UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatische Verzamelingentheorie

2005/2006; 2nd Semester
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Homework Set # 6 *Deadline:* March 23rd, 2006

Exercise 16 (total of five points).

Let X be a set. Define by transfinite recursion

$$\begin{aligned} S_0(X) &:= X \\ S_{n+1}(X) &:= \bigcup S_n(X) \\ S_\omega(X) &:= \bigcup \{S_n(X)\,;\, n \in \omega\}. \end{aligned}$$

We call $\mathrm{tcl}(X) := S_\omega(X)$ the **transitive closure** of X. Prove that $\mathrm{tcl}(X)$ is a transitive set (*i.e.*, every element of it is a subset of it; 2 points) and that every transitive superset of X contains $\mathrm{tcl}(X)$ (3 points).

Exercise 17 (total of ten points).

A set X is called **finite** if there is some $n \in \omega$ and some bijection $f : X \to n$. A set X is called **hereditarily finite** if $\mathrm{tcl}(X)$ is finite. Let **HF** be the class of all hereditarily finite sets. A set X is called **countable** if there is an injection from X into ω. A set X is called **hereditarily countable** if $\mathrm{tcl}(X)$ is countable. Let **HC** be the class of all hereditarily countable sets.

- Prove that $\mathbf{V}_\omega = \mathbf{HF}$ (3 points).
- Prove that $\mathbf{V}_{\omega+1} \subseteq \mathbf{HC}$ (3 points).
- Prove that $\mathbf{V}_{\omega+2} \not\subseteq \mathbf{HC}$ (2 points).
- Prove that $\mathbf{HC} \not\subseteq \mathbf{V}_{\omega+2}$ (2 points).

Exercise 18 (total of thirteen points).

Let $\mathsf{Z}^{\mathrm{F}}_{-}$ be the set of axioms consisting of the empty set axiom, the pairing axiom, the union axiom, the axiom of foundation, and the axiom scheme of separation. Let Pow be the power set axiom, Inf be the axiom of infinity and Refl the set of instances of the axiom scheme of replacement.
We write

$$\begin{aligned} \mathsf{Z}_{-} &:= \mathsf{Z}^{\mathrm{F}}_{-} + \mathsf{Inf} \\ \mathsf{ZF}^{\mathrm{F}} &:= \mathsf{Z}^{\mathrm{F}}_{-} + \mathsf{Repl} + \mathsf{Pow} \end{aligned}$$

- Prove that if λ is a limit ordinal, then $\mathbf{V}_\lambda$ is a model of $\mathsf{Z}^{\mathrm{F}}_{-} + \mathsf{Pow}$ (½ point for empty set, pairing, union and foundation, 1 point for separation: 3 points).
- Prove that $\mathbf{HF}$ is not a model of Inf (2 points).
- Prove that $\mathbf{HF}$ is a model of ZF^{F} (2 points).
- Prove that $\mathbf{HC}$ is a model of the pairing axiom (1 point).
- Prove that $\mathbf{HC}$ is not a model of Pow (3 points).
  Hint. It is not enough to observe that $\omega \in \mathbf{HC}$ and $\wp(\omega) \notin \mathbf{HC}$. You have to argue that nothing else can play the role of the power set of ω in $\mathbf{HC}$.
- Is $\mathbf{V}_{\omega+\omega+\omega}$ a model of Refl? (2 points)