UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatische Verzamelingentheorie

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dr Benedikt Löwe

Homework Set # 3 *Deadline:* March 2nd, 2006

Exercise 8 (total of six points).
If $\mathbf{X}_0 = \langle X_0, \leq_0 \rangle$ and $\mathbf{X}_1 = \langle X_1, \leq_1 \rangle$ are tosets, then we define $\mathbf{X}_0 \oplus \mathbf{X}_1$ as follows:

$$X := (\{0\} \times X_0) \cup (\{1\} \times X_1),$$
$$\langle i, x \rangle \leq \langle j, y \rangle :\iff (i = j \wedge x \leq_i y) \vee (i = 0 \wedge j = 1), \text{ and}$$
$$\mathbf{X}_0 \oplus \mathbf{X}_1 := \langle X, \leq \rangle.$$

Why do we bother with the cartesian products? In other words: what would be wrong with the definition $X := X_0 \cup X_1$? (1 point)
Prove that if $\mathbf{X}_0$ and $\mathbf{X}_1$ are wosets, then so is $\mathbf{X}_0 \oplus \mathbf{X}_1$ (5 points).

Exercise 9 (total of ten points).
By the representation theorem for wellorders, every woset $\mathbf{W}$ is isomorphic to a unique ordinal. We write $\mathrm{o.t.}(\mathbf{W})$ for it (pronounced: “the order type of $\mathbf{W}$”).
If α and β are ordinals, then $\alpha \oplus \beta$ is a woset by Exercise 8. We define an operation $+$ on ordinals by

$$\alpha + \beta := \mathrm{o.t.}(\alpha \oplus \beta).$$

By ω, we denote the set of natural numbers as an ordinal, *i.e.*, the set $\{0, 1, 2, ...\}$ where $n = \{0, 1, ..., n-1\}$.
Describe the ordinal $\omega + n$ as a set (1 point). List all of the elements of $\omega + \omega$ (1 point).
For ordinals α, β and γ, prove that

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \text{ (5 points).}$$

List all of the elements of $\omega + \omega + \omega$ (3 points).

Exercise 10 (total of nine points).

We give an example of a computation of the order type of a well-order: Consider $\langle \mathbb{N}, \leq \rangle$ with the usual order and $E := \{n \in \mathbb{N}\,;\, n \text{ is even}\}$. Then $\mathbf{E} := \langle E, \leq \rangle$ is a woset (why?). The ordinal $\mathrm{o.t.}(\mathbf{E})$ is ω as the following argument shows:

> The function $d : E \to \mathbb{N}$ defined by $d(n) := \frac{n}{2}$ is an order isomorphism between $\mathbf{E}$ and $\langle \mathbb{N}, \leq \rangle = \omega$ (check that!). By the representation theorem, there is a unique ordinal isomorphic to any wellorder, so $\mathrm{o.t.}(\mathbf{E}) = \omega$.

Compute the order types of the following wosets $\mathbf{O}$ (1 point), $\mathbf{W}_0$ (2 points), $\mathbf{W}_1$ (2 points), $\mathbf{W}_0 \oplus \mathbf{W}_1$ (4 points).

(1) Let $\langle \mathbb{N}, \leq \rangle$ be the usual ordering of $\mathbb{N}$, $O := \{n \in \mathbb{N}\,;\, n \text{ is odd and } n \geq 17\}$, and $\mathbf{O} := \langle O, \leq \rangle$.

(2) Let $\leq_0$ be defined on the natural numbers as follows: Let $X := \{0, 70, 72\}$.

$$\begin{aligned} n \leq_0 m \quad :\Longleftrightarrow \quad & (n \notin X \wedge m \notin X \wedge n \leq m) \vee \\ & (n \notin X \wedge m \in X) \vee \\ & (m = 0) \vee (n \in X \wedge m \in X \wedge n \neq 0 \wedge n \leq m). \end{aligned}$$

Let $\mathbf{W}_0 := \langle \mathbb{N}, \leq_0 \rangle$.

(3) Let $\leq_1$ be defined on the natural numbers as follows: Let $Y := \{0, 1, 2\}$.

$$\begin{aligned} n \leq_1 m \quad :\Longleftrightarrow \quad & (n \notin Y \wedge m \notin Y \wedge n \in E \wedge m \in E \wedge n \leq m) \vee \\ & (n \notin Y \wedge m \notin Y \wedge n \notin E \wedge m \notin E \wedge n \leq m) \vee \\ & (n \notin Y \wedge m \notin Y \wedge n \notin E \wedge m \in E \wedge \frac{n-1}{2} \leq \frac{m}{2}) \vee \\ & (n \notin Y \wedge m \notin Y \wedge n \in E \wedge m \notin E \wedge \frac{n}{2} < \frac{m-1}{2}) \vee \\ & (n \notin Y \wedge m \in Y) \vee \\ & (n \in Y \wedge m \in Y \wedge m \leq n). \end{aligned}$$

Let $\mathbf{W}_1 := \langle \mathbb{N}, \leq_1 \rangle$.