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Differential Topology

Problem Set 2

1. What are the possible degrees of maps from T^2 to itself? Are homotopy classes of such maps classified by their degree?

2. De Rham cohomology of a manifold M is defined for any $k \geq 0$ as

$$H^k_{\mathrm{dR}}(M) := \frac{\ker d : \Omega^k(M) \to \Omega^{k+1}(M)}{\operatorname{Im} d : \Omega^{k-1}(M) \to \Omega^k(M)}.$$

Here $\Omega^k(M)$ denotes the real vector space of smooth differential k-forms on M. The aim of this exercise is to prove that integration provides an isomorphism of the top-dimensional de Rham cohomology group $H^n_{\mathrm{dR}}(M)$ of a closed, connected and oriented n-dimensional manifold with $\mathbb{R}$. This fact was used in Problem 15 on the first problem set.

a) Prove by induction on n that if $f : \mathbb{R}^n \to \mathbb{R}$ is a function with compact support and $\int_{\mathbb{R}^n} f(x)\,dx_1 \ldots dx_n = 0$, then there exist functions $u_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \ldots, n\}$ with compact support such that $f = \sum_i \frac{\partial u_i}{\partial x_i}$.
Hint: The case $n = 1$ is an easy consequence of the fundamental theorem of calculus. For the induction step consider the auxiliary function

$$g(x_2, \ldots, x_n) := \int_{\mathbb{R}} f(x_1, x_2, \ldots, x_n)\,dx_1,$$

and observe that by Fubini's theorem one can apply the induction hypothesis to obtain $u_2 \ldots, u_n$. To get the remaining function u_1, adjust

$$w_1(x_1, \ldots, x_n) := \int_{-\infty}^{x_1} f(t, x_2, \ldots, x_n)\,dt$$

by subtracting a suitably cut off version of g.

b) Deduce from this that every compactly supported form $\omega \in \Omega^n(\mathbb{R}^n)$ with vanishing integral is the differential of a compactly supported form $\eta \in \Omega^{n-1}(\mathbb{R}^n)$.

c) Prove that for a closed connected oriented manifold M there are finitely many open sets $U_0, U_1, \ldots, U_r$ diffeomorphic to balls and covering M and diffeomorphisms $\varphi_i : M \to M$ isotopic to the identity with $\varphi_i(U_0) = U_i$.

d) Prove that for any closed form $\alpha \in \Omega^n(M)$ with compact support in some U_i, $\varphi_i^*\alpha$ and α are cohomologous.
Hint: Consider an isotopy $\Phi_t : M \to M$, $t \in [0,1]$ with $\Phi_0 = id_M$ and $\Phi_1 = \varphi_i$. Now argue that for $t, t' \in [0,1]$ sufficiently close, $\Phi_t^*\alpha$ and $\Phi_{t'}^*\alpha$ will both have support in $\Phi_t^{-1}(U_i)$ and have the same integral, and so by part **b)** *they must be cohomologous. Finish with a standard open-and-closed argument.*

e) Now complete the proof of the original claim by using a partition of unity subordinate to the cover $\{U_i\}_{i=0,\dots,r}$ of M from part **c)** to break up a given form $\omega \in \Omega^n(M)$ whose integral over M vanishes into components ω_i with support in U_i and applying the result of part **b)** to the form

$$\tilde{\omega} = \sum_{i=0}^{n} \varphi_i^* \omega_i$$

with support in U_0, which by part **d)** is cohomologous to ω.

3. Suppose $p : E \to B$ is a vector bundle over a compact base B.

a) Prove that for $N \in \mathbb{N}$ sufficiently large there exists a surjective bundle morphism $\underline{\mathbb{R}}^N \to E$.

b) Prove that for $K \in \mathbb{N}$ sufficiently large there exists an injective bundle morphism $E \to \underline{\mathbb{R}}^K$.

4. Prove the *collar neighborhood theorem*: If M is a smooth manifold with compact boundary $B = \partial M$, then B has a neighborhood $N \subseteq M$ in M diffeomorphic to $B \times [0, 1)$ via a diffeomorphism sending $p \in B$ to $(p, 0) \in B \times [0, 1)$.

