CS:ST Lecture X [Part 2]

2 October 2020

A few comments.

<u>LORENZO</u>: $\mathrm{Det}(\mathrm{BOR}) \Longrightarrow \mathrm{PSP}(\mathrm{BOR})$

$\mathrm{ZFC} \Longrightarrow \mathrm{PSP}(\underset{\sim}{\Sigma}^1_1)$

If A has p.s.p. $\Longrightarrow$ A is either ctble or $|A| = 2^{\aleph_0}$.

<u>Corollary</u> ZFC $\Longrightarrow$ no Borel or analytic set can be a ctrex of CH.

What about $\underset{\sim}{\Pi}^1_1$? [Next big goal: prove from extra ass. $\mathrm{Det}(\underset{\sim}{\Pi}^1_1)$.

Then: Extra ass. $\Longrightarrow$ $\underset{\sim}{\Pi}^1_1$ sets cannot be ctrex to CH.]

<u>Without extra assumptions</u>

<u>Proposition</u> (LUZIN-SIERPIŃSKI)

Every $\underset{\sim}{\Pi}^1_1$ set is a union of $\aleph_1$ many Borel sets.

Proof Remember:

(1) WO is $\underset{\sim}{\Pi}^1_1$-hard

(2) WO_α is $\underset{\sim}{\Delta}^1_1 = BOR$

Take $A \in \Pi^1_1$ arbitrary. By (1) find cts f s.t.

$$A = f^{-1}[WO].$$

$$WO = \bigcup_{\alpha < \omega_1} WO_\alpha \quad \text{(by (2))}.$$

$$A = f^{-1}\left[\bigcup_{\alpha<\omega_1} WO_\alpha\right]$$

$$A = \bigcup_{\alpha<\omega_1} \underbrace{f^{-1}[WO_\alpha]}_{\underset{\sim}{\Delta}^1_1}$$

since cts preimage of Borel set.

q.e.d.

Corollary (ZFC):
If A is $\underset{\sim}{\Pi}^1_1$, then

$$|A| = \begin{cases} \leq \aleph_0 \\ \aleph_1 \\ 2^{\aleph_0} \end{cases}$$

Proof. By L-S, we know $A = \bigcup_{\alpha<\omega_1} A_\alpha$
where A_α is Borel.
By PSP(BOR) each of these is either countable or contains a non-empty perfect subset.

Case 1 One of the A_α contains a non-empty pf. subset.

$$|A| = \Big|\bigcup_{\alpha<\omega_1} A_\alpha\Big| \geq |A_\alpha| = 2^{\aleph_0}.$$

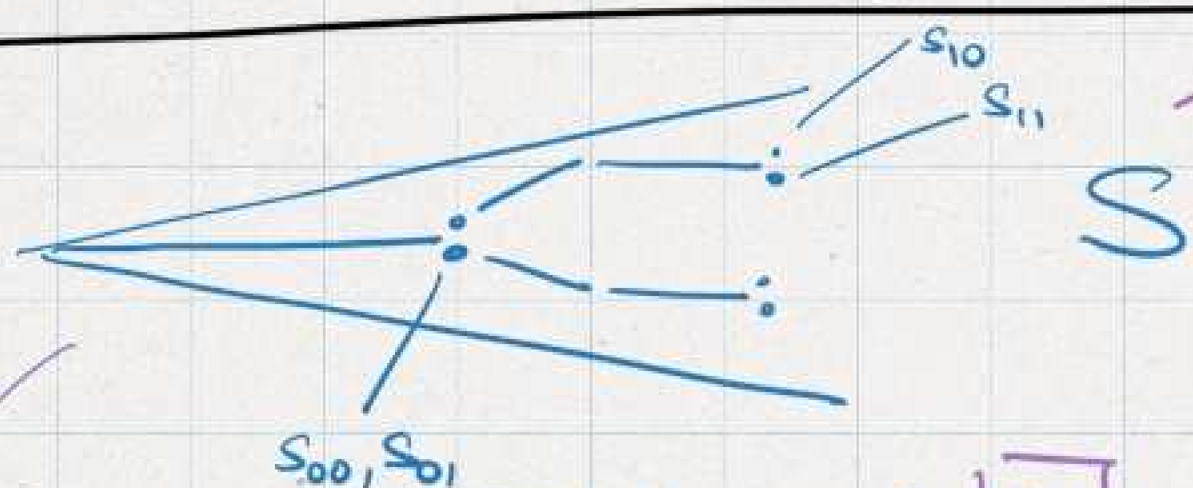
Case 2 None of them do.

