MINIMUM VERTEX DEGREE CONDITIONS FOR LOOSE HAMILTON CYCLES IN 3-UNIFORM HYPERGRAPHS

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Abstract. We investigate minimum vertex degree conditions for 3-uniform hypergraphs which ensure the existence of loose Hamilton cycles. A loose Hamilton cycle is a spanning cycle in which only consecutive edges intersect and these intersections consist of precisely one vertex.

We prove that every 3-uniform n-vertex (n even) hypergraph $\mathcal{H}$ with minimum vertex degree $\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16}+o(1)\right)\binom{n}{2}$ contains a loose Hamilton cycle. This bound is asymptotically best possible.

§1. Introduction

We consider k-uniform hypergraphs $\mathcal{H} = (V, E)$ with vertex sets $V = V(\mathcal{H})$ and edge sets $E = E(\mathcal{H}) \subseteq \binom{V}{k}$, where $\binom{V}{k}$ denotes the family of all k-element subsets of the set V. We often identify a hypergraph $\mathcal{H}$ with its edge set, i.e., $\mathcal{H} \subseteq \binom{V}{k}$, and for an edge $\{v_1, \dots, v_k\} \in \mathcal{H}$ we often suppress the enclosing braces and write $v_1 \dots v_k \in \mathcal{H}$ instead. Given a k-uniform hypergraph $\mathcal{H} = (V, E)$ and a set $S = \{v_1, \dots, v_s\} \in \binom{V}{s}$ let $\deg(S) = \deg(v_1, \dots, v_s)$ denote the number of edges of $\mathcal{H}$ containing the set S and let $N(S) = N(v_1, \dots, v_s)$ denote the set of those $(k-s)$-element sets $T \in \binom{V}{k-s}$ such that $S \cup T$ forms an edge in $\mathcal{H}$. We denote by $\delta_s(\mathcal{H})$ the minimum s-degree of $\mathcal{H}$, i.e., the minimum of $\deg(S)$ over all s-element sets $S \subseteq V$. For $s = 1$ the corresponding minimum degree $\delta_1(\mathcal{H})$ is referred to as minimum vertex degree whereas for $s = k-1$ we call the corresponding minimum degree $\delta_{k-1}(\mathcal{H})$ the minimum collective degree of $\mathcal{H}$.

We study sufficient minimum degree conditions which enforce the existence of spanning, so-called Hamilton cycles. A k-uniform hypergraph $\mathcal{C}$ is called an *ℓ-cycle* if there is a cyclic ordering of the vertices of $\mathcal{C}$ such that every edge consists of k consecutive vertices, every vertex is contained in an edge and two consecutive edges (where the ordering of the edges

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is inherited by the ordering of the vertices) intersect in exactly ℓ vertices. For $\ell = 1$ we call the cycle *loose* whereas the cycle is called *tight* if $\ell = k - 1$. Naturally, we say that a k-uniform, n-vertex hypergraph $\mathcal{H}$ contains a Hamilton ℓ-cycle if there is a subhypergraph of $\mathcal{H}$ which forms an ℓ-cycle and which covers all vertices of $\mathcal{H}$. Note that a Hamilton ℓ-cycle contains exactly $n/(k - \ell)$ edges, implying that the number of vertices of $\mathcal{H}$ must be divisible by $(k - \ell)$ which we indicate by $n \in (k - \ell)\mathbb{N}$.

Minimum collective degree conditions which ensure the existence of tight Hamilton cycles were first studied in [6] and in [12, 13]. In particular, in [12, 13] Rödl, Ruciński, and Szemerédi found asymptotically sharp bounds for this problem.

Theorem 1. *For every $k \geqslant 3$ and $\gamma > 0$ there exists an n_0 such that every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geqslant (1/2 + \gamma)n$ contains a tight Hamilton cycle.* □

The corresponding question for loose cycles was first studied by Kühn and Osthus. In [10] they proved an asymptotically sharp bound on the minimum collective degree which ensures the existence of loose Hamilton cycles in 3-uniform hypergraphs. This result was generalised to higher uniformity by the last two authors [4] and independently by Keevash, Kühn, Osthus and Mycroft in [7].

Theorem 2. *For all integers $k \geqslant 3$ and every $\gamma > 0$ there exists an n_0 such that every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $n \in (k - 1)\mathbb{N}$ and $\delta_{k-1}(\mathcal{H}) \geqslant (\frac{1}{2(k-1)} + \gamma)n$ contains a loose Hamilton cycle.* □

Indeed, in [4] asymptotically sharp bounds for Hamilton ℓ-cycles for all $\ell < k/2$ were obtained. Later this result was generalised to all $0 < \ell < k$ by Kühn, Mycroft, and Osthus [9]. These results are asymptotically best possible for all k and $0 < \ell < k$. Hence, asymptotically, the problem of finding Hamilton ℓ-cycles in uniform hypergraphs with large minimum *collective* degree is solved.

We focus on minimum *vertex* degree conditions which ensures the existence of Hamilton cycles. For $\delta_1(\mathcal{H})$ very few results on spanning subhypergraphs are known (see e.g. [3, 11]). In this paper we give an asymptotically sharp bound on the minimum vertex degree in 3-uniform hypergraphs which enforces the existence of loose Hamilton cycles.

Theorem 3 (Main result)**.** *For all $\gamma > 0$ there exists an n_0 such that the following holds. Suppose $\mathcal{H}$ is a 3-uniform hypergraph on $n > n_0$ with $n \in 2\mathbb{N}$ and*

$$\delta_1(\mathcal{H}) > \left(\frac{7}{16} + \gamma\right)\binom{n}{2}.$$

Then $\mathcal{H}$ contains a loose Hamilton cycle.

In the proof we apply the so-called absorbing technique. In [12] Rödl, Ruciński, and Szemerédi introduced this elegant approach to tackle minimum degree problems for spanning graphs and hypergraphs. In our case it reduces the problem of finding a loose Hamilton cycle to the problem of finding a *nearly* spanning loose path and indeed, finding such a path will be the main obstacle to Theorem 3.

As mentioned above, Theorem 3 is best possible up to the error constant γ as seen by the following construction from [10].

Fact 4. *For every $n \in 2\mathbb{N}$ there exists a 3-uniform hypergraph $\mathcal{H}_3 = (V, E)$ on $|V| = n$ vertices with $\delta_1(\mathcal{H}_3) \geqslant \frac{7}{16}\binom{n}{2} - O(n)$, which does not contain a loose Hamilton cycle.*

Proof. Consider the following 3-uniform hypergraph $\mathcal{H}_3 = (V, E)$. Let $A \mathbin{\dot\cup} B = V$ be a partition of V with $|A| = \lfloor \frac{n}{4} \rfloor - 1$ and let E be the set of all triplets from V with at least one vertex in A. Clearly, $\delta_1(\mathcal{H}_3) = \binom{|A|}{2} + |A|(|B| - 1) = \frac{7}{16}\binom{n}{2} - O(n)$. Now consider an arbitrary cycle in $\mathcal{H}_3$. Note that every vertex, in particular every vertex from A, is contained in at most two edges of this cycle. Moreover, every edge of the cycle must intersect A. Consequently, the cycle contains at most $2|A| < n/2$ edges and, hence, cannot be a Hamilton cycle. $\square$

We note that the construction $\mathcal{H}_3$ in Fact 4 satisfies $\delta_2(\mathcal{H}_3) \geqslant n/4 - 1$ and indeed, the same construction proves that the minimum collective degree condition given in Theorem 2 is asymptotically best possible for the case $k = 3$.

This leads to the following conjecture for minimum vertex degree conditions enforcing loose Hamilton cycles in k-uniform hypergraphs. Let $k \geqslant 3$ and let $\mathcal{H}_k = (V, E)$ be the k-uniform, n-vertex hypergraph on $V = A \mathbin{\dot\cup} B$ with $|A| = \frac{n}{2(k-1)} - 1$. Let E consist of all k-sets intersecting A in at least one vertex. Then $\mathcal{H}_k$ does not contain a loose Hamilton cycle and we believe that any k-uniform, n-vertex hypergraph $\mathcal{H}$ which has minimum vertex degree $\delta_1(\mathcal{H}) \geqslant \delta_1(\mathcal{H}_k) + o(n^2)$ contains a loose Hamilton cycle. Indeed, Theorem 3 verifies this for the case $k = 3$.

§2. Proof of the main result

The proof of Theorem 3 will be given in Section 2.3. It uses several auxiliary lemmas which we introduce in Section 2.2. We start with an outline of the proof.

2.1. **Outline of the proof.**

We will build a loose Hamilton cycle by connecting loose paths. Formally, a 3-uniform hypergraph $\mathcal{P}$ is a *loose path* if there is an ordering $(v_1, \dots, v_t)$ of its vertices such that every edge consists of three consecutive vertices, every vertex is contained in an edge and two consecutive edges intersect in exactly one vertex. The elements v_1 and v_t are called the *ends* of $\mathcal{P}$.

The Absorbing Lemma (Lemma 7) asserts that every 3-uniform hypergraph $\mathcal{H} = (V, E)$ with sufficiently large minimum vertex degree contains a so-called *absorbing* loose path $\mathcal{P}$, which has the following property: For every set $U \subset V \setminus V(\mathcal{P})$ with $|U| \in 2\mathbb{N}$ and $|U| \leqslant \beta n$ (for some appropriate $0 < \beta < \gamma$) there exists a loose path $\mathcal{Q}$ with the same ends as $\mathcal{P}$, which covers precisely the vertices $V(\mathcal{P}) \cup U$.

The Absorbing Lemma reduces the problem of finding a loose Hamilton cycle to the simpler problem of finding an almost spanning loose cycle, which contains the absorbing path $\mathcal{P}$ and covers at least $(1-\beta)n$ of the vertices. We approach this simpler problem as follows. Let $\mathcal{H}'$ be the induced subhypergraph $\mathcal{H}$, which we obtain after removing the vertices of the absorbing path $\mathcal{P}$ guaranteed by the Absorbing Lemma. We remove from $\mathcal{H}'$ a "small" set R of vertices, called the *reservoir* (see Lemma 6), which has the property that many loose paths can be connected to one loose cycle by using the vertices of R only.

Let $\mathcal{H}''$ be the remaining hypergraph after removing the vertices from R. We will choose $\mathcal{P}$ and R small enough, so that $\delta_1(\mathcal{H}'') \geqslant (\frac{7}{16} + o(1))\binom{|V(\mathcal{H}'')|}{2}$. The third auxiliary lemma, the Path-tiling Lemma (Lemma 10), asserts that all but $o(n)$ vertices of $\mathcal{H}''$ can be covered by a family of pairwise disjoint loose paths and, moreover, the number of those paths will be constant (independent of n). Consequently, we can connect those paths and $\mathcal{P}$ to form a loose cycle by using exclusively vertices from R. This way we obtain a loose cycle in $\mathcal{H}$, which covers all but the $o(n)$ left-over vertices from $\mathcal{H}''$ and some left-over vertices from R. We will ensure that the number of those yet uncovered vertices will be smaller than βn and, hence, we can appeal to the absorption property of $\mathcal{P}$ and obtain a Hamilton cycle.

As indicated earlier, among the auxiliary lemmas mentioned above the Path-tiling Lemma is the only one for which the full strength of the condition $(\frac{7}{16} + o(1))\binom{n}{2}$ is required and indeed, we consider Lemma 10 to be the main obstacle to proving Theorem 3. For the other lemmas we do not attempt to optimise the constants.

2.2. **Auxiliary lemmas.**

In this section we introduce the technical lemmas needed for the proof of the main theorem.

We start with the connecting lemma which is used to connect several "short" loose paths to a long one. Let $\mathcal{H}$ be a 3-uniform hypergraph and $(a_i, b_i)_{i\in[k]}$ a set consisting of k mutually disjoint pairs of vertices. We say that a set of triples $(x_i, y_i, z_i)_{i\in[k]}$ *connects* $(a_i, b_i)_{i\in[k]}$ if

- $\big|\bigcup_{i\in[k]}\{a_i, b_i, x_i, y_i, z_i\}\big| = 5k$, i.e. the pairs and triples are all disjoint,
- for all $i \in [k]$ we have $\{a_i, x_i, y_i\}, \{y_i, z_i, b_i\} \in \mathcal{H}$.

