UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatic Set Theory

(Axiomatische Verzamelingentheorie)

2003/2004; 2nd Trimester
dr Benedikt Löwe

Homework Set # 9 Deadline: Thursday, March 18th, 2004

Exercise 9.1 (Trees).

As defined in the lecture course, we say that $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a **tree** if it is closed under initial segments, *i.e.*, if $s \in T$ and $n \in \omega$, then $s{\upharpoonright}n \in T$. Let T be an arbitrary subset of $\mathbb{N}^{<\mathbb{N}}$; then we can define a binary relation E on T by

$$s \, E_T \, t \iff \exists m \, (t = s^\frown\langle m \rangle).$$

With this relation, $\langle T, E_T \rangle$ forms a directed graph.

(1) Show that T is a tree if and only if $\langle T, E_T \rangle$ is connected, acyclic, and contains $\varnothing$.

(2) Call a graph $\langle G, E \rangle$ **countably branching** if for each $v \in G$, the set $\{w \, ; \, v \, E \, w\}$ is countable. Show (using AC) that for each connected, acyclic, countably branching graph G there is a tree T such that $\langle G, E \rangle$ is isomorphic to $\langle T, E_T \rangle$.

Note. Before starting, fix the notion of "isomorphic" that you want to use in this exercise.

(3) Let $s \in \mathbb{N}^{<\mathbb{N}}$ and $T_s := \{t \, ; \, s \subseteq t \, \vee \, t \subseteq s\}$. Show that $[T_s] = [s]$.

(4) Let $x \in \mathbb{N}^{\mathbb{N}}$ and $T_x := \{x{\upharpoonright}n \, ; \, n \in \mathbb{N}\}$. Show that $[T_x] = \{x\}$.

Exercise 9.2 (Countable ordinals and the reals)

If $x, y \in \mathbb{N}^{\mathbb{N}}$ and $x \neq y$, let $n_{x,y}$ be the least n such that $x(n) \neq y(n)$. Define the following relation $<$ on $\mathbb{N}^{\mathbb{N}}$ (this is the lexicographic ordering):

$$x < y :\iff x \neq y \,\&\, x(n_{x,y}) < y(n_{x,y}).$$

Show that $<$ is a strict linear ordering.

Prove that for an ordinal α the following are equivalent:

(i) The ordinal α is countable, *i.e.*, $\alpha < \aleph_1$, and

(ii) there is an order-preserving injection $f : \langle \alpha, \in \rangle \to \langle \mathbb{N}^{\mathbb{N}}, < \rangle$.

(In other words: The countable ordinals are exactly those that are homeomorphic to a set of reals. Why is this statement a reformulation of the above equivalence?)