UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatische Verzamelingentheorie

2005/2006; 2nd Semester
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Homework Set # 5 *Deadline:* March 16th, 2006

Exercise 13 (total of eight points).

$$\begin{aligned} &(\text{Found}) && \forall x \exists y (x \cap y = \varnothing) \\ &(\text{vN}) && \forall x \exists \alpha (x \in \mathbf{V}_\alpha) \end{aligned}$$

Prove in ZF^- that (Found) and (vN) are equivalent.

Exercise 14 (total of nine points).

Work in ZF. Use the *Recursion Theorem* to give precise definitions of the following informally defined class $\mathbf{W}$:

- $\mathbf{W}_0 := \varnothing$,
- $\mathbf{W}_{\alpha+1} := \wp(\wp(\mathbf{W}_\alpha))$,
- $\mathbf{W}_\lambda := \bigcup_{\alpha<\lambda} \mathbf{W}_\alpha$, and
- $\mathbf{W} := \bigcup_{\alpha\in\text{Ord}} \mathbf{W}_\alpha$.

This means: Find the LAST-formula Φ that you need in order to apply the Recursion Theorem to get arbitrarily large germs, and then give the LAST-formula that defines the class based on the conclusion of the Recursion Theorem (2 points). Be precise!

In addition, define a class $\mathbf{X}$ by

- $\mathbf{X}_0 := \omega$,
- $\mathbf{X}_{\alpha+1} := \wp(\mathbf{X}_\alpha)$,
- $\mathbf{X}_\lambda := \bigcup_{\alpha<\lambda} \mathbf{X}_\alpha$, and
- $\mathbf{X} := \bigcup_{\alpha\in\text{Ord}} \mathbf{X}_\alpha$.

Prove that $\mathbf{W} = \mathbf{V}$ (3 points) and that $\mathbf{X} = \mathbf{V}$ (4 points).

Exercise 15 (total of eight points).

Work in ZF. The Mirimanoff rank $\varrho(x) := \min\{\alpha\,;\, x \in \mathbf{V}_{\alpha+1}\}$ is defined for all sets. Let x and y be sets with $\varrho(x) = \alpha \leq \varrho(y) = \beta$. Compute the Mirimanoff rank of $\{x\}$ (1 point), $\{x, y\}$ (1 point), $\langle x, y\rangle$ (ordered pair, 2 points), $x \times y$ (2 points), $\bigcup x$ (2 points).