Very large cardinals and ordinal definability

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Kurt Gödel Research Center – Research Seminar in Set Theory

29. October 2024

Joint work in progress
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Introduction

Our starting point is the following naïve question:

Question

Are there natural extensions of ZFC that prove the existence of sets that are not ordinal definable?

All standard large cardinal axioms are compatible with the assumption that $\mathrm{V} = \mathrm{HOD}$ and therefore do not provide affirmative answers to this question.

If we instead ask for extensions of ZF, then large cardinals beyond choice (e.g., Reinhardt cardinals) provide trivial affirmative answers.

As we will see below, there are more interesting things to say about the relationship between V and HOD in this setting.

Proposition (ZF_j)

If $j : \mathrm{V} \longrightarrow \mathrm{V}$ is an elementary embedding with first non-trivial fixed point λ, then λ is a cardinal of countable cofinality that is regular in HOD.

Proof.

First, observe that for all $\alpha < \lambda < \beta$, there is a non-trivial elementary embedding $i : \mathrm{V}_\beta \longrightarrow \mathrm{V}_\beta$ such that $\mathrm{crit}(i) > \alpha$ and λ is the first non-trivial fixed point of i.

Assume, towards a contradiction, that λ is singular in HOD.

Proof (cont.).

Pick $\beta > \lambda$ such that V_β is sufficiently elementary in V.

Then there is an elementary embedding $i : \mathrm{V}_\beta \longrightarrow \mathrm{V}_\beta$ such that $\mathrm{cof}(\lambda)^{\mathrm{HOD}} < \mathrm{crit}(i)$ and λ is the first non-trivial fixed point of i.

Let $c : \mathrm{cof}(\lambda)^{\mathrm{HOD}} \longrightarrow \lambda$ be the least cofinal function in the canonical well-ordering of HOD.

Then the set $\{c\}$ is definable from the parameter λ and hence $i(c) = c$.

Pick $\alpha < \mathrm{cof}(\lambda)^{\mathrm{HOD}}$ with $c(\alpha) > \mathrm{crit}(i)$. Then

$$c(\alpha) \;<\; i(c(\alpha)) \;=\; i(c)(i(\alpha)) \;=\; c(\alpha),$$

a contradiction. $\square$

Definition (Woodin, ZF)

A cardinal λ is *Berkeley* if for every transitive set M with $\lambda \in M$ and every $\alpha < \lambda$ there is a non-trivial elementary embedding $j : M \longrightarrow M$ with $\alpha < \mathrm{crit}(j) < \lambda$.

Definition (Goldberg & Schlutzenberg, ZF)

A cardinal λ is *rank-Berkeley* if for all $\alpha < \lambda < \beta$, there is a non-trivial elementary embedding $j : \mathrm{V}_\beta \longrightarrow \mathrm{V}_\beta$ with the property that $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Exacting cardinals

We now want to isolate canonical weakenings of rank-Berkeleyness that are compatible with the Axiom of Choice and still allow us to run the above argument.

The starting point for finding these weakenings is the following classical result of Magidor:

Lemma (Magidor)

The following statements are equivalent for every cardinal κ:

- κ is a supercompact cardinal.
- For all ordinals $\zeta > \kappa$, there exists
  - an ordinal $\eta < \kappa$,
  - a cardinal $\bar{\kappa} < \eta$, and
  - a non-trivial elementary embedding $j : \mathrm{V}_\eta \longrightarrow \mathrm{V}_\zeta$ with $\mathrm{crit}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$.

Theorem (L.)

The following statements are equivalent for every cardinal κ:

- For all cardinals $\zeta > \kappa$, there exists
  - an ordinal $\eta < \kappa$,
  - a cardinal $\bar{\kappa} < \eta$,
  - an elementary submodel X of V_η with $\mathrm{V}_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, and
  - an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ with $\mathrm{crit}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$.
- The cardinal κ is a strongly unfoldable cardinal.
- The cardinal κ is shrewd.

Definition (A.–B.–L.)

A cardinal λ is *exacting* if for all $\alpha < \lambda < \beta$, there exists

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Theorem (A.–B.–L.)

If $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ is an I0-embedding, then there exists a set-sized transitive model M of ZFC such that $\lambda \in M$ and the cardinal λ is exacting in M.

Proposition (A.–B.–L.)

Exacting cardinals are regular in HOD.

Proof.

Assume that λ is an exacting cardinal that is singular in HOD.

Pick $\beta > \lambda$ such that V_β is sufficiently elementary in V.

Then there is $X \prec \mathrm{V}_\beta$ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j : X \longrightarrow V_\beta$ with $\mathrm{cof}(\lambda)^{\mathrm{HOD}} < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Then the Kunen Inconsistency implies that λ is the first non-trivial fixed point of j.

Let $c : \mathrm{cof}(\lambda)^{\mathrm{HOD}} \longrightarrow \lambda$ be the least cofinal function with respect to the canonical well-ordering of HOD.

Then $c \in X$ with $j(c) = c$ and, for $\alpha < \mathrm{cof}(\lambda)^{\mathrm{HOD}}$ with $c(\alpha) > \mathrm{crit}(j)$, we have $c(\alpha) < j(c(\alpha)) = j(c)(j(\alpha)) = c(\alpha)$, a contradiction.

Exacting cardinals can also be characterized through a natural strengthening of a classical model-theoretic reflection principle.

Remember that a cardinal λ is *Jónsson* if every structure in a countable first-order language whose domain has cardinality λ has a proper elementary substructure of cardinality λ.

Theorem (A.–B.–L.)