Suppose that a and b are ends of two disjoint loose paths not intersecting $\{x, y, z\}$ and suppose that (x, y, z) connects (a, b). Then this connection would join the two paths to one loose path. The following lemma states that several paths can be connected, provided the minimum vertex degree is sufficiently large.

Lemma 5 (Connecting lemma)**.** *Let $\gamma > 0$, let $m \geqslant 1$ be an integer, and let $\mathcal{H} = (V, E)$ be a 3-uniform hypergraph on n vertices with $\delta_1(\mathcal{H}) \geqslant \left(\frac{1}{4} + \gamma\right)\binom{n}{2}$ and $n \geqslant \gamma m/12$.*

For every set $(a_i, b_i)_{i\in[m]}$ of mutually disjoint pairs of distinct vertices, there exists a set of triples $(x_i, y_i, z_i)_{i\in[m]}$ connecting $(a_i, b_i)_{i\in[m]}$.

Proof. We will find the triples (x_i, y_i, z_i) to connect a_i with b_i for $i \in [m]$ inductively as follows. Suppose, for some $j < k$ the triples (x_i, y_i, z_i) with $i < j$ are constructed so far and for $(a, b) = (a_j, b_j)$ we want to find a triple (x, y, z) to connect a and b. Let

$$U = V \smallsetminus \left(\bigcup_{i=1}^{m}\{a_i, b_i\} \cup \bigcup_{i=1}^{j-1}\{x_i, y_i, z_i\}\right)$$

and for a vertex $u \in V$ let $L_u = (V \smallsetminus \{u\}, E_u)$ be the *link graph* of v defined by

$$E_u = \{vw \colon uvw \in E(\mathcal{H})\}\,.$$

We consider $L_a[U]$ and $L_b[U]$, the subgraphs of L_a and L_b induced on U. Owing to the minimum degree condition of $\mathcal{H}$ and to the assumption $m \leqslant \gamma n/12$, we have

$$e(L_a[U]) \geqslant \left(\frac{1}{4} + \gamma\right)\binom{n}{2} - 5m(n-1) \geqslant \left(\frac{1}{4} + \frac{\gamma}{6}\right)\binom{n}{2} \tag{1}$$

and the same lower bound also holds for $e(L_b[U]))$. Note that any pair of edges $xy \in L_a[U]$ and $yz \in L_b[U]$ with $x \neq z$ leads to a connecting triple (x, y, z) for (a, b). Thus, if no connecting triple exists, then for every vertex $u \in U$ one of the following must hold: either u is isolated in $L_a[U]$ or $L_b[U]$ or it is adjacent to exactly one vertex w in both graphs $L_a[U]$ and $L_b[U]$. In other words, any vertex not isolated in $L_a[U]$ has at most one neighbour in $L_b[U]$. Let I_a be the set of isolated vertices in $L_a[U]$. Since $e(L_a[U]) > \frac{1}{4}\binom{n}{2}$ we have $|I_a| < n/2$. Consequently,

$$\begin{aligned} e(L_b[U]) &\leqslant \binom{|I_a|}{2} + |\{uw \in E(L_b[U]) \colon u \in U \smallsetminus I_a\}| \\ &\leqslant \binom{|I_a|}{2} + (|U| - |I_a|) < \binom{\lfloor n/2 \rfloor}{2} + n\,. \end{aligned}$$

Using $\gamma \leqslant 3/4$ and $n \geqslant \gamma/12$ we see that this upper bound violates the lower bound on $e(L_b[U])$ from (1). $\square$

When connecting several paths to a long one we want to make sure that the vertices used for the connection all come from a small set, called reservoir, which is disjoint to the paths. The existence of such a set is guaranteed by the following.

Lemma 6 (Reservoir lemma)**.** *For all $0 < \gamma < 1/4$ there exists an n_0 such that for every 3-uniform hypergraph $\mathcal{H} = (V, E)$ on $n > n_0$ vertices with minimum vertex degree $\delta_1(\mathcal{H}) \geqslant \left(\frac{1}{4} + \gamma\right)\binom{n}{2}$ there is a set R of size at most γn with the following property: For every system $(a_i, b_i)_{i\in[k]}$ consisting of $k \leqslant \gamma^3 n/12$ mutually disjoint pairs of vertices from V there is a triple system connecting $(a_i, b_i)_{i\in[k]}$ which, moreover, contains vertices from R only.*

Proof. We shall show that a random set R has the required properties with positive probability. For this proof we use some exponential tail estimates. Here we will follow a basic technique described in [5, Section 2.6]. Alternatively Lemma 6 could be deduced more directly from Janson's inequality.

For given $0 < \gamma < 1/4$ let n_0 be sufficiently large. Let $\mathcal{H}$ be as stated in the lemma and $v \in V(\mathcal{H})$. Let $L(v)$ be the link graph defined on the vertex set $V(\mathcal{H}) \setminus \{v\}$, having the edges $e \in E(L(v))$ if $e \cup \{v\} \in \mathcal{H}$. Note that $L(v)$ contains $\deg_{\mathcal{H}}(v)$ edges. Since the edge set of the omplete graph K_n can be decomposed into $n-1$ edge disjoint matchings, we can decompose the edge set of L into $i_0 = i_0(v) < n$ pairwise edge disjoint matchings. We denote these matchings by $M_1(v), \ldots, M_{i_0}(v)$.

We randomly choose a vertex set V_p from V by including each vertex $u \in V$ into V_p with probability $p = \gamma - \gamma^3$ independently. For every $i \in [i_0]$ let

$$X_i(v) = \left| M_i(v) \cap \binom{V_p}{2} \right|$$

denote the number of edges $e \in M_i(v)$ contained in V_p. This way $X_i(v)$ is a binomially distributed random variable with parameters $|M_i(v)|$ and p^2. Using the following Chernoff bounds for $t > 0$ (see, e.g., [5, Theorem 2.1])

$$\mathbb{P}\left(\mathrm{Bin}(m, \zeta) \geqslant m\zeta + t\right) < e^{-t^2/(2\zeta m + t/3)} \tag{2}$$

$$\mathbb{P}\left(\mathrm{Bin}(m, \zeta) \leqslant m\zeta - t\right) < e^{-t^2/(2\zeta m)} \tag{3}$$

we see that

$$\gamma\frac{n}{2} \leqslant |V_p| \leqslant pn + (3n \ln 20)^{1/2} \leqslant \gamma n - 2k \tag{4}$$

with probability at least $9/10$.

Further, using (3) and $|M_i(v)| \leqslant n/2$ we see that with probability at most n^{-2} there exists an index $i \in [i_0]$ such that $X_i(v) \leqslant |M_i(v)|p^2 - (3n \ln n)^{1/2}$. Using $\sum_{i\in[i_0]} |M_i(v)| = \deg_{\mathcal{H}}(v)$

and recalling that $\deg_{V_p}(v)$ denotes the degree of v in $\mathcal{H}[V_p \cup \{v\}]$ we obtain that

$$\deg_{V_p}(v) = \sum_{i \in [i_0]} X_i(v) \geqslant p^2 \deg_{\mathcal{H}}(v) - n(3n \ln n)^{1/2}\,, \tag{5}$$

holds with probability at least $1 - n^{-2}$.

Repeating the same argument for every vertex $v \in V$ we infer from the union bound that (5) holds for all vertices $v \in V$ simultaneously with probability at least $1 - 1/n$. Hence, with positive probability we obtain a set R satisfying (4) and (5) for all $v \in V$.

Let $(a_i, b_i)_{i \in [k]}$ be given and let $S = \bigcup_{i \in [k]} \{a_i, b_i\}$. Then we have $|R \cup S| \leqslant \gamma n$ and

$$\deg_{R \cup S}(v) \geqslant \deg_R(v) \geqslant \left(\frac{1}{4} + \gamma^2\right)\binom{\gamma n}{2} \geqslant \left(\frac{1}{4} + \gamma^2\right)\binom{|R \cup S|}{2}$$

for all $v \in V$. Thus, we can appeal to the Connecting Lemma (Lemma 5) to obtain a triple system which connects $(a_i, b_i)_{i \in [k]}$ and which consists of vertices from R only. $\square$

Next, we introduce the Absorbing Lemma which asserts the existence of a "short" but powerful loose path $\mathcal{P}$ which can absorb any small set $U \subset V \smallsetminus V(\mathcal{P})$. In the following note that $\left(\frac{5}{8}\right)^2 < \frac{7}{16}$.

Lemma 7 (Absorbing lemma)**.** *For all $\gamma > 0$ there exist $\beta > 0$ and n_0 such that the following holds. Let $\mathcal{H} = (V, E)$ be a 3-uniform hypergraph on $n > n_0$ vertices which satisfies $\delta_1(\mathcal{H}) \geqslant \left(\frac{5}{8} + \gamma\right)^2 \binom{n}{2}$. Then there is a loose path $\mathcal{P}$ with $|V(\mathcal{P})| \leqslant \gamma^7 n$ such that for all subsets $U \subset V \smallsetminus V(\mathcal{P})$ of size at most βn and $|U| \in 2\mathbb{N}$ there exists a loose path $\mathcal{Q} \subset \mathcal{H}$ with $V(\mathcal{Q}) = V(\mathcal{P}) \cup U$ and $\mathcal{P}$ and $\mathcal{Q}$ have exactly the same ends.*

The principle used in the proof of Lemma 7 goes back to Rödl, Ruciński, and Szemerédi. They introduced the concept of *absorption*, which, roughly speaking, stands for a local extension of a given structure, which preserves the global structure. In our context of loose cycle we say that a 7-tuple $(v_1, \dots, v_7)$ *absorbs* the two vertices $x, y \in V$ if

- $v_1v_2v_3, v_3v_4v_5, v_5v_6v_7 \in \mathcal{H}$ and
- $v_2xv_4,\ v_4yv_6 \in \mathcal{H}$

are guaranteed. In particular, $(v_1, \dots, v_7)$ and $(v_1, v_3, v_2, x, v_4, y, v_6, v_5, v_7)$ both form loose paths which, moreover, have the same ends. The proof of Lemma 7 relies on the following result which states that for each pair of vertices there are many 7-tuples absorbing this pair, provided the minimum vertex degree of $\mathcal{H}$ is sufficiently large.

Proposition 8. *For every $\gamma \in (0, 3/8)$ there exists an n_0 such that the following holds. Suppose $\mathcal{H} = (V, E)$ is a 3-uniform hypergraph on $n > n_0$ vertices with $\delta_1(\mathcal{H}) \geqslant \left(\frac{5}{8} + \gamma\right)^2 \binom{n}{2}$. Then for every pair of vertices $x, y \in V$ the number of 7-tuples absorbing x and y is at least $(\gamma n)^7/8$.*

Proof. For given $\gamma > 0$ we choose $n_0 = 168/\gamma^7$. First we show the following.

Claim 9. *For every pair $x, y \in V(\mathcal{H})$ of vertices there exists a set $D = D(x,y) \subset V$ of size $|D| = \gamma n$ such that one of the following holds:*

- $\deg(x,d) \geqslant \gamma n$ *and* $\deg(y,d) \geqslant \frac{3}{8}n$ *for all* $d \in D$ *or*
- $\deg(y,d) \geqslant \gamma n$ *and* $\deg(x,d) \geqslant \frac{3}{8}n$ *for all* $d \in D$.

Proof of Claim 9. By assuming the contrary there exist two vertices x and y such that no set $D = D(x,y)$ fulfills Claim 9.

