UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Advanced Topics in Set Theory

2004/2005; 1st Semester
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Homework Set # 9 *Deadline:* December 2nd, 2004

Exercise A (Ulam).

Work in ZFC. A cardinal is **measurable** if it carries a κ-complete nonprincipal ultrafilter. Show that every measurable cardinal is strongly inaccessible, *i.e.*, it is regular and a strong limit cardinal.

Hint. Assume for some $\lambda < \kappa$ that $2^\lambda \geq \kappa$. By AC, find a subset $S \subseteq \{f\,;\, f : \lambda \to 2\}$ of cardinality κ. Take an arbitrary κ-complete ultrafilter on S and show that it is principal.

Exercise B.

If $\{X_\alpha\,;\,\alpha < \kappa\}$ is a family of subsets of κ, we defined the **diagonal intersection** as follows:

$$\triangle X_\alpha := \{\xi \in \kappa\,;\, \xi \in \bigcap_{\alpha<\xi} X_\alpha\}.$$

Prove that the diagonal intersection of closed unbounded sets is closed unbounded in κ.

Exercise C.

Work in ZFC. Find a subset of $\aleph_1$ such that neither A nor $\aleph_1 \backslash A$ is closed unbounded in $\aleph_1$. Deduce that $\mathcal{C}_{\aleph_1}$ is not an ultrafilter.

Exercise D (Kleinberg).

For $\lambda < \kappa$ regular, let $\mathcal{C}_\kappa^\lambda$ be the filter generated by

$$\{C \cap \mathrm{Cof}(\lambda)\,;\, C \text{ is club in } \kappa\}.$$

Work in ZF without the Axiom of Choice. Suppose that

- for all regular $\lambda < \kappa$, $\mathcal{C}_\kappa^\lambda$ is an ultrafilter, and
- the set of regular cardinals below κ is not stationary in κ.

Let $\mathcal{U}$ be an ultrafilter containing $\mathcal{C}_\kappa$ which is closed under taking diagonal intersections. Prove that there is some regular $\lambda < \kappa$ such that $\mathcal{U} = \mathcal{C}_\kappa^\lambda$.