The following are equivalent for each cardinal λ with $\mathrm{V}_\lambda \prec_{\Sigma_1} \mathrm{V}$:

- The cardinal λ is exacting.
- For every class $\mathcal{C}$ of structures in a countable first-order language that is definable by a formula with parameters contained in $V_\lambda \cup \{\lambda\}$, every structure of cardinality λ in $\mathcal{C}$ contains a proper elementary substructure of cardinality λ that is isomorphic to a structure in $\mathcal{C}$.

The HOD Dichotomy

Following Woodin, seminal results of Jensen can be formulated as the following dichotomy:

Theorem (Jensen)

Exactly one of the following statements holds:

- For every singular cardinal λ, the cardinal λ is singular in L and $(\lambda^+)^{\mathrm{L}} = \lambda^+$ holds ("L *is close to* V").
- Every uncountable cardinal is inaccessible in L ("L *is far from* V").

Woodin proved a surprising result that shows that strong large cardinal axioms imply an analog of Jensen's dichotomy for the inner model HOD:

Theorem (Woodin)

If δ is an extendible cardinal, then exactly one of the following statements holds:

- For every singular cardinal $\lambda > \delta$, the cardinal λ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$ holds ("HOD *is close to* V").
- Every regular cardinal greater than or equal to δ is measurable in HOD ("HOD *is far from* V").

The following notion plays a central role in the proof of Woodin's result:

Definition (Woodin)

An uncountable regular cardinal κ is *ω-strongly measurable in HOD* if there is a cardinal $\delta < \kappa$ such that $(2^\delta)^{\mathrm{HOD}} < \kappa$ and HOD contains no partition of the set E^κ_ω of all elements of κ with cofinality ω into δ-many sets that are all stationary in V.

Lemma (Woodin)

If a cardinal κ is ω-strongly measurable in HOD, then κ is a measurable cardinal in HOD.

Theorem (Woodin)

If δ is an extendible cardinal, then exactly one of the following statements holds:

- No regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD and for every singular cardinal $\lambda > \delta$, the cardinal λ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$ holds.
- Every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.

The Weak HOD Conjecture (Woodin)

The theory

$$\mathrm{ZFC} + \text{“}\textit{There is a huge cardinal above an extendible cardinal}\text{”}$$

proves that a proper class of regular cardinals is not ω-strongly measurable in HOD.

Corollary (A.–B.–L.)

If δ is an extendible cardinal below an exacting cardinal, then every regular cardinal greater or equal to δ is ω-strongly measurable in HOD.

Corollary (A.–B.–L.)

If the theory

$$\mathrm{ZFC} + \text{“\textit{There is an exacting cardinal above an extendible cardinal}”}$$

is consistent, then the Weak HOD Conjecture is false.

Theorem (A.–B.–L., ZF_j)

If $j : \mathrm{V} \longrightarrow \mathrm{V}$ is an elementary embedding with $\mathrm{V}_{\mathrm{crit}(j)} \prec_{\Sigma_3} \mathrm{V}$ and there is a supercompact cardinal greater than its least non-trivial fixed point, then there exists a set-sized model of the theory

ZFC + "There is an exacting cardinal that is a limit of extendible cardinals".

Ultraexacting cardinals

Definition (A.–B.–L.)

A cardinal λ is *ultraexacting* if for all $\alpha < \lambda < \beta$, there exist

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $j \restriction \mathrm{V}_\lambda \in X$.

Theorem (A.–B.–L.)

If $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ is an I0-embedding and G is $\mathrm{Add}(\lambda^+, 1)$-generic over V, then the cardinal λ is ultraexacting in $L(V_{\lambda+1}, G)$.

Theorem (A.–B.–L., ZF_j)

If $j : \mathrm{V} \longrightarrow \mathrm{V}$ is an elementary embedding with $\mathrm{V}_{\mathrm{crit}(j)} \prec_{\Sigma_3} \mathrm{V}$ and there is a supercompact cardinal greater than its least non-trivial fixed point, then there exists a set-sized model of the theory

ZFC + "There is an ultraexacting cardinal that is a limit of extendible cardinals".

Lemma

Assume that $\lambda < \zeta$ are cardinals with $\mathrm{V}_\zeta \prec_{\Sigma_2} \mathrm{V}$, X is an elementary submodel of V_ζ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$ and $j : X \longrightarrow \mathrm{V}_\zeta$ is an elementary embedding with $\mathrm{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $j \restriction \mathrm{V}_\lambda \in X$.

Then X contains a function $j_+ : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$ with

$$j_+ \restriction (\mathrm{V}_{\lambda+1} \cap X) \ = \ j \restriction (\mathrm{V}_{\lambda+1} \cap X).$$

Theorem (A.–B.–L.)

If λ is an ultraexacting cardinal, then λ^+ is ω-strongly measurable in HOD.

Theorem (A.–B.–L.)

If λ is an ultraexacting cardinal with the property that $\mathrm{V}^{\#}_{\lambda+1}$ exists, then there is an I0-embedding $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$.

Definition

Given an inner model N and ordinals $\lambda < \vartheta$, we say that the ordinal λ is *N-ϑ-Berkeley* if for every $\alpha < \lambda$ and every transitive set M in N that contains λ as an element and has cardinality less than ϑ in N, there exists a non-trivial elementary embedding $j : M \longrightarrow M$ with $\alpha < \mathrm{crit}(j) < \lambda$.

Remember that, given a set A, we let Θ_A denote the least ordinal ζ with the property that there is no surjection from A onto ζ.

Theorem (A.–B.–L.)

Every ultraexacting cardinal λ is HOD-$\Theta^{L(\mathrm{V}_{\lambda+1})}_{\mathrm{V}_{\lambda+1}}$-Berkeley.

Thank you for listening!