5. Prove that for any vector bundle $E \to B$ the vector bundle $E \oplus E \to B$ is orientable. Deduce as a consequence that the total space TM of the tangent bundle of any smooth manifold M is orientable, regardless of the orientability of M itself.

6. Let $f : M \to M'$ be a smooth map between smooth manifolds which is transverse to the submanifold $Z' \subseteq M'$. We already know that in this case the inverse image $Z := f^{-1}(Z') \subseteq M$ is a smooth submanifold of the same codimension as Z'. Prove that the normal bundle of $Z \subseteq M$ is the pullback of the normal bundle of $Z' \subseteq M'$.
Note that for $Z = \{q\}$ a regular value this implies that $f^{-1}(q) \subseteq M$ has trivial normal bundle. So while many interesting submanifolds arise as preimages of regular values under a smooth map, "most" submanifolds cannot be obtained in this way, simply because normal bundles are typically not trivial.

7. **a)** Prove that S^n admits a nonvanishing vector field if and only if n is odd.
Hint: For the "only if" part, use such a vector field to construct a homotopy from the identity to the antipodal map.

b) Suppose M and N are manifolds of positive dimension such that $TM \oplus \underline{\mathbb{R}}$ and $TN \oplus \underline{\mathbb{R}}$ are trivial, and assume that TM has a nonvanishing section. Prove that under these assumptions $T(M \times N)$ is a trivial bundle.

c) Deduce that a product of two or more spheres with positive dimensions has trivial tangent bundle if and only if at least one of them has odd dimension.

d) Illustrate your proof by constructing explciti trivializations of $T(S^1 \times S^2)$ and $T(S^2 \times S^5)$.

8. The aim of this exercise is to complete the proof of the proposition formulated in class that the connected sum of connected manifolds of the same dimension $n > 0$ is well-defined up to diffeomorphism.

a) Use the isotopy extension trick (suitably applied to $\beta^{-1} \circ \alpha$) to prove that for any two orientation-reversing diffeomorphisms $\alpha : (0,1) \to (0,1)$ and $\beta : (0,1) \to (0,1)$ there exists a diffeomorphism $g : (0,1) \to (0,1)$ with compact support such that α and $\beta \circ g$ agree on the interval $(\frac{1}{4}, \frac{3}{4})$.

b) Now use this together with the fact proved in class that the identifications in performing the connected sum can be done on a smaller closed annulus inside $B(0,1)\backslash\{0\}$ to prove that for ball embeddings $h_1 : B(0,1) \to M_1$ and $h_2 : B(0,1) \to M_2$ there is a diffeomorphism

$$M_1 \#_{(h_1,h_2,\alpha)} M_2 \cong M_1 \#_{(h_1,h_2,\beta)} M_2$$

9. Prove that if M is any closed oriented manifold, then $M\#(-M)$ bounds a compact oriented manifold of one dimension higher.

10. **a)** Use van Kampen's theorem to prove that for connected manifolds M_1 and M_2 of dimension $n \geq 3$ we have

$$\pi_1(M_1 \# M_2) = \pi_1(M_1) \star \pi_1(M_2),$$

where $\star$ denotes free product of groups.

b) Use this to prove that for $n \geq 3$ the k-fold connected sums

$$\#^k (S^1 \times S^{n-1})$$

for different values of k are pairwise non-diffeomorphic.

c) *(Assuming you know how to compute homology or cohomology.)* Prove more generally that for $n \geq 2$ and integers $0 < \ell_i < n$ and $k_i \in \mathbb{N}$ there is a diffeomorphism

$$\#^{k_1}(S^{\ell_1} \times S^{n-\ell_1}) \cong \#^{k_2}(S^{\ell_2} \times S^{n-\ell_2})$$

if and only if $k_1 = k_2$ and either $\ell_1 = \ell_2$ or $\ell_1 = n - \ell_2$.