colouring, but fails to satisfy $|\operatorname{im}(\gamma)| \leq t_0$. Then Theorem 2.2 immediately implies that any collection of $|\operatorname{im}(\gamma)|$ mutually different coloured hyperedges of $K_n^{(r)}$ contains an injective (ℓ, k)-sunflower for some $\ell \in [0, r-1]$, which is a contradiction to the assumption that γ is k-local.

The following lemma then forms the heart of the proof of Theorem 1.2.

Lemma 2.3 *Suppose $s_H \geq 2$. For all integers $\tilde{k} > 0$ and $i \in [0, r-1]$ there exists some integer $k = k(\tilde{k}, i)$ such that if $\gamma \in \mathcal{L}_n^{(r)}(H, s_H)$ yields an injective (i, k)-sunflower, then it yields an injective $(j, \tilde{k})$-sunflower for some $j > i$.*

Moreover, there exists some integer $\hat{k} = \hat{k}(H) > 0$ so that every $\gamma \in \mathcal{L}_n^{(r)}(H, s_H)$ yields no injective $(r-1, \hat{k})$-sunflower.

Let us first see how this lemma implies (2). In view of Proposition 2.1 it suffices to show that every (H, s_H)-local colouring $\gamma \in \mathcal{L}_n^{(r)}(H, s_H)$ is k-local for some constant $k = k(H)$. Suppose for a contradiction that it were not k-local for some large k.. Then a repeated application of Lemma 2.3 shows that γ must have an injective $(r-1, \tilde{k})$-sunflower for some (arbitrarily large) $\tilde{k}$. But as Lemma 2.3 also bounds the maximum size of an injective sunflower of type $r-1$ by some absolute constant $\hat{k}$, this yields a contradiction.

Proof of Lemma 2.3 (sketch) The proof splits into two parts. First one shows that if there is no injective $(j, \tilde{k})$-sunflower in $K_n^{(r)}$ for $j > i$, then H must have a special structure. More precisely, H contains a subhypergraph $H' = S' + e'$, where S' is an (i, s_H)-sunflower and e' intersects at least two petals of S' and contains at least i vertices outside the petals. The fact that H must contain an (i, s_H)-sunflower S' follows from the definition of s_H in (2). Moreover, since $s_H \geq 2$ it follows by an averaging argument that there exists an $e' \in E(H) \setminus E(S')$ with at least i vertices outside the petals of S'. It then follows by some case analysis that e' must intersect at least two petals of S' or otherwise one could show that $\gamma \notin \mathcal{L}_n^{(r)}(H, s_H)$. In particular, this proves the moreover part of the lemma, since an edge e' with $r-1$ vertices outside the petals can intersect at most one petal.

In the second part of the proof we use the special structure of $H' \subseteq H$ (especially the properties of e') combined with the right (sufficiently large) choice of $k = k(\tilde{k}, i)$ to ensure the existence of an injective $(j, \tilde{k})$-sunflower in

$K_n^{(r)}$. The proof of this part relies on the fact that whenever we find an edge e in $K_n^{(r)}$ which intersects an appropriate (i, s_H)-subsunflower S of the given injective (i,k)-sunflower in the same way as e' intersects S', then $\gamma(e) \in \gamma(S)$. (Otherwise we find a copy of H' in $K_n^{(r)}$ which picks up $s_H + 1$ colours.) Iterating this observation over the 'right' choices of e then yields an injective $(j, \tilde{k})$-sunflower in $K_n^{(r)}$.

3 Related results

Let $k = \mathrm{Fin}(H) + 1$ for some hypergraph H. Then by definition there are (H,k)-local colourings which use an unbounded total number of colours, like the colouring γ_{match} in our introductory example with $r = 2$, $H = K_5$ and $k = 3$. Note however, that γ_{match} exhibits this richness in colours only on a vanishing proportion of the edges: the deletion of a suitable set of $o(n^2)$ edges would lead to a bounded number of remaining colours (in fact, only one). This gives rise to the question whether being $(K_5, 3)$-local forces every colouring to be limited to an 'essentially bounded' total number of colours?

The answer is yes. More generally for an arbitrary r-uniform hypergraph H, denote by $\mathrm{EssFin}(H)$ the maximal integer k, such that there exists an integer t_0 so that for every (H,k)-local colouring γ we can find a set $E' \subseteq E(K_n^{(r)})$ with

$$|E'| = (1 - o(1))\binom{n}{r} \quad \text{and} \quad |\gamma(E')| \leq t_0.$$

In other words, $\mathrm{EssFin}(H)$ is the largest integer such that every (H,k)-local colouring can only use an essentially bounded number of colours in total. The forthcoming paper [2] (and building on the work from [1]) gives a characterization of $\mathrm{EssFin}(H)$ for any hypergraph H. Very roughly spoken, the proof of this result is based on showing that any essentially *un*bounded colouring must be at least as colourful as a non-monochromatic canonical colouring.

Let us return to our example for a last time. From the result in [2] it follows that $\mathrm{EssFin}(K_5) > \mathrm{Fin}(K_5) = 2$. Moreover, it is easy to see that $\mathrm{EssFin}(K_5) < 4$ by considering the colouring where each edge $\{x,y\}$ with $x < y$ is coloured with colour x, thus $\mathrm{EssFin}(K_5) = 3$ as claimed earlier.

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