Let $A(z) = \{d \in V\colon \deg(z,d) < \gamma n\}$ and let $a = |A(x)|/n$ and $b = |A(y)|/n$. Without loss of generality we assume $a \leqslant b$. There are at most $(a+\gamma)n$ vertices $v \in V$ satisfying $\deg(y,v) \geqslant \frac{3}{8}n$. Let $B(y) = \{v \in V\colon \deg(y,v) \geqslant 3n/8\}$ and note that $|B(y)| < (a+\gamma)n$. Hence, the number of ordered pairs (u,v) such that $u \in B(y)$ and $\{u,v,y\} \in \mathcal{H}$ is at most

$$|B(y)|(n - |A(y)|) + |A(y)|\gamma n \leqslant (a+\gamma)(1-b)n^2 + b\gamma n^2.$$

Consequently, with $2\deg(y)$ being the number of ordered pairs (u,v) such that $\{u,v,y\} \in \mathcal{H}$ we have

$$\begin{aligned}\left(\left(\frac{5}{8}\right)^2 + \frac{9\gamma}{8}\right)n^2 \leqslant 2\deg(y) &\leqslant (1-b-a-\gamma)\frac{3}{8}n^2 + (a+\gamma)(1-b)n^2 + 2b\gamma n^2\\ &\leqslant \frac{n^2}{8}(5a - 3b - 8ab) + \frac{(3+8\gamma)n^2}{8},\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\left(\left(\frac{5}{8}\right)^2 + \frac{9\gamma}{8}\right)n^2 \leqslant 2\deg(y) &\leqslant \frac{3n^2}{8}(1-b) + (a+\gamma)\left(\frac{5}{8} - b\right)n^2 + 2b\gamma n^2\\ &\leqslant \frac{n^2}{8}(5a - 3b - 8ab) + \frac{(3+8\gamma)n^2}{8},\end{aligned}$$

where in the last inequality we use the fact that $b \leqslant 3/8$ which is a direct consequence of the condition on $\delta_1(\mathcal{H})$. It is easily seen that this maximum is attained by $a = b = 1/8$, for which we would obtain

$$\deg(y) \leqslant \left(\left(\frac{5}{8}\right)^2 + \gamma\right)n^2,$$

a contradiction. □

We continue the proof of Proposition 8. For a given pair $x, y \in V$ we will select the tuple $v_1, \dots, v_7$ such that the edges

- $v_1v_2v_3, v_3v_4v_5, v_5v_6v_7 \in \mathcal{H}$ and
- $v_2xv_4,\ v_4yv_6 \in \mathcal{H}$

are guaranteed. Note that $(v_1, \dots, v_7)$ forms a loose path with the ends v_1 and v_7 and $(v_1, v_3, v_2, x, v_4, y, v_6, v_5, v_7)$ also forms a loose path with the same ends, showing that $(v_1, \dots, v_7)$ is indeed an absorbing tuple for the pair a, b. Moreover, we will show that the number of choices for each v_i will give rise to the number of absorbing tuples stated in the proposition.

First, we want to choose v_4 and let $D(x, y)$ be a set with the properties stated in Claim 9. Without loss of generality we may assume that $|N(y, d)| \geqslant \frac{3}{8}n$ for all $d \in D(x, y)$. Fixing some $v_4 \in D(x, y)$ We choose $v_2 \in N(x, v_4)$ for which there are γn choices. This gives rise to to hyperegde $v_2 x v_4 \in \mathcal{H}$. and applying Claim 9 to v_2 and v_4 we obtain a set $D(v_2, v_4)$ with the properties stated in Claim 9 and we choose $v_3 \in D(v_2, v_4)$. We choose $v_1 \in N(v_2, v_3)$ to obtain the edge $v_1 v_2 v_3 \in \mathcal{H}$. Note that $|N(v_2, v_3)| \geqslant \gamma n$. Next, we choose v_5 from $N(v_3, v_4)$ which has size $|N(v_3, v_4)| \geqslant \gamma n$. This gives rise to the edge $v_3 v_4 v_5 \in \mathcal{H}$. We choose v_6 from the set $N(y, v_4)$ with the additional property that $\deg(v_5, v_6) \geqslant \gamma n/2$. Hence, we obtain $v_4 y v_6 \in \mathcal{H}$ and we claim that there are at least $\gamma n/2$ such choices. Otherwise at least $(|N(y, v_4)| - \gamma n/2)$ vertices $v \in V$ satisfy $\deg(v_5, v) < \gamma n/2$, hence

$$\deg(v_5) < \frac{3\gamma}{16} n^2 + \binom{(\frac{5}{8} + \frac{\gamma}{2})n}{2} < \delta(\mathcal{H}),$$

which is a contradiction. Lastly we choose $v_7 \in N(v_5, v_6)$ to obtain the edge $v_5 v_6 v_7 \in \mathcal{H}$ which completes the absorbing tuple $(v_1, \dots, v_7)$.

The number of choices for $v_1, \dots, v_7$ is at least $(\gamma n)^7/4$ and there are at most $\binom{7}{2} n^6$ choices such that $v_i = v_j$ for some $i \neq j$. Hence, we obtain at least $(\gamma n)^7/8$ absorbing 7-tuples for the pair x, y. □

With Proposition 8 and the connecting lemma (Lemma 5) at hand the proof of the absorbing lemma follows a scheme which can be found in [4, 12]. We choose a family $\mathcal{F}$ of 7-tuples by selecting each 7-tuples with probability $p = \gamma^7 n^{-6}/448$ independently. Then, it is easily shown that with non-zero probability the family $\mathcal{F}$ satisfies

- $|\mathcal{F}| \leqslant \gamma^7 n/12$,
- for all pairs $x, y \in V$ there are at least $p\gamma^7 n^7/16$ tuples in $\mathcal{F}$ which absorbs x, y
- the number of intersecting pairs of 7-tuples in $\mathcal{F}$ is at most $p\gamma^7 n^7/32$

We eliminate intersecting pairs of 7-tuples by deleting one tuple for each such pair. By definition each for the remaining 7-tuples $(v_1^i, \dots, v_7^i)_{i \in [k]}$ with $k \leqslant \gamma^7 n/12$ forms a loose path with ends v_1^i and v_7^i and appealing to Lemma 5 we can connect them to one loose path which can absorb any $p\gamma^7 n^7/32 = \beta$ pairs of vertices, proving the lemma. To avoid unnecessary calculations we omit the details here. □

The next lemma is the main obstacle when proving Theorem 3. It asserts that the vertex set of a 3-uniform hypergraph $\mathcal{H}$ with minimum vertex degree $\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16} + o(1)\right)\binom{n}{2}$ can be almost perfectly covered by a constant number of vertex disjoint loose paths.

Lemma 10 (Path-tiling lemma). *For all $\gamma > 0$ and $\alpha > 0$ there exist integers p and n_0 such that for $n > n_0$ the following holds. Suppose $\mathcal{H}$ is a 3-uniform hypergraph on n vertices with minimum vertex degree*

$$\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16} + \gamma\right)\binom{n}{2}.$$

Then there is a family of p disjoint loose paths in $\mathcal{H}$ which covers all but at most αn vertices of $\mathcal{H}$.

The proof of Lemma 10 uses the weak regularity lemma for hypergraphs and will be given in Section 3.

2.3. **Proof of the main theorem.** In this section we give the proof of the main result, Theorem 3. The proof is based on the three auxiliary lemmas introduced in Section 2.2 and follows the outline given in Section 2.1.

Proof of Theorem 3. For given $\gamma > 0$ we apply the Absorbing Lemma (Lemma 7) with $\gamma/8$ to obtain $\beta > 0$ and n_7. We apply the Reservoir Lemma (Lemma 6) for $\gamma' = \min\{\beta/3, \gamma/8\}$ to obtain n_6 which is n_0 of Lemma 6. Finally, we apply the Path-tiling Lemma (Lemma 10) with $\gamma/2$ and $\alpha = \beta/3$ to obtain p and n_{10}. The n_0 of Theorem 3 is chosen by

$$n_0 = \max\{n_7, 2n_6, 2n_{10}, 24(p+1)/{\gamma'}^3\}.$$

Now let $n \geqslant n_0$, $n \in 2\mathbb{N}$ and let $\mathcal{H} = (V, E)$ be a 3-uniform hypergraph on n vertices with

$$\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16} + \gamma\right)\binom{n}{2}.$$

Let $\mathcal{P}_0 \subset \mathcal{H}$ be the absorbing path guaranteed by Lemma 7. Let a_0 and b_0 be the ends of $\mathcal{P}_0$ and note that

$$|V(\mathcal{P}_0)| \leqslant \gamma' n < \gamma n/8\,.$$

Moreover, the path $\mathcal{P}_0$ has the absorption property, i.e., for all $U \subset V \setminus V(\mathcal{P}_0)$ with $|U| \leqslant \beta n$ and $|U| \in 2\mathbb{N}$ there exists

$$\text{a loose path } \mathcal{Q} \subset \mathcal{H} \text{ s.t. } V(\mathcal{Q}) = V(\mathcal{P}_0) \cup U \text{ and } \mathcal{Q} \text{ has the ends } a_0 \text{ and } b_0. \tag{6}$$

Let $V' = (V \setminus V(\mathcal{P}_0)) \cup \{a_0, b_0\}$ and let $\mathcal{H}' = \mathcal{H}[V'] = (V', E(\mathcal{H}) \cap \binom{V'}{3})$ be the induced subhypergraph of $\mathcal{H}$ on V'. Note that $\delta_1(\mathcal{H}') \geqslant (\frac{7}{16} + \frac{3}{4}\gamma)\binom{n}{2}$.

Due to Lemma 6 we can choose a set $R \subset V'$ of size at most $\gamma'|V'| \leqslant \gamma' n$ such that for every system consisting of at most $(\gamma')^3|V'|/12$ mutually disjoint pairs of vertices from V can be connected using vertices from R only.

Set $V'' = V \smallsetminus (V(\mathcal{P}_0) \cup R)$ and let $\mathcal{H}'' = \mathcal{H}[V'']$ be the induced subhypergraph of $\mathcal{H}$ on V''. Clearly,

$$\delta(\mathcal{H}'') \geqslant \left(\frac{7}{16} + \frac{\gamma}{2}\right)\binom{n}{2}$$

Consequently, Lemma 10 applied to $\mathcal{H}''$ (with γ_{10} and α) yields a loose path tiling of $\mathcal{H}''$ which covers all but at most $\alpha|V''| \leqslant \alpha n$ vertices from V'' and which consists of at most p paths. We denote the set of the uncovered vertices in V'' by T. Further, let $\mathcal{P}_1, \mathcal{P}_2 \dots, \mathcal{P}_q$ with $q \leqslant p$ denote the paths of the tiling. By applying the reservoir lemma appropriately we connect the loose paths $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_q$ to one loose cycle $\mathcal{C} \subset \mathcal{H}$.

Let $U = V \smallsetminus V(\mathcal{C})$ be the set of vertices not covered by the cycle $\mathcal{C}$. Since $U \subseteq R \cup T$ we have $|U| \leqslant (\alpha + \gamma_6)n \leqslant \beta n$. Moreover, since $\mathcal{C}$ is a loose cycle and $n \in 2\mathbb{N}$ we have $|U| \in 2\mathbb{N}$. Thus, using the absorption property of $\mathcal{P}_0$ (see (6)) we can replace the subpath $\mathcal{P}_0$ in $\mathcal{C}$ by a path $\mathcal{Q}$ (since $\mathcal{P}_0$ and $\mathcal{Q}$ have the same ends) and since $V(\mathcal{Q}) = V(\mathcal{P}_0) \cup U$ the resulting cycle is a loose Hamilton cycle of $\mathcal{H}$. □

§3. Proof of the Path-tiling Lemma

In this section we give the proof of the Path-tiling Lemma, Lemma 10. Lemma 10 will be derived from the following lemma. Let $\mathcal{M}$ be the 3-uniform hypergraph defined on the vertex set $\{1, \dots, 8\}$ with the edges $123, 345, 456, 678 \in \mathcal{M}$. We will show that the condition $\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16} + o(1)\right)\binom{n}{2}$ will ensure an almost perfect $\mathcal{M}$-*tiling of* $\mathcal{H}$, i.e., a family of vertex disjoint copies of $\mathcal{M}$, which covers almost all vertices.

Lemma 11. *For all $\gamma > 0$ and $\alpha > 0$ there exists n_0 such that the following holds. Suppose $\mathcal{H}$ is a 3-uniform hypergraph on $n > n_0$ vertices with minimum vertex degree*

$$\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16} + \gamma\right)\binom{n}{2}.$$

Then there is an $\mathcal{M}$-tiling of $\mathcal{H}$ which covers all but at most αn vertices of $\mathcal{H}$.

The proof of Lemma 11 requires the regularity lemma which we introduce in Section 3.1. Sections 3.2 and 3.3 are devoted to the proof of Lemma 11 and finally, in Section 3.4, we deduce Lemma 10 from Lemma 11 by making use of the regularity lemma.

3.1. **The weak regularity lemma and the cluster hypergraph.** In this section we introduce the *weak hypergraph regularity lemma*, a straightforward extension of Szemerédi's regularity lemma for graphs [15]. Since we only apply the lemma to 3-uniform hypergraphs we will restrict the introduction to this case.

