UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Core Logic

2005/2006; 1st Semester
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Homework Set # 9 *Deadline:* November 15th, 2005

Exercise 29 (total of six points).

Let $\mathcal{L} = \{R\}$ be a language with one binary relation symbol. Consider the following seven $\mathcal{L}$-sentences:

$$\begin{aligned}
\varphi_{\text{(i)}} &:\Leftrightarrow \forall x \neg Rxx \\
\varphi_{\text{(ii)}} &:\Leftrightarrow \forall x \forall y (x \neq y \rightarrow (Rxy \vee Ryx)) \\
\varphi_{\text{(iii)}} &:\Leftrightarrow \forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz) \\
\varphi_{\text{(iv)}} &:\Leftrightarrow \forall x \exists y \exists z (Ryx \wedge Rxz) \\
\varphi_{\text{ME}} &:\Leftrightarrow \exists x \forall y (Ryx \vee x = y) \\
\varphi_{\text{LEP}} &:\Leftrightarrow \forall x \exists y \forall z (Rxz \rightarrow (Rzy \vee y = z))
\end{aligned}$$

Check whether the following sets of sentences are consistent. If they are, give a model. If they aren't, derive a contradiction (2 points each).

(1) $\{\varphi_{\text{(i)}}, \varphi_{\text{(iii)}}, \varphi_{\text{(iv)}}, \varphi_{\text{ME}}\}$,
(2) $\{\varphi_{\text{(i)}}, \varphi_{\text{(iii)}}, \varphi_{\text{LEP}}, \neg\varphi_{\text{ME}}\}$,
(3) $\{\varphi_{\text{(i)}}, \varphi_{\text{(ii)}}, \varphi_{\text{(iii)}}, \varphi_{\text{LEP}}, \neg\varphi_{\text{ME}}\}$,

Exercise 30 (total of nine points).

We are modelling Achilles and the turtle as a transfinite process on the real line $\mathbb{R}$. Please give arguments for all answers.

(1) Achilles' position at time t is given by A_t, the turtle's position is given by T_t. We start with $A_0 := 0$ and $T_0 := 1$. For every index i, we define $A_{i+1} := A_i + |T_i - A_i|$, $T_{i+1} := T_i + \frac{1}{2} \cdot |T_i - A_i|$, and

$$T_\infty := \lim_{i \in \mathbb{N}} T_i,$$

$$A_\infty := \lim_{i \in \mathbb{N}} A_i,$$

$$T_{\infty+\infty} := \lim_{i \in \mathbb{N}} T_{\infty+i}, \text{ and}$$

$$A_{\infty+\infty} := \lim_{i \in \mathbb{N}} A_{\infty+i}.$$

Determine the least index i such that $A_i = T_i$ (1 point). Where is Achilles at time $\infty + \infty$ (2 points)?

(2) Now the positions are given by A^*_t and T^*_t defined as follows. For each index $i \in \{0, 1, 2, ..., \infty, \infty+1, \infty+2, \infty+3, ...\}$, we define the *value* $\mathrm{v}(i)$ as follows:

$$\mathrm{v}(i) := n \text{ if } i = n \text{ or } i = \infty + n.$$

We start with $A^*_0 := 0$ and $T^*_0 := 1$. For every index i, we define $A^*_{i+1} := A^*_i + \frac{1}{2^{\mathrm{v}(i)}}$, $T^*_{i+1} := T^*_i + \frac{1}{2^{\mathrm{v}(i)+1}}$, and

$$T^*_\infty := \lim_{i\in\mathbb{N}} T^*_i,$$
$$A^*_\infty := \lim_{i\in\mathbb{N}} A^*_i,$$
$$T^*_{\infty+\infty} := \lim_{i\in\mathbb{N}} T^*_{\infty+i}, \text{ and}$$
$$A^*_{\infty+\infty} := \lim_{i\in\mathbb{N}} A^*_{\infty+i}.$$

Show that for every natural number n, we have $T_n = T^*_n = A_{n+1} = A^*_{n+1}$ (2 points). Compute $A^*_{\infty+5}$, $T^*_{\infty+12}$, $A^*_{\infty+\infty}$ and $T^*_{\infty+\infty}$ (1 point each).

Exercise 31 (total of seven points).

Consider the language of arithmetic $\mathcal{L} = \{\dot{+}, \dot{\times}, \dot{S}, \dot{0}, \dot{1}, \dot{<}, \dot{=}\}$ and its standard model $\mathbf{N} := \langle \mathbb{N}, +, \cdot, \mathrm{succ}, 0, 1, <, = \rangle$. (Here succ is the successor function $n \mapsto n + 1$.) The language of arithmetic allows to define formulas that describe the natural numbers:

$$\chi_n(x) :\simeq x \dot{=} \underbrace{\dot{S} \ldots \dot{S}}_{\text{n times}} \dot{0}$$

We say that a set of $\mathcal{L}$-sentences T is an *arithmetic* if $\mathbf{N} \models T$. Prove that every arithmetic has a model which is not isomorphic to $\mathbf{N}$ (7 points).

Hint. Define an extension $\mathcal{L}^* := \mathcal{L} \cup \{\dot{c}\}$ of $\mathcal{L}$ where $\dot{c}$ is a constant symbol and look at the theory $T^* := T \cup \{\neg\chi_n(\dot{c})\,;\, n \in \mathbb{N}\}$. Prove that there is no value c of $\dot{c}$ such that $\langle \mathbf{N}, c\rangle$ is a model of T^*. Prove that T^* is consistent by using the compactness theorem. Use these two facts to prove the claim. (You may use that isomorphic models satisfy the same sentences.)

For students with a mathematical logic background: If T is sufficiently strong, then you can say that the interpretation of $\dot{c}$ must be an "infinite element". Make this statement mathematically precise and prove it (3 extra points).