Universität Hamburg Winter 2023/24
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Geometric Topology

Problem Set 4

1. Prove that every vector bundle whose rank is bigger than the dimension of the base manifold admits a nowhere vanishing section.

2. In this exercise, we denote by $\underline{\mathbb{R}}^k$ the trivial bundle of rank k over the given base.

a) Prove that S^n admits a non-vanishing vector field if and only if n is odd.
Hint: Use such a vector field to construct a homotopy from the identity to the antipodal map.

b) Suppose M and N are manifolds of positive dimension such that $TM \oplus \underline{\mathbb{R}}^1$ and $TN \oplus \underline{\mathbb{R}}^1$ are trivial and assume that TM has a nonvanishing section. Prove that $T(M \times N)$ is a trivial bundle.

c) Deduce that a product of two or more spheres has trivial tangent bundle if and only if at least one of them has odd dimension.

d) Illustrate your proof by giving an explicit trivialization of $T(S^2 \times S^5)$.
(If you find this too hard, try $T(S^1 \times S^2)$ first.)

3. A Lie group is a smooth manifold G which is also a group, and such that the group multiplication $\mu : G \times G \to G$ and the inversion $\iota : G \to G$ are smooth maps. Prove that every Lie group has trivial tangent bundle.

4. Let $p : E \to B$ be a real vector bundle of rank r over B. Prove that E is an orientable vector bundle if and only if $\Lambda^r E \to B$ admits a section (so it is a trivial bundle).

5. Let E_i be vector bundles over the same base B. A sequence of vector bundle morphisms, all covering the identity map on B,

$$\dots \xrightarrow{F_{i-2}} E_{i-1} \xrightarrow{F_{i-1}} E_i \xrightarrow{F_i} E_{i+1} \xrightarrow{F_{i+1}} \dots$$

is called exact, if for each $b \in B$ and each index i we have

$$\operatorname{image}(F_{i-1})_b = \ker(F_i)_b.$$

a) Prove that in every exact sequence of vector bundles all maps have constant rank over each connected component of B.

A short exact sequence is an exact sequence of the form

$$0 \to E_1 \xrightarrow{F_1} E_2 \xrightarrow{F_2} E_3 \to 0.$$

b) State explicitly which properties exactness of the sequence implies for each of the maps F_1 and F_2.

c) Prove that in a short exact sequence as above, E_2 is isomorphic to the direct sum $E_1 \oplus E_3$.