Let $\mathcal{H} = (V, E)$ be a 3-uniform hypergraph and let A_1, A_2, A_3 be mutually disjoint non-empty subsets of V. We define $e(A_1, A_2, A_3)$ to be the number of edges with one vertex in each A_i, $i \in [3]$, and the density of $\mathcal{H}$ with respect to (A_1, A_2, A_3) as

$$d(A_1, A_2, A_3) = \frac{e_{\mathcal{H}}(A_1, A_2, A_3)}{|A_1||A_2||A_3|}\,.$$

We say the triple (V_1, V_2, V_3) of mutually disjoint subsets $V_1, V_2, V_3 \subseteq V$ is (ε, d)-*regular*, for constants $\varepsilon > 0$ and $d \geqslant 0$, if

$$|d(A_1, A_2, A_3) - d| \leqslant \varepsilon$$

for all triple of subsets $A_i \subset V_i$, $i \in [3]$, satisfying $|A_i| \geqslant \varepsilon |V_i|$. The triple (V_1, V_2, V_3) is called ε-*regular* if it is (ε, d)-regular for some $d \geqslant 0$. It is immediate from the definition that an (ε, d)-regular triple (V_1, V_2, V_3) is (ε', d)-regular for all $\varepsilon' > \varepsilon$ and if $V_i' \subset V_i$ has size $|V_i'| \geqslant c|V_i|$, then (V_1', V_2', V_3') is $(\varepsilon/c, d)$-regular.

Next we show that regular triples can be almost perfectly covered by copies of $\mathcal{M}$ provided the sizes of the partition classes obey certain restrictions. First note that $\mathcal{M}$ is a subhypergraph of a tight path. The latter is defined similarly to a loose path, i.e. there is an ordering $(v_1, \ldots, v_t)$ of the vertices such that every edge consists of three consecutive vertices, every vertex is contained in an edge and two consecutive edges intersect in exactly two vertices.

Proposition 12. *Suppose $\mathcal{H}$ is a 3-uniform hypergraph on m vertices with at least dm^3 edges. Then there is a tight path in $\mathcal{H}$ which covers at least $2(dm+1)$ vertices. In particular, if $\mathcal{H}$ is 3-partite with the partition classes V_1, V_2, V_3 and $2dm > 10$ then for each $i \in [3]$ there is a copy of $\mathcal{M}$ in $\mathcal{H}$ which intersects V_i in exactly two vertices and the other partition classes in three vertices.*

Proof. Starting from $\mathcal{H}$ we remove all edges containing u, v for each pair $u, v \in V$ of vertices such that $0 < \deg(u, v) < 2dm$. We keep doing this until every pair u, v satisfies $\deg(u, v) = 0$ or $\deg(u, v) \geqslant 2dm$ in the current hypergraph $\mathcal{H}'$. Since less than

$$\binom{m}{2} \cdot 2dm < dm^3 \leqslant e(\mathcal{H})$$

edges were removed during the process we know that $\mathcal{H}'$ is not empty. Hence we can pick a maximal non-empty tight path $(v_1, v_2, \ldots, v_t)$ in $\mathcal{H}'$. Since the pair v_1, v_2 is contained in an edge in $\mathcal{H}'$ it is contained in $2dm$ edges and since the path was chosen to be maximal all these vertices must lie in the path. Hence, the chosen tight path contains at least $2(dm+1)$ vertices. This completes the first part of the proof.

For the second part, note that there is only one way to embed a tight path into a 3-partite 3-uniform hypergraph once the two starting vertices are fixed. Since $\mathcal{M}$ is a subhypergraph of the tight path on eight vertices we obtain the second part of the statement by possibly deleting up to two starting vertices. $\square$

Proposition 13. *Suppose the triple (V_1, V_2, V_3) is (ε, d)-regular with $d \geqslant 2\varepsilon$ and suppose the sizes of the partition classes satisfy*

$$m = |V_1| \geqslant |V_2| \geqslant |V_3| \text{ with } 5|V_1| \leqslant 3(|V_2| + |V_3|) \tag{7}$$

and $2\varepsilon^2 m > 7$. Then there is an $\mathcal{M}$-tiling of (V_1, V_2, V_3) leaving at most $3\varepsilon m$ vertices uncovered.

Proof. Note that if we take a copy of $\mathcal{M}$ intersecting V_i, $i \in [3]$ in exactly two vertices then this copy intersects the other partition classes in exactly three vertices. We define

$$t_i = (1-\varepsilon)\frac{1}{8}\left(3|V_j| + 3|V_k| - 5|V_i|\right) \quad \text{where} \quad i, j, k \in [3] \text{ are distinct.}$$

Due to our assumption all t_i are non-negative and we choose t_i copies of $\mathcal{M}$ intersecting V_i in exactly two vertices. This would leave $|V_i| - (2t_i + 3t_j + 3t_k) = \varepsilon|V_i|$ vertices in V_i uncovered, hence at most $3\varepsilon m$ in total.

To complete the proof we exhibit a copy of $\mathcal{M}$ in all three possible types in the remaining hypergraph, hence showing that the choices of the copies above are indeed possible. To this end, from the remaining vertices of each partition class V_i take a subset U_i, $i \in [3]$ of size $\varepsilon|V_i|$. Due to the regularity of the triple (V_1, V_2, V_3) we have $e(U_1, U_2, U_3) \geqslant (d-\varepsilon)(\varepsilon m)^3$. Hence, by Proposition 12 there is a copy of $\mathcal{M}$ (of each type) in (U_1, U_2, U_3). $\square$

The connection of regular partitions and dense hypergraphs is established by regularity lemmas. The version introduced here is a straightforward generalisation of the original regularity lemma to hypergraphs (see, e.g., [1, 2, 14]).

Theorem 14. *For all $t_0 \geqslant 0$ and $\varepsilon > 0$, there exist $T_0 = T_0(t_0, \varepsilon)$ and $n_0 = n_0(t_0, \varepsilon)$ so that for every 3-uniform hypergraph $\mathcal{H} = (V, E)$ on $n \geqslant n_0$ vertices, there exists a partition $V = V_0 \,\dot\cup\, V_1 \,\dot\cup\, \ldots \,\dot\cup\, V_t$ such that*

(i) $t_0 \leqslant t \leqslant T_0$,
(ii) $|V_1| = |V_2| = \cdots = |V_t|$ *and* $|V_0| \leqslant \varepsilon n$,
(iii) *for all but at most* $\varepsilon\binom{t}{3}$ *sets* $\{i_1, i_2, i_3\} \in \binom{[t]}{3}$, *the triple* $(V_{i_1}, V_{i_2}, V_{i_3})$ *is* ε*-regular.* $\square$

A partition as given in Theorem 14 is called an (ε, t)-*regular partition* of $\mathcal{H}$. For an (ε, t)-regular partition of $\mathcal{H}$ and $d \geqslant 0$ we refer to $\mathcal{Q} = (V_i)_{i\in[t]}$ as the family of *clusters* (note that the exceptional vertex set V_0 is excluded) and define the *cluster hypergraph* $\mathcal{K} = \mathcal{K}(\varepsilon, d, \mathcal{Q})$ with vertex set $[t]$ and $\{i_1, i_2, i_3\} \in \binom{[t]}{3}$ being an edge if and only if $(V_{i_1}, V_{i_2}, V_{i_3})$ is ε-regular and $d(V_{i_1}, V_{i_2}, V_{i_3}) \geqslant d$.

In the following we show that the cluster hypergraph almost inherits the minimum vertex degree of the original hypergraph. The proof which we give for completeness is standard and can be found e.g. in [8] for the case of graphs.

Proposition 15. *For all $\gamma > d > \varepsilon > 0$ and all t_0 there exist T_0 and $n_0 \in \mathbb{N}$ such that the following holds.*

If $\mathcal{H}$ is a 3-uniform hypergraph on $n > n_0$ vertices with $\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16} + \gamma\right)\binom{n}{2}$, then there exists an (ε, t)-regular partition $\mathcal{Q}$ with $t_0 < t < T_0$ such that the cluster hypergraph $\mathcal{K} = \mathcal{K}(\varepsilon, d, \mathcal{Q})$ has minimum vertex degree $\delta_1(\mathcal{K}) \geqslant \left(\frac{7}{16} + \gamma - \varepsilon - d\right)\binom{t}{2}$.

Proof. Let $\gamma > d > \varepsilon$ and t_0 be given. We apply the regularity lemma with $\varepsilon' = \varepsilon^2/144$ and $t_0' = \max\{2t_0, 10/\varepsilon\}$ to obtain T_0' and n_0'. We set $T_0 = T_0'$ and $n_0 = n_0'$. Let $\mathcal{H}$ be a 3-uniform hypergraph on $n > n_0$ vertices which satisfies $\delta(\mathcal{H}) \geqslant (7/16+\gamma)\binom{n}{2}$. By applying the regularity lemma we obtain an (ε', t')-regular partition $V_0' \cup V_1 \cup \ldots \cup V_{t'}$ of V and let $m = |V_1| = (1-\varepsilon')n/t'$ denote the size of the partition classes.

Let $I = \{i \in [t'] \colon V_i$ is contained in more than $\varepsilon\binom{t'}{2}/8$ non ε'-regular triples$\}$ and observe that $|I| < 8\varepsilon' t'/\varepsilon$ due to the property (iii) of Theorem 14. Set $V_0 = V_0' \cup \bigcup_{i\in I} V_i$ and let $J = [t'] \smallsetminus I$ and $t = |J|$. We now claim that V_0 and $\mathcal{Q} = (V_j)_{j\in J}$ is the desired partition. Indeed, we have $T_0 > t' \geqslant t > t'(1 - 8\varepsilon'/\varepsilon) \geqslant t_0$ and $|V_0| < \varepsilon' n + 8\varepsilon' n/\varepsilon \leqslant \varepsilon n/16$. The property (iii) follows directly from Theorem 14. For a contradiction, assume now that $\deg_{\mathcal{K}}(V_j) < (\frac{7}{16} + \gamma - \varepsilon - d)\binom{t}{2}$ for some $j \in J$. Let x_j denote the number of edges which intersect V_j in exactly one vertex and each other V_i, $i \in J$, in at most one vertex. Then, the assumption yields

$$\begin{aligned} x_j &\leqslant |V_j| \left[\left(\frac{7}{16} + \gamma - \varepsilon - d\right)\binom{t}{2} m^2 + \frac{\varepsilon}{8}\binom{t'}{2} m^2 + \frac{\varepsilon}{16} n^2 + d\binom{t}{2} m^2 \right] \\ &\leqslant |V_j| \frac{n^2}{2}\left(\frac{7}{16} + \gamma - \frac{\varepsilon}{2}\right) \end{aligned}$$

On the other hand, from the minimum degree of $\mathcal{H}$ we obtain

$$x_j \geqslant |V_j|\left(\frac{7}{16} + \gamma\right)\binom{n}{2} - 2\binom{|V_j|}{2} n - 3\binom{|V_j|}{3} \geqslant |V_j|\binom{n}{2}\left(\frac{7}{16} + \gamma - \frac{4}{t'}\right)$$

a contradiction. □

3.2. **Fractional** $\hom(\mathcal{M})$**-tiling.** To obtain a large $\mathcal{M}$-tiling in the hypergraph $\mathcal{H}$, we consider weighted homomorphisms from $\mathcal{M}$ into the cluster hypergraph $\mathcal{K}$. To this purpose, we define the following.

Definition 16. *Let $\mathcal{L}$ be a 3-uniform hypergraph. A function $h\colon V(\mathcal{L}) \times E(\mathcal{L}) \to [0,1]$ is called a fractional* $\hom(\mathcal{M})$*-tiling of $\mathcal{L}$ if*

(*a*) $h(v,e) \neq 0 \Rightarrow v \in e$,
(*b*) $h(v) = \sum_{e\in E(\mathcal{L})} h(v,e) \leqslant 1$,
(*c*) *for every $e \in E(\mathcal{L})$ there exists a labeling of the vertices of $e = uvw$ such that*

$$h(u,e) = h(v,e) \geqslant h(w,e) \geqslant \frac{2}{3}h(u,e)$$

By $h_{\min}$ we denote the smallest non-zero value of $h(v,e)$ (and we set $h_{\min} = \infty$ if $h \equiv 0$) and the sum over all values is the weight $w(h)$ *of h*

$$w(h) = \sum_{(v,e)\in V(\mathcal{L})\times E(\mathcal{L})} h(v,e)\,.$$

The allowed values of h are based on the homomorphisms from $\mathcal{M}$ to a single edge, hence the term $\hom(\mathcal{M})$-tiling. Given one such homomorphism, assign each vertex in the image the number of vertices from $\mathcal{M}$ mapped to it. In fact, for any such homomorphism the preimage of one vertex has size two, while the preimages of the other two vertices has size three. Consequently, for any family of homomorphisms of $\mathcal{M}$ into a single edge the smallest and the largest class of preimages can differ by a factor of $2/3$ at most and this observation is the reason for condition (*c*) in Definition 16. We also note the following.