$$|A| \leq \aleph_1 \cdot \aleph_0 = \aleph_1.$$

Consequence

If $2^{\aleph_0} > \aleph_2$,

then you can't have a Π^1_1 set with card. $\aleph_2$.



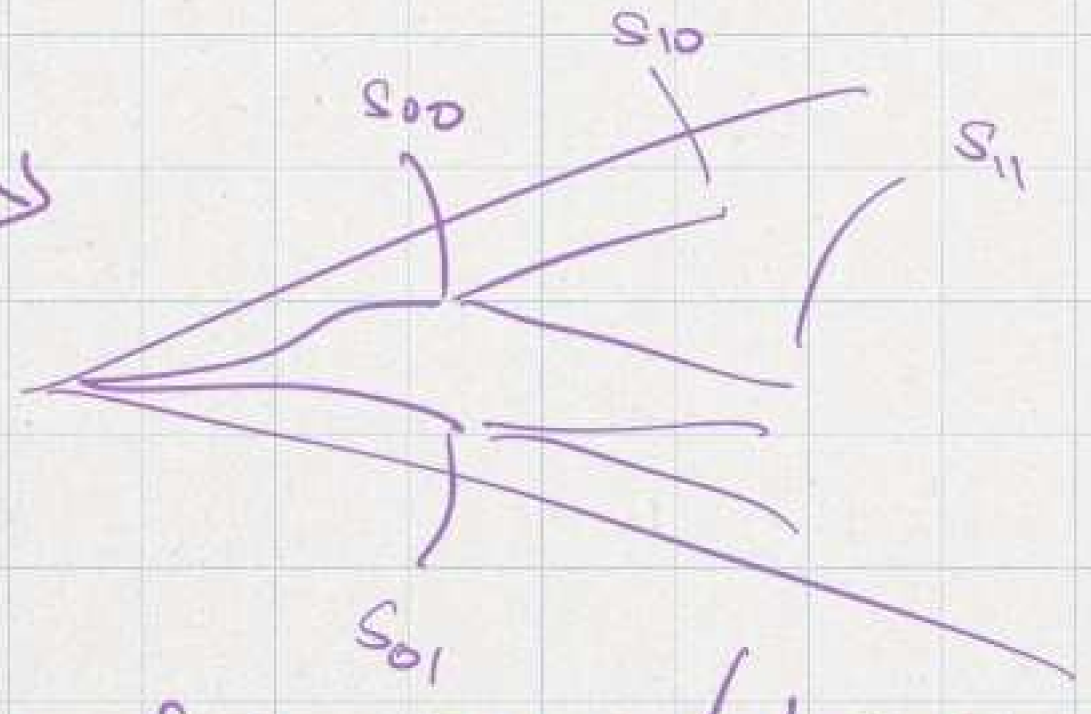
$\lim(\sigma * z) = x_z$

$z \in 2^\omega$

$(s_{00}, s_{01}), \boxed{i_0}, (s_{10}, s_{11}) \boxed{i_1} \ldots$

$x_z \in [T]$

$T \subseteq \omega^{<\omega}$



$T := \{ t ;\ \exists i,j \left(t \subseteq s_{i,j} \right) \text{ and } s_{i,j} \in S \}$

T is <u>perfect</u> iff

$\forall s \in T \ \exists t_0, t_1 \in T$ s.t.

$s \subseteq t_0$ & t_0, t_1 are
$s \subseteq t_1$ incompatible

A is closed & has no isolated pts

$\Longleftrightarrow$

$$\boxed{\begin{array}{l} T_A \text{ is perfect} \\ \text{and } A = [T_A]. \end{array}}$$