UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Core Logic

2006/2007; 1st Semester
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Homework Set # 8 *Deadline:* November 15th, 2006

Exercise 25 (5 points).
REPEATED FROM HOMEWORK SET # 7.

We are considering a system reminiscent of Leibniz' attempts to arithmetize language. In the lecture, we introduced a system based on the divisor structure of the natural numbers, but this system was too simple as it didn't allow proper discussion of negative statements. Therefore, we add a number that should take care of the negative statements to the system. (The rough idea is: If 2 is *animal*, 3 is *rationalis* and 7 is *asinarius* ("donkey-like"), then $\langle 6, 7\rangle$ would represent *homo* (to preclude the option of constructing a *homo asinarius*) and $\langle 14, 3\rangle$ would represent *asinus* (to preclude the option of constructing an *asinus rationalis*.)

Formally: Call a pair $X := \langle \mathrm{p}_X, \mathrm{n}_X\rangle$ a **pseudo-Leibniz predicate** (**PLP**) if p_X and n_X are both positive natural numbers ≥ 2. We write $n|m$ for "n divides m" (*i.e.*, there is a $k \geq 1$ such that $nk = m$) and $n \perp m$ for "n and m are coprime" (*i.e.*, if $k|n$ and $k|m$, then $k = 1$). We define the following semantics for categorical propositions using PLPs:

$$\begin{aligned} X\mathrm{a}Y &:\equiv \mathrm{p}_X|\mathrm{p}_Y \;\&\; \mathrm{p}_Y \perp \mathrm{n}_X \\ X\mathrm{i}Y &:\equiv \exists k \geq 1(\mathrm{p}_X|k\cdot\mathrm{p}_Y \;\&\; k\cdot\mathrm{p}_Y \perp \mathrm{n}_X) \\ X\mathrm{e}Y &:\equiv \forall k \geq 1(\neg(\mathrm{p}_X|k\cdot\mathrm{p}_Y) \;\vee\; \neg(k\cdot\mathrm{p}_Y \perp \mathrm{n}_X)) \end{aligned}$$

In this semantics, **Barbara** can be expressed as:

$$\forall X, Y, Z\left((\mathrm{p}_X|\mathrm{p}_Y \;\&\; \mathrm{p}_Y|\mathrm{p}_Z \;\&\; \mathrm{p}_Y \perp \mathrm{n}_X \;\&\; \mathrm{p}_Z \perp \mathrm{n}_Y) \to \mathrm{p}_X|\mathrm{p}_Z \;\&\; \mathrm{p}_Z \perp \mathrm{n}_X\right).$$

(1) Define a semantics for $X\mathrm{o}Y$ such that this is contradictory to $X\mathrm{a}Y$ (½ point).
(2) Give an example of a PLP that shows that **Barbara** is not valid with this semantics (2 points).
(3) Prove that **Celarent** is valid with this semantics (2½ points).

Exercise 26 (6 points).

Let $\mathbf{B} = \langle B, 0, 1, +, \cdot, -\rangle$ be a Boolean algebra. Define an operation $\star$ by $x \star y := -(x+y)$ (the `NOR` or Sheffer operation).

(1) Give formulas $\varphi_{\mathrm{mult}}, \varphi_{\mathrm{add}}, \varphi_{\mathrm{comp}}$ in the language just containing $\star$, $=$ and parentheses such that

$$\begin{aligned} \varphi_{\mathrm{mult}}(x,y,z) &\equiv x\cdot y = z \\ \varphi_{\mathrm{add}}(x,y,z) &\equiv x + y = z \\ \varphi_{\mathrm{comp}}(x,z) &\equiv -x = z \end{aligned}$$

(1 point each). (In other words, the $\star$-language is expressive enough to define the language of Boolean algebras.)

(2) Prove that the following three so-called "Sheffer axioms" hold for $\star$ (1 point each):

$$(x \star x) \star (x \star x) = x$$

$$x \star (y \star (y \star y)) = x \star x$$

$$(x \star (y \star z)) \star (x \star (y \star z)) = ((y \star y) \star x) \star ((z \star z) \star x)$$

Exercise 27 (2 points).

Give the names of the following people (1 point each):

- X was a Aristotelian philosopher from Constantinople who lived in Italy most of his life. From 1456 to 1458, he was the professor for rhetoric and poetics at the *studio fiorentino* and one of the teachers of Lorenzo de'Medici (*il Magnifico*).
- Y was one of the authors of *La logique, ou l'art de penser*. He was called "the Great" to distinguish him from his father who had the same name.

Exercise 28 (9 points).

Translated into the language of Boolean algebras, **Celarent** became

For all a, b, and c, if $ba = \mathbf{0}$ and $c(\mathbf{1} - b) = \mathbf{0}$, then $ca = \mathbf{0}$.

Rephrase **Baraco**, **Darapti**, and **Felapton** in a similar way in the language of Boolean algebras (1 point each). Find out whether these statements are true in Boolean algebras. If they are, prove it from the axioms of Boolean algebras. If not, give a counterexample. (2 points each).