Fact 17. *There is a fractional* $\hom(\mathcal{M})$*-tiling h of the hypergraph $\mathcal{M}$ which has $h_{\min} \geqslant 1/3$ and weight $w(h) = 8$.*

Proof. Let x_1, x_2, w_1, y_1, y_2, w_2, z_1, and z_2 be the vertices of $\mathcal{M}$ and let

$$x_1x_2w_1\,, \quad w_1y_1y_2\,, \quad y_1y_2w_2\,, \quad \text{and} \quad w_2z_1z_2$$

be the edges of $\mathcal{M}$. On the edges $x_1x_2w_1$ and $z_1z_2w_2$ we assign the vertex weights $(1,1,2/3)$, where the weight $2/3$ is assigned to w_1 and w_2. The vertex weights for edges $y_1y_2w_1$ and $y_1y_2w_2$ are $(1/2,1/2,1/3)$, where w_1 and w_2 get the weight $1/3$. It is easy to see that those vertex weights give rise to a $\hom(\mathcal{M})$-tiling h on $\mathcal{M}$ with $h_{\min} = 1/3$ and $w(h) = 8$. □

The notion $\hom(\mathcal{M})$-tiling is also motivated by the following proposition which shows that such a fractional $\hom(\mathcal{M})$-tiling in a cluster hypergraph can be "converted" to an integer $\mathcal{M}$-tiling in the original hypergraph.

Proposition 18. *Let $\mathcal{Q}$ be an (ε, t)-regular partition of a 3-uniform, n-vertex hypergraph $\mathcal{H}$ with $n > 21\varepsilon^{-2}$ and let $\mathcal{K} = \mathcal{K}(\varepsilon, 6\varepsilon, \mathcal{Q})$ be the corresponding cluster hypergraph. Furthermore, let $h\colon V(\mathcal{K}) \times E(\mathcal{K}) \to [0,1]$ be a fractional* $\hom(\mathcal{M})$*-tiling of $\mathcal{K}$ with $h_{\min} \geqslant 1/3$. Then there exists an $\mathcal{M}$-tiling of $\mathcal{H}$ which covers all but at most $(w(h) - 27t\varepsilon)|V_1|$ vertices.*

Proof. We restrict our consideration to the subhypergraph $\mathcal{K}' \subset \mathcal{K}$ consisting of the hyperedges with positive weight, i.e., $e = abc \in \mathcal{K}$ with $h(a), h(b), h(c) \geqslant h_{\min}$. For each $a \in V(\mathcal{K}')$ let V_a be the corresponding partition class in $\mathcal{Q}$. Due to the property (b) of Definition 16 we can subdivide V_a (arbitrarily) into a collection of pairwise disjoint sets $(U_a^e)_{a \in e \in \mathcal{K}}$ of size $|U_a^e| = h(a,e)|V_a|$. Note that every edge $e = abc \in \mathcal{K}$ corresponds to the $(\varepsilon, 6\varepsilon)$-regular triplet (V_a, V_b, V_c). Hence we obtain from the definition of regularity and $h_{\min} \geqslant 1/3$ that the triplet (U_a^e, U_b^e, U_c^e) is $(3\varepsilon, 6\varepsilon)$-regular. From the property (c) in Definition 16 and Proposition 13 we obtain an $\mathcal{M}$-tiling of (U_a^e, U_b^e, U_c^e) incorporating at least $\big(h(a,e) + h(b,e) + h(c,e) - 9\varepsilon\big)|V_a|$ vertices. Applying this to all hyperedges of $\mathcal{K}'$ we obtain an $\mathcal{M}$-tiling incorporating at least

$$\left(\sum_{abc=e\in\mathcal{K}'} h(a,e) + h(b,e) + h(c,e) - 9\varepsilon\right)|V_a| \geqslant \big(w(h) - 9|\mathcal{K}'|\varepsilon\big)|V_a|$$

vertices. Noting that $|\mathcal{K}'| \leqslant 3t$ (because of $h_{\min} \geqslant 1/3$) and $|V_a| \geqslant |V_1|$ we obtain the proposition. □

Owing to Proposition 18, we are given a connection between fractional $\hom(\mathcal{M})$-tilings of the cluster hypergraph $\mathcal{K}$ of $\mathcal{H}$ and $\mathcal{M}$-tilings in $\mathcal{H}$. A vertex $i \in V(\mathcal{K})$ corresponds to a class of vertices V_i in the regular partition of $\mathcal{H}$. The total vertex weight $h(i)$ essentially translates to the proportion of vertices of V_i which can be covered by the corresponding $\mathcal{M}$-tilings in $\mathcal{H}$. Consequently, $w(h)$ essentially translates to the proportion of vertices covered by the corresponding $\mathcal{M}$-tiling in $\mathcal{H}$. This reduces our task to finding a fractional $\hom(\mathcal{M})$-tiling with weight greater than the number of vertices previously covered in $\mathcal{K}$.

The following lemma (Lemma 19), which is the main tool for the proof of Lemma 11, follows the idea discussed above. In the proof of Lemma 11 we fix a maximal $\mathcal{M}$-tiling in the cluster hypergraph $\mathcal{K}$ of the given hypergraph $\mathcal{H}$. Owing to the minimum degree condition of $\mathcal{H}$ and Proposition 15, a typical vertex in the cluster hypergraph $\mathcal{K}$ will be contained in at least $(7/16 + o(1))\binom{|V(\mathcal{K})|}{2}$ hyperedges of $\mathcal{K}$. We will show that a typical vertex u of $\mathcal{K}$ which is not covered by the maximal $\mathcal{M}$-tiling of $\mathcal{K}$, has the property that $(7/16 + o(1)) \cdot 64 > 28$ of the edges incident to u intersect some pair of copies of $\mathcal{M}$ from the $\mathcal{M}$-tiling of $\mathcal{K}$. Lemma 19 asserts that two such vertices and the pair of copies of $\mathcal{M}$ can be used to obtain a fractional $\hom(\mathcal{M})$-tiling with a weight significantly larger than 16, the number of vertices of the two copies of $\mathcal{M}$. This lemma will come in handy in the proof of Lemma 11, where it is used to show that one can cover a higher proportion of the vertices of $\mathcal{H}$ than the proportion of vertices covered by the largest $\mathcal{M}$-tiling in $\mathcal{K}$.

We consider a set of hypergraphs $\mathscr{L}_{29}$ definied as follows: Every $\mathcal{L} \in \mathscr{L}_{29}$ consists of two (vertex disjoint) copies of $\mathcal{M}$, say $\mathcal{M}_1$ and $\mathcal{M}_2$, and two additional vertices u and v such that all edges incident to u or v contain precisely one vertex from $V(\mathcal{M}_1)$ and one vertex from $V(\mathcal{M}_2)$. Moreover, $\mathcal{L}$ satisfies the following properties

- for every $a \in V(\mathcal{M}_1)$ and $b \in V(\mathcal{M}_2)$ we have $uab \in E(\mathcal{L})$ iff $vab \in E(\mathcal{L})$
- $\deg(u) = \deg(v) \geqslant 29$.

Lemma 19. *For every $\mathcal{L} \in \mathscr{L}_{29}$ there exists a fractional* $\hom(\mathcal{M})$*-tiling h with $h_{\min} \geqslant 1/3$ and $w(h) \geqslant 16 + \frac{1}{3}$.*

The following proof of Lemma 19 is based on straightforward, but somewhat tedious case distinction.

Proof. For the proof we fix the following labeling of the vertices of the two disjoint copies of $\mathcal{M}$. Let

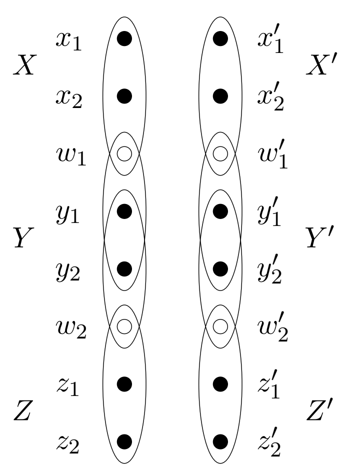
$$V(\mathcal{M}_1) = \{x_1, x_2, w_1, y_1, y_2, w_2, z_1, z_2\} \quad \text{and} \quad E(\mathcal{M}_1) = \big\{x_1x_2w_1,\ w_1y_1y_2,\ y_1y_2w_2,\ w_2z_1z_2\big\}$$

be the vertices and edges of the first copy of $\mathcal{M}$. Analogously, let

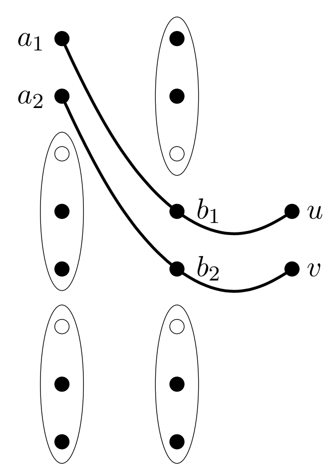
$$V(\mathcal{M}_2) = \{x'_1, x'_2, w'_1, y'_1, y'_2, w'_2, z'_1, z'_2\} \quad \text{and} \quad E(\mathcal{M}_2) = \big\{x'_1x'_2w'_1,\ w'_1y'_1y'_2,\ y'_1y'_2w'_2,\ w'_2z'_1z'_2\big\}$$

be the vertices and edges of the other copy of $\mathcal{M}$ (see Figure 1.a). Moreover, we set $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2\}$ and, let X', Y', and Z' be defined analogously for $\mathcal{M}_2$.

FIGURE 1. Labels and case: a_1b_1, $a_2b_2 \in L_1$ with $\{b_1, b_2\} \in \{X', Y', Z'\}$



1.a: Vertex labels of $\mathcal{M}_1$ and $\mathcal{M}_2$ in $\mathcal{L}$



1.b: All edges are (a1)-edges

The proof of Lemma 19 proceeds in two steps. First, we show that in any possible counterexample $\mathcal{L}$, the edges incident to u and v which do not contain any vertex from $\{w_1, w_2, w_1', w_2'\}$ form a subgraph of $K_{2,3,3}$ (see Claim 20). In the second step we show that every edge contained in this subgraph of $K_{2,3,3}$ forbids too many other edges incident to u and v, which will yield a contradiction to the condition $\deg(u) = \deg(v) \geqslant 29$ of $\mathcal{L}$ (see Claim 21).

We introduce the following notation to simplify later arguments. For a given $\mathcal{L} \in \mathscr{L}_{29}$ with $\mathcal{M}_1$ and $\mathcal{M}_2$ being the copies of $\mathcal{M}$, let L be set of the pairs $(a, b) \in V(\mathcal{M}_1) \times V(\mathcal{M}_2)$ such that $uab \in E(\mathcal{L})$. We split L into $L_1 \cup L_2$ according to

$$(a, b) \in \begin{cases} L_1, & \text{if } \{a, b\} \cap \{w_1, w_2, w_1', w_2'\} = \varnothing, \\ L_2, & \text{otherwise.} \end{cases}$$

It will be convenient to view L_1 and L_2 as bipartite graphs with vertex classes $V(\mathcal{M}_1)$ and $V(\mathcal{M}_2)$.

We split the proof of Lemma 19 into the following two claims.

Claim 20. *For all $\mathcal{L} \in \mathscr{L}_{29}$ without a fractional* $\hom(\mathcal{M})$*-tiling with $h_{\min} \geqslant 1/3$ and $w(h) \geqslant 16 + 1/3$, we have $L_1 \subseteq K_{3,3}$, where each of the sets X, Y, Z and X', Y', Z' contains precisely one of the vertices of the $K_{3,3}$.*

Claim 20 will be used in the proof of the next claim, which clearly implies Lemma 19.

Claim 21. *Let $F = \big\{a'b' \in V(\mathcal{M}_1) \times V(\mathcal{M}_2)\colon\ a' \in \{w_1, w_2\} \text{ or } b' \in \{w_1', w_2'\}\big\}$ and for every edge $ab \in L_1$ let $\mathcal{F}(a, b) \subseteq F$ be the set of those $e \in F$, whose appearance in $\mathcal{L}$ (i.e. $e \in L_2$) implies the existence of a fractional* $\hom(\mathcal{M})$*-tiling h with $h_{\min} \geqslant 1/3$ and $w(h) \geqslant 16 + 1/3$. Then there is an injection $f\colon L_1 \to F$ such that $f(a, b) \in \mathcal{F}(a, b)$ for every pair $ab \in L_1$.*

Clearly, $|F| = 28$ and $L_2 \subset F$. Hence, from $|L_2| + |f(L_1)| = |L_2| + |L_1| \geqslant 29$ we derive that L_2 and $f(L_1)$ must intersect. By Claim 21 this yields the desired fractional $\hom(\mathcal{M})$-tiling and Lemma 19 follows. □

In the proofs of Claim 20 and Claim 21 we will consider fractional $\hom(\mathcal{M})$-tilings h which use vertex weights of special types. In fact, for an edge $e = a_1a_2a_3$, the weights $h(a_1, e)$, $h(a_2, e)$, and $h(a_3, e)$ will be of the following forms

(a1) $h(a_1, e) = h(a_2, e) = h(a_3, e) = 1$

(a2) $h(a_1, e) = h(a_2, e) = h(a_3, e) = \frac{1}{2}$

(a3) $h(a_1, e) = h(a_2, e) = h(a_3, e) = \frac{1}{3}$

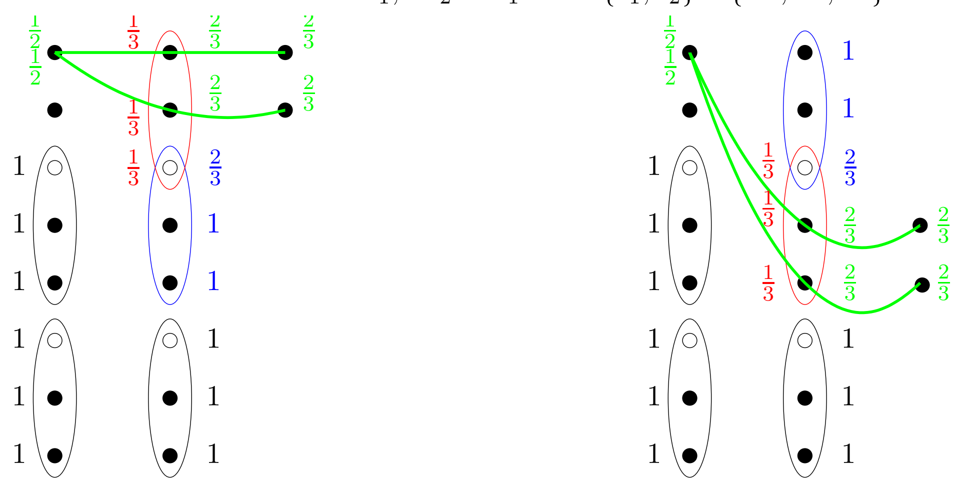
(b1) $h(a_1, e) = h(a_2, e) = 1$ and $h(a_3, e) = \frac{2}{3}$

(b2) $h(a_1, e) = h(a_2, e) = \frac{1}{2}$ and $h(a_3, e) = \frac{1}{3}$
(b3) $h(a_1, e) = h(a_2, e) = \frac{2}{3}$ and $h(a_3, e) = \frac{1}{2}$

An edge that satisfies (a1) is called an (a1)-edge, etc. Note that all these types satisfy condition (c) of Definition 16.

Proof of Claim 20. Given $\mathcal{L} \in \mathscr{L}_{29}$ satisfying the assumptions of the claim and with the labeling from Figure 1.a. Observe that for any $A \in \{X, Y, Z\}$, the hypergraph $\mathcal{M}_1 - A$ contains two disjoint edges. Similarly, for every $B \in \{X', Y', Z'\}$, $\mathcal{M}_2 - B$ contains two disjoint edges.

FIGURE 2. Case: ab_1, $ab_2 \in L_1$ with $\{b_1, b_2\} \in \{X', Y', Z'\}$



2.a: (a1)-edges $w_1y_1y_2$, $w_2z_1z_2$, and $w_2'z_1'z_2'$, (b3)-edges $ax_1'u$ and $ax_2'v$, (b1)-edge $w_1'y_1'y_2'$, and (a3)-edge $x_1'x_2'w_1$.

2.b: (a1)-edges $w_1y_1y_2$, $w_2z_1z_2$, and $w_2'z_1'z_2'$, (b3)-edges $ay_1'u$ and $ay_2'v$, (b1)-edge $x_1'x_2'w_1'$, and (a3)-edge $w_1'y_1'y_2'$.

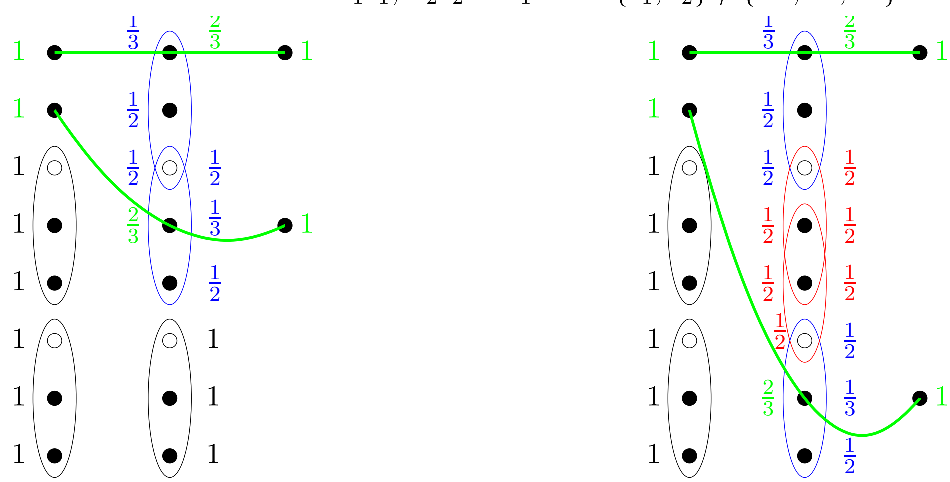
First we exclude the case that there is a matching $\{a_1b_1, a_2b_2\}$ of size two in L_1 between some $\{a_1, a_2\} = A \in \{X, Y, Z\}$ and some $\{b_1, b_2\} = B \in \{X', Y', Z'\}$. In this case we can construct a fractional hom($\mathcal{M}$)-tiling h as follows: Choose two edges ua_1b_1, va_2b_2. Using these and the four disjoint edges in $(\mathcal{M}_1 - A) \cup (\mathcal{M}_2 - B)$, we obtain six disjoint edges (see Figure 1.b). Letting all these six edges be (a1)-edges, we obtain a fractional hom($\mathcal{M}$)-tiling h with $h_{\min} = 1$ and $w(h) = 18$.

We show that every $a \in A \in \{X, Y, Z\}$ has at most one neighbour in each $B \in \{X', Y', Z'\}$. Assuming the contrary, let $a \in A \in \{X, Y, Z\}$ and $\{b_1, b_2\} = B \in \{X', Y', Z'\}$, with ab_1, $ab_2 \in L_1$. For symmetry reasons, we only have to consider the case $B = X'$ and $B = Y'$. The case $B = Z'$ is symmetric to $B = X'$. In those cases, we choose h as shown in Figure 2.a ($B = X'$) and Figure 2.b ($B = Y'$) and in either case we find a fractional hom($\mathcal{M}$)-tiling h satisfying $h_{\min} = 1/3$ and $w(h) = 16 + 1/3$. Note that the cases $A = Y$

and $A = Z$ can be treated in the same manner since the only condition needed to define h is that $\mathcal{M}_1 - A$ contains two disjoint edges.

To show that L_1 is indeed contained in a $K_{3,3}$ it remains to verify that every a_1b_1, a_2b_2 with $\{a_1, a_2\} = A \in \{X, Y, Z\}$, $b_1 \in B_1 \in \{X', Y', Z'\}$, and $b_2 \in B_2 \in \{X', Y', Z'\} \smallsetminus B_1$ guarantees the existence of a fractional hom$(\mathcal{M})$-tiling h with $h_{\min} \geqslant 1/3$ and $w(h) \geqslant 16 + 1/3$. Again owing to the symmetry, the only cases we need to consider are $B_1 = X', B_2 = Y'$ (see Figure 3.a) and $B_1 = X', B_2 = Z'$ (see Figure 3.b). In fact, the fractional hom$(\mathcal{M})$-tilings h given in Figure 3.a and Figure 3.b satisfy $h_{\min} \geqslant 1/3$ and $w(h) = 17$. Again the cases $A = Y$ and $A = Z$ can be treated in the same manner. This concludes the proof of Claim 20. $\square$

FIGURE 3. Case: a_1b_1, $a_2b_2 \in L_1$ with $\{b_1, b_2\} \notin \{X', Y', Z'\}$



3.a: (a1)-edges $w_1y_1y_2$, $w_2z_1z_2$, and $w_2'z_1'z_2'$, (b1)-edges $a_1x_1'u$ and $a_2y_1'v$, and (b2)-edges $x_1'x_2'w_1'$ and $w_1'y_1'y_2'$.

3.b: (a1)-edges $w_1y_1y_2$ and $w_2z_1z_2$, (b1)-edges $a_1x_1'u$ and $a_2z_1'v$, (b2)-edges $x_1'x_2'w_1'$ and $w_2'z_1'z_2'$, and (a2)-edges $w_1'y_1'y_2'$ and $y_1'y_2'w_2'$.

To complete the proof of Lemma 19 it is left to prove Claim 21.

Proof of Claim 21. Before defining the injection $f\colon L_1 \to F$ we collect some information about $\mathcal{F}(a, b)$ with $ab \in L_1$. Owing to Claim 20, we may assume without loss of generality that x_1, y_1, z_1 and x_1', y_1', z_1' are the vertices which span all edges of L_1. First we consider $e = y_1y_1'$. As shown in Figure 4.a the appearance of $w_1y_2' \in L_1$ would give rise to a fractional hom$(\mathcal{M})$-tiling h with $h_{\min} \geqslant 1/3$ and $w(h) = 16.5$. Consequently, we have

$$w_1y_2' \in \mathcal{F}(y_1, y_1').$$

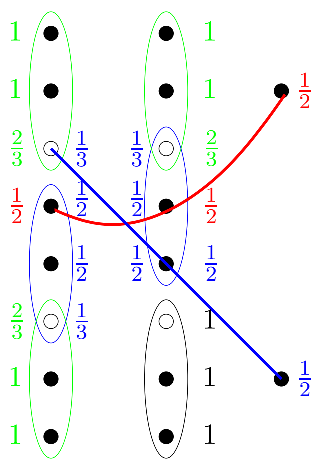
For the case $e = x_1x_1' \in L_1$, Figure 4.b, shows that $x_2w_1' \in \mathcal{F}(x_1, x_1')$ and by symmetry, it follows that

$$\{x_2w_1', w_1x_2'\} \subseteq \mathcal{F}(x_1, x_1')\,.$$

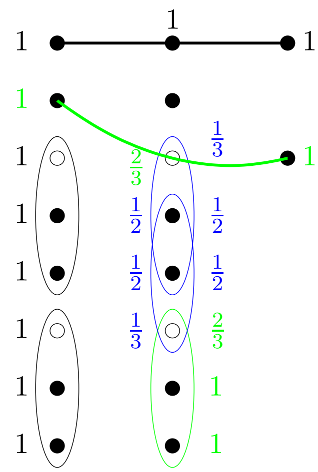
By applying appropriate automorphisms to $\mathcal{M}_1$ and $\mathcal{M}_2$ we immediately obtain information on $\mathcal{F}(x_1, z_1')$, $\mathcal{F}(z_1, x_1')$, and $\mathcal{F}(z_1, z_1')$. Indeed, we have

$$\{x_2w_2', w_1z_2'\} \subseteq \mathcal{F}(x_1, z_1')\,,\ \{w_2x_2', z_2w_1'\} \subseteq \mathcal{F}(z_1, x_1')\,,\ \{z_2w_2', w_2z_2'\} \subseteq \mathcal{F}(z_1, z_1')\,.$$

FIGURE 4. $\mathcal{F}(y_1, y_1')$ and $\mathcal{F}(x_1, x_1')$



4.a: (a1)-edge $w_2'z_1'z_2'$, (a2)-edge $y_1y_1'u$, (b1)-edges $x_1x_2w_1$, $w_1z_1z_2$, and $x_1'x_2'w_1'$, and (b2)-edges $w_1y_2'v$, $y_1y_2w_2$, and $w_1'y_1'y_2'$.



4.b: (a1)-edges $x_1x_1'u$, $w_1y_1y_2$, and $w_2z_1z_2$, (b1)-edges $x_2w_1'v$ and $w_2'z_1'z_2'$, and (b2)-edges $w_1'y_1'y_2'$ and $y_1'y_2'w_2'$.

Next we consider $e = y_1x_1'$. In this case Figure 5.a shows that $y_2w_1' \in \mathcal{F}(y_1, x_1')$. Moreover, as shown in Figure 5.b we also have $w_1x_2' \in \mathcal{F}(y_1, x_1')$ and, consequently, we obtain

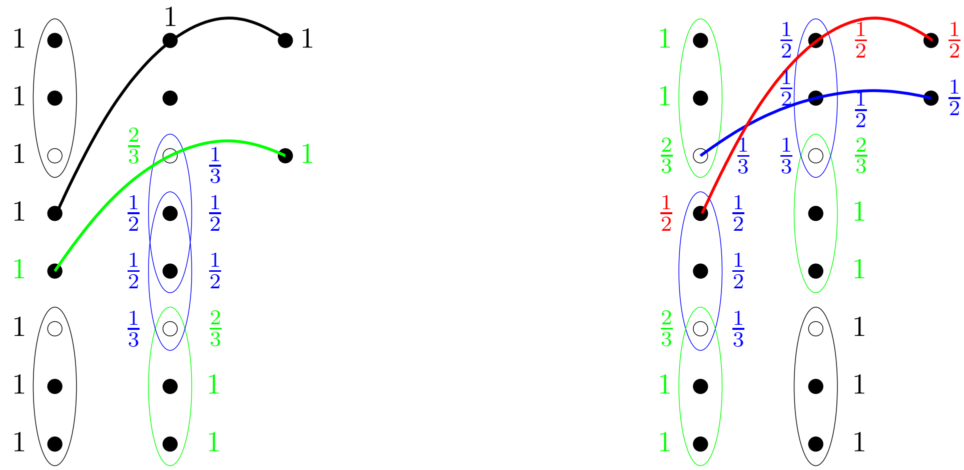
$$\{y_2w_1', w_1x_2'\} \subseteq \mathcal{F}(y_1, x_1')\,.$$

Again applying appropriate automorphisms to $\mathcal{M}_1$ and $\mathcal{M}_2$ we immediately obtain information on $\mathcal{F}(x_1, y_1')$, $\mathcal{F}(y_1, z_1')$, and $\mathcal{F}(z_1, y_1')$. Indeed one can show

$$\{w_1y_2', x_2w_1'\} \subseteq \mathcal{F}(x_1, y_1')\,,\ \{y_2w_2', w_1z_2'\} \subseteq \mathcal{F}(y_1, z_1')\,,\ \{w_2y_2', z_2w_1'\} \subseteq \mathcal{F}(z_1, y_1')\,.$$

Finally, we define an injection $f\colon L_1 \to F \supseteq L_2$ such that $f(a, b) \in \mathcal{F}(a, b)$ for every pair $ab \in L_1$. Recall that due to Claim 20 we have $L_1 \subseteq \{x_1, y_1, z_1\} \times \{x_1', y_1', z_1'\}$ and it

FIGURE 5. $\mathcal{F}(y_1x_1')$



5.a: (a1)-edges y_1x_1u, $x_1x_2w_1$, and $w_2z_1z_2$, (b1)-edges $y_2w_1'v$ and $w_2'z_1'z_2'$, and (b2)-edges $w_1'y_1'y_2'$ and $y_1'y_2'w_2'$.

5.b: (a1)-edge $w_2'z_1'z_2'$, (a2)-edge $y_1x_1'u$, (b1)-edges $x_1x_2w_1$, $w_2z_1z_2$, and $w_1'y_1'y_2'$, and (b2)-edges $w_1x_2'v$, $y_1y_2w_2$, and $x_1'x_2'w_1'$.

follows from the discussion above that we can fix f as follows

$$
\begin{aligned}
f(x_1,x_1') &= w_1x_2'\,, & f(x_1,y_1') &= x_2w_1'\,, & f(x_1,z_1') &= x_2w_2'\,,\\
f(y_1,x_1') &= y_2w_1'\,, & f(y_1,y_1') &= w_1y_2'\,, & f(y_1,z_1') &= w_1z_2'\,,\\
f(z_1,x_1') &= w_2x_2'\,, & f(z_1,y_1') &= w_2y_2'\,, & f(z_1,z_1') &= z_2w_2'\,.
\end{aligned}
$$

Consequently, $|L_1| \leqslant |F| - |L_2|$ and Claim 21 follows from $|L_2| + |L_1| \leqslant |F| \leqslant 28$. □

3.3. **Proof of the $\mathcal{M}$-tiling Lemma.** In this section we prove Lemma 11. Let $\mathcal{H}$ be a 3-uniform hypergraph on n vertices. We say $\mathcal{H}$ has a β-*deficient* $\mathcal{M}$-tiling if there exists a family of pairwise disjoint copies of $\mathcal{M}$ in $\mathcal{H}$ leaving at most βn vertices uncovered.

Proposition 22. *For all $1/2 > d > 0$ and all $\beta, \delta > 0$ the following holds. Suppose there exists an n_0 such that every 3-uniform hypergraph $\mathcal{H}$ on $n > n_0$ vertices with minimum vertex degree $\delta_1(\mathcal{H}) \geqslant d\binom{n}{2}$ has a β-deficient $\mathcal{M}$-tiling. Then every 3-uniform hypergraph $\mathcal{H}'$ on $n' > n_0$ vertices with $\delta_1(\mathcal{H}') \geqslant (d-\delta)\binom{n'}{2}$ has a $(\beta + 25\sqrt{\delta})$-deficient $\mathcal{M}$-tiling.*

Proof. Given a 3-uniform hypergraph $\mathcal{H}'$ on $n' > n_0$ vertices with $\delta_1(\mathcal{H}') \geqslant (d-\delta)\binom{n'}{2}$. By adding a set A of $3\sqrt{\delta}n'$ new vertices to $\mathcal{H}'$ and adding all triplets to $\mathcal{H}'$ which intersect A we obtain a new hyperpgraph $\mathcal{H}$ on $n = n' + |A|$ vertices which satisfies $\delta_1(\mathcal{H}) \geqslant d\binom{n}{2}$. Consequently, $\mathcal{H}$ has a β-deficient $\mathcal{M}$-tiling and by removing the $\mathcal{M}$-copies intersecting A, we obtain a $(\beta + 25\sqrt{\delta})$-deficient $\mathcal{M}$-tiling of $\mathcal{H}'$. □

Proof of Lemma 11. Let $\gamma > 0$ be given and we assume that there is an $\alpha > 0$ such that for all n_0' there is a 3-uniform hypergraph $\mathcal{H}$ on $n > n_0'$ vertices which satisfies $\delta_1(\mathcal{H}) \geqslant (\frac{7}{16} + \gamma)\binom{n}{2}$ but which does not contain an α-deficient $\mathcal{M}$-tiling. Let α_0 be the supremum of all such α and note that α_0 is bounded away from one due to Proposition 12.

We choose $\varepsilon = (\gamma\alpha_0/2^{100})^2$. By definition of α_0, there is an n_0 such that all 3-uniform hypergraphs $\mathcal{H}$ on $n > n_0$ vertices satisfying $\delta_1(\mathcal{H}) \geqslant (\frac{7}{16} + \gamma)n$ have an $(\alpha_0 + \varepsilon)$-deficient $\mathcal{M}$-tiling. Hence, by Proposition 22 all 3-uniform hypergraphs $\mathcal{H}$ on $n > n_0$ vertices satisfying $\delta_1(\mathcal{H}) \geqslant (\frac{7}{16} + \gamma - \varepsilon)\binom{n}{2}$ have an $(\alpha_0 + \varepsilon + 25\sqrt{\varepsilon})$-deficient $\mathcal{M}$-tiling. We will show that there exists an n_1 (to be chosen) such that all 3-uniform hypergraphs $\mathcal{H}$ on $n > n_1$ vertices satisfying $\delta_1(\mathcal{H}) \geqslant (\frac{7}{16} + \gamma)\binom{n}{2}$ have an $(\alpha_0 - \varepsilon)$-deficient $\mathcal{M}$-tiling, contradicting the definition of α_0.

We apply Proposition 15 with the constants γ, $\varepsilon/12$, $d = \varepsilon/2$ and $t_{15} = \max\{n_0, (\varepsilon/12)^{-3}\}$ to obtain an n_{15} and T_{15}. Let $n_1 \geqslant \max\{n_{15}, n_0\}$ be sufficiently large and let $\mathcal{H}$ be an arbitrary 3-uniform hypergraph on $n > n_1$ vertices which satisfies $\delta_1(\mathcal{H}) \geqslant (\frac{7}{16} + \gamma)\binom{n}{2}$ but which does not contain an α_0-deficient $\mathcal{M}$-tiling. We apply Proposition 15 to $\mathcal{H}$ with the constants chosen above and obtain a cluster hypergraph $\mathcal{K} = \mathcal{K}(\varepsilon/12, \varepsilon/2, \mathcal{Q})$ on $t > t_{15}$ vertices which satisfies $\delta_1(\mathcal{K}) \geqslant (\frac{7}{16} + \gamma - \varepsilon)\binom{t}{2}$. Taking $\mathscr{M}$ to be the largest $\mathcal{M}$-tiling in $\mathcal{K}$ we know by the definition of α_0 and by Proposition 22 that $\mathscr{M}$ is an α_1-deficient $\mathcal{M}$-tiling of $\mathcal{K}$, for some $\alpha_1 \leqslant \alpha_0 + 26\sqrt{\varepsilon}$.

We claim that $\mathscr{M}$ is not $(\alpha_0/2)$-deficient and for a contradiction, assume the contrary. Then, from Fact 17, we know that for each $\mathcal{M}_j \in \mathscr{M}$ there is a fractional hom$(\mathcal{M})$-tiling h^j of $\mathcal{M}_j$ with $h^j_{\min} \geqslant 1/3$ and weight $w(h^j) = 8$. Hence, the union of all these fractional hom$(\mathcal{M})$-tiling gives rise to a fractional hom$(\mathcal{M})$-tiling h of $\mathcal{K}$ with $h_{\min} \geqslant 1/3$ and weight

$$w(h) \geqslant 8|\mathscr{M}| \geqslant t(1 - \alpha_0/2).$$

By applying Proposition 18 to the fractional hom$(\mathcal{M})$-tiling h (and recalling that the vertex classes $V_1, \dots, V_t$ of the regular partition has the same size, which was at least $(1 - \varepsilon/12)n/t$) we obtain an $\mathcal{M}$-tiling of $\mathcal{H}$ which covers at least

$$\big(w(h) - 3t\varepsilon\big)\left(1 - \frac{\varepsilon}{12}\right)\frac{n}{t} \geqslant (1 - \alpha_0 + \varepsilon)n$$

vertices of $\mathcal{H}$. This, however, yields a $(\alpha_0 - \varepsilon)$ deficient $\mathcal{M}$-tiling of $\mathcal{H}$ contradicting the choice of $\mathcal{H}$. Hence, $\mathscr{M}$ is not $(\alpha_0/2)$-deficient from which we conclude that X, the set of vertices in $\mathcal{K}$ not covered by $\mathscr{M}$, has size

$$|X| \geqslant \frac{\alpha_0 t}{2}. \tag{8}$$

For a pair $\mathcal{M}_i\mathcal{M}_j \in \binom{\mathscr{M}}{2}$ the edge $e \in \mathcal{K}$ is ij-crossing if $|e \cap V(\mathcal{M}_i)| = |e \cap V(\mathcal{M}_j)| = 1$.

Claim 23. *Let $\mathcal{C}$ be the set of all triples xij such that $x \in X$, $\mathcal{M}_i\mathcal{M}_j \in \binom{\mathscr{M}}{2}$ and there are at least 29 ij-crossing edges containing x. Then we have $|\mathcal{C}| \geqslant \gamma\binom{t}{2}|X|/72$.*

Proof. Let $\mathcal{A}$ be the set of those hyperedges in $\mathcal{K}$ which are completely contained in X and let $\mathcal{B}$ be the set of all the edges with exactly two vertices in X. Then it is sufficient to show that

$$|\mathcal{A}| \leqslant \frac{7}{16}\binom{|X|}{3} \quad \text{and} \quad |\mathcal{B}| \leqslant \frac{7}{2}\binom{|X|}{2}|\mathscr{M}|. \tag{9}$$

Indeed, assuming (9) and $|\mathcal{C}| \leqslant \gamma\binom{t}{2}|X|/72$ we obtain the following contradiction

$$\begin{aligned}
\sum_{x\in X}\deg(x) &\leqslant 3|\mathcal{A}| + 2|\mathcal{B}| + 28\left(|X|\binom{|\mathscr{M}|}{2} - |\mathcal{C}|\right) + 64|\mathcal{C}| + \binom{8}{2}|\mathscr{M}||X| \\
&\leqslant |X|\left[\frac{7}{16}\binom{|X|}{2} + \frac{7}{2}|X||\mathscr{M}| + 28\binom{|\mathscr{M}|}{2} + \frac{36}{72}\gamma\binom{t}{2} + \binom{8}{2}|\mathscr{M}|\right] \\
&\leqslant |X|\left[\left(\frac{7}{16} + \frac{\gamma}{2}\right)\binom{t}{2} + \binom{8}{2}|\mathscr{M}|\right] \\
&< |X| \cdot \delta_1(\mathcal{K})
\end{aligned}$$

where in the third inequality we used $\binom{t}{2} = \binom{|X|}{2} + 8|X||\mathscr{M}| + \binom{8|\mathscr{M}|}{2}$.

Note that the first part of (9) trivially holds since in the opposite case, using the first part of Proposition 12 we obtain a tight path in X of length at least eight. However, this path contains a copy of $\mathcal{M}$ as a subhypergraph which yields a contradiction to the maximality of $\mathscr{M}$.

To complete the proof let us assume $|\mathcal{B}| > \frac{7}{2}\binom{|X|}{2}|\mathscr{M}|$ from which we deduce that there is an $\mathcal{M}' \in \mathscr{M}$ such that $V(\mathcal{M}')$ intersects at least $\frac{7}{2}\binom{|X|}{2}$ edges from $\mathcal{B}$. From $V(\mathcal{M}')$ we remove the vertices which are contained in less than $13|X|$ edges from $\mathcal{B}$. Note that there are at least four vertices, say $v_1, \dots, v_4$, and at least $(3+\varepsilon)\binom{|X|}{2}$ edges from $\mathcal{B}$ left which intersect these vertices. Hence, there exists a pair x_1, x_2 such that $\{x_1, x_2, v_i\} \in \mathcal{H}$ for all $i = 1, \dots, 4$. Removing all edges intersecting x_1, x_2 we still have at least $(3+\varepsilon/2)\binom{|X|}{2}$ edges intersecting $v_1, \dots, v_4$ and we can find another pair x_3, x_4 disjoint from x_1, x_2 with $\{x_3, x_4, v_i\} \in \mathcal{H}$ for $i = 1, \dots, 4$. For each v_i we can find another edge from $\mathcal{B}$ containing v_i, keeping them all mutually disjoint and also disjoint from $\{x_1, x_2\}$ and $\{x_3, x_4\}$. This is possible since each v_i is contained in more than $13|X|$ edges from $\mathcal{B}$. This, however, yields two copies of $\mathcal{M}$ which contradicts the fact that $\mathscr{M}$ was a largest possible $\mathcal{M}$-tiling. □

The set X will be used to show that there is an $\mathcal{L} \in \mathscr{L}_{29}$ such that $\mathcal{K}$ contains many copies of $\mathcal{L}$.

Claim 24. *There is an element $\mathcal{L} \in \mathscr{L}_{29}$ and a family $\mathscr{L}$ of vertex disjoint copies of $\mathcal{L}$ in the cluster hypergraph $\mathcal{K}(\varepsilon/12, \varepsilon/2, \mathcal{Q})$ such that $|\mathscr{L}| \geqslant \gamma\alpha_0 t/2^{75}$.*

Proof. We consider the 3-uniform hypergraph $\mathcal{C}$ (as given from Claim 23) on the vertex set $X \cup \mathscr{M}$. Note that for fixed ij a vertex x is contained in at most 64 ij-crossing edges, thus there are at most 2^{64} different hypergraphs with the property that x is contained in at least 29 edges which are ij-crossing. We colour each edge xij by one of the 2^{64} colours, depending on the 3-partite hypergraph induced on x, $\mathcal{M}_i, and \mathcal{M}_j$. On the one hand, we observe that a monochromatic tight path consisting of the two edges $xij, x'ij \in \mathcal{C}$ corresponds to a copy of $\mathcal{L}$. On the other hand, Claim 23 implies that there is a colour such that at least

$$\frac{|\mathcal{C}|}{2^{64}} \geqslant \frac{\gamma\binom{t}{2}|X|}{72 \cdot 2^{64}} \overset{(8)}{\geqslant} \frac{\alpha_0 \gamma t^3}{2^{73}}$$

edges in $\mathcal{C}$ are coloured by it. Hence, by Proposition 12 there is a tight path with $\alpha_0\gamma t/2^{72}$ vertices using edges of this colour only. Note that in this tight path every three consecutive vertices contain one vertex from X and the other two vertices are from $\mathscr{M}$. Thus, this path gives rise to at least $\alpha_0\gamma t/2^{75}$ pairwise vertex disjoint tight paths on four vertices such that the ends are vertices from X. □

For any $\mathcal{L}_i \in \mathscr{L}$ we know from Lemma 19 that there is a fractional hom($\mathcal{M}$)-tiling h^i of $\mathcal{L}_i$ with $h^i_{\min} \geqslant 1/3$ and weight $w(h^i) \geqslant 16 + 1/3$. Furthermore, for every $\mathcal{M}_j \in \mathscr{M}$ which is not contained in any $\mathcal{L}_i \in \mathscr{L}$ we know from Fact 17 that there is a fractional hom($\mathcal{M}$)-tiling of h^j of $\mathcal{M}_j$ with $h^j_{\min} \geqslant 1/3$ and weight $w(h^j) = 8$. Hence, the union of all these fractional hom($\mathcal{M}$)-tiling gives rise to a fractional hom($\mathcal{M}$)-tiling h of $\mathcal{K}$ with $h_{\min} \geqslant 1/3$ and weight

$$w(h) \geqslant \left(16 + \frac{1}{3}\right)|\mathscr{L}| + 8(|\mathscr{M}| - 2|\mathscr{L}|) = 8|\mathscr{M}| + \frac{|\mathscr{L}|}{3}.$$

By applying Proposition 18 to the fractional hom($\mathcal{M}$)-tiling h (and recalling that the vertex classes $V_1, \ldots, V_t$ of the regular partition has the same size which was at least $(1 - \varepsilon/12)n/t$) we obtain an $\mathcal{M}$-tiling of $\mathcal{H}$ which covers at least

$$\big(w(h) - 3t\varepsilon\big)\left(1 - \frac{\varepsilon}{12}\right)\frac{n}{t} \geqslant \left(8|\mathscr{M}| + \frac{|\mathscr{L}|}{3} - 3t\varepsilon\right)\left(1 - \frac{\varepsilon}{12}\right)\frac{n}{t}$$

vertices of $\mathcal{H}$.

Since $\mathscr{M}$ was an $(\alpha_0 + 26\sqrt{\varepsilon})$-deficient $\mathcal{M}$-tiling of $\mathcal{K}$, the tiling we obtained above is an $(\alpha_0 - \varepsilon)$-deficient $\mathcal{M}$-tiling of $\mathcal{H}$ due to the choice of ε. This, however, is a contradiction to the fact that $\mathcal{H}$ does not permit an $(\alpha_0 - \varepsilon)$-deficient $\mathcal{M}$-tiling. □

3.4. **Proof of the path-tiling lemma.** In this section we prove Lemma 10. The proof will use the following proposition which has been proven in [4] (see Lemma 20) in an even more general form, hence we omit the proof here.

Proposition 25. *For all d and $\beta > 0$ there exist $\varepsilon > 0$, integers p and m_0 such that for all $m > m_0$ the following holds. Suppose $\mathcal{V} = (V_1, V_2, V_3)$ is an (ε, d)-regular triple with $|V_i| = 3m$ for $i = 1, 2$ and $|V_3| = 2m$. Then there there is a loose path tiling of $\mathcal{V}$ which consists of at most p pairwise vertex disjoint paths and which covers all but at most βm vertices of $\mathcal{V}$.*

With this result at hand one can easily derive the path-tiling lemma (Lemma 10) from the $\mathcal{M}$-tiling lemma (Lemma 11).

Proof of Lemma 10. Given $\gamma > 0$ and $\alpha > 0$ we first apply Proposition 25 with $d = \gamma/3$ and $\beta = \alpha/4$ to obtain $\varepsilon' > 0$, p', and m_0. Next, we apply Lemma 11 with $\gamma/2$ and $\alpha/2$ to obtain n_{11}. Then we apply Proposition 15 with γ, d and $\varepsilon = \frac{1}{3}\min\{d/2, \varepsilon', \alpha/8\}$ from above and $t_0 = n_{11}$ to obtain T_0 and n_{15}. Lastly we set $n_0 = \max\{n_{15}, 2T_0m_0\}$ and $p = p'T_0$.

Given a 3-uniform hypergraph $\mathcal{H}$ on $n > n_0$ vertices which satisfies $\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16} + \gamma\right)\binom{n}{2}$. By applying Proposition 15 with the constants chosen above we obtain an (ε, t)-regular partition $\mathcal{Q}$. Furthermore, we know that the corresponding cluster hypergraph $\mathcal{K} = \mathcal{K}(\varepsilon, d, \mathcal{Q})$ satisfies $\delta_1(\mathcal{K}) \geqslant (7/16 + \gamma/2)\binom{t}{2}$. Hence, by Lemma 11 we know that there is an $\mathcal{M}$-tiling $\mathscr{M}$ of $\mathcal{K}$ which covers all but at most $\alpha t/2$ vertices of $\mathcal{K}$. Note that the corresponding vertex classes in $\mathcal{H}$ contain all but at most $\alpha n/2 + |V_0|$ vertices.

We want to apply Proposition 25 to each copy $\mathcal{M}' \in \mathscr{M}$ of $\mathcal{M}$. To this end, let $\{1, \dots, 8\}$ denote the vertex set of such an copy $\mathcal{M}'$ and let $123, 345, 456, 678$ denote the edges of $\mathcal{M}'$. Further, for each $a \in V(\mathcal{M}')$ let V_a denote the corresponding partition class in $\mathcal{H}$. We split V_i, $i = 3, 4, 5, 6$, into two disjoint sets V_i^1 and V_i^2 of sizes $|V_i^1| = 2|V_i|/3$ and $|V_i^2| = |V_i|/3$ for $i = 3, 6$ and $|V_i^1| = |V_i^2| = |V_i|/2$ for $i = 4, 5$. Then the tuples (V_1, V_2, V_3^1), (V_8, V_7, V_6^1) and (V_3^2, V_4^1, V_5^1), (V_4^2, V_5^2, V_6^2) all satisfy the condition of Proposition 25, hence, there is a path tiling of these tuples consisting of at most $4p'$ paths which covers all but at most $12\beta n/t$ vertices of $V_1, \dots, V_8$.

Since $\mathscr{M}$ contains at most $t/8$ elements we obtain a path tiling which consists of at most $4p't/8 \leqslant p'T_0/2 \leqslant p$ paths which covers all but at most $12\beta n/t \times t/8$ vertices. Consequently, the total number of vertices in $\mathcal{H}$ not covered by the path tiling is at most $3\beta n/2 + \alpha n/2 + |V_0| \leqslant \alpha n$. This completes the proof of Lemma 10. □

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