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Projective Hausdorff Gaps

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Abstract In [5] Todorčević shows that there is no Hausdorff gap (A, B) if A is analytic. In this note we extend the result by showing that the assertion "there is no Hausdorff gap (A, B) if A is coanalytic" is equivalent to "there is no Hausdorff gap (A, B) if A is $\boldsymbol{\Sigma}^1_2$", and equivalent to $\forall r\ (\aleph_1^{L[r]} < \aleph_1)$.

We also consider real-valued games corresponding to Hausdorff gaps, and show that $\mathsf{AD}_\mathbb{R}$ for pointclasses $\boldsymbol{\Gamma}$ implies that there are no Hausdorff gaps (A, B) if $A \in \boldsymbol{\Gamma}$.

Keywords Hausdorff gaps · descriptive set theory · infinite games

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1 Introduction

In the parlance of gaps in $\mathcal{P}(\omega)/\text{fin}$, two sets $a, b \in [\omega]^\omega$ are called *orthogonal* ($a \perp b$) if $|a \cap b| < \omega$. If $B \subseteq [\omega]^\omega$, then a is *orthogonal to* B ($a \perp B$) if $a \perp b$ for every $b \in B$. Two sets $A, B \subseteq [\omega]^\omega$ are *orthogonal* ($A \perp B$) if $a \perp b$ for every $a \in A$ and $b \in B$.

A pair (A, B) of orthogonal subsets of $[\omega]^\omega$ is called a *pre-gap*. There is a trivial way of constructing a pre-gap: take any infinite, co-infinite set $c \in [\omega]^\omega$ and pick any A and B in such a way that $\forall a \in A : a \subseteq^* c$ and $\forall b \in B : c \cap b =^* \varnothing$ (here $\subseteq^*$ and $=^*$ denote the subset and equality relations modulo finite). A set c as above is said to *separate*, or *interpolate*, the pre-gap (A, B). The objects worth of study are pre-gaps (A, B) that are not constructed in this trivial fashion.

Definition 1.1. A pre-gap (A, B) is called a *gap* if there is no c which separates A from B.

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An early result of Hadamard [1] already established that there cannot be a gap (A, B) if both A and B are countable. Hausdorff, in his well-known paper [2], constructed an "(ω_1, ω_1)-gap", i.e., a gap (A, B) where both A and B are well-ordered by $\subseteq^*$ with length ω_1. This construction proceeds by induction on $\alpha < \omega_1$, and the sets A and B are in general not definable.

Stevo Todorčević, on the other hand, showed in [5, p 56–57] that if one drops the well-ordering requirement, gaps in $\mathcal{P}(\omega)/\mathrm{fin}$ can be quite explicitly defined. For example, if $A := \{\{x{\restriction}n \mid x(n) = 0\} \mid x \in 2^\omega\}$ and $B := \{\{x{\restriction}n \mid x(n) = 1\} \mid x \in 2^\omega\}$ then (identifying $2^{<\omega}$ with ω) it is easy to see that (A, B) is a gap and that A and B are both perfect sets in the standard topology on $[\omega]^\omega$. However, as also shown in [5], such explicit counterexamples cease to exist if one assumes that A and B are well-ordered. In fact, a sufficient condition is that A and B are σ-directed.

Definition 1.2. A set $A \subseteq [\omega]^\omega$ is *σ-directed* if for every countable collection $\{a_n \in A \mid n \in \omega\}$, there exists $a \in A$ such that $a_n \subseteq^* a$ for all n.

If (A, B) is a gap and both A and B are σ-directed, we will call it a *Hausdorff gap*[1]

Theorem 1.3 (Todorčević, [5, Corollary 1])**.** *There cannot be a Hausdorff gap (A, B) if A is analytic.*

A variation of the proof shows that in the Solovay model (the $L(\mathbb{R})$ of the Lévy collapse of an inaccessible cardinal) there cannot be any Hausdorff gap. A detailed argument of this fact was provided in [3, Theorem 4.4.1], but it also follows from [6, Theorem 1].

The main purpose of this note is to extend Todorčević's result to the next level of the projective hierarchy. We use the following notation: if $\boldsymbol{\Gamma}$ is a projective pointclass, we say that (A, B) is a *$(\boldsymbol{\Gamma}, \boldsymbol{\Gamma})$-Hausdorff gap* if both A and B are in $\boldsymbol{\Gamma}$, and a *$(\boldsymbol{\Gamma}, \cdot)$-Hausdorff gap* if $A \in \boldsymbol{\Gamma}$ and B is arbitrary.

Proposition 1.4. *The following are equivalent:*

1. *there are no $(\boldsymbol{\Sigma}^1_2, \cdot)$-Hausdorff gaps,*
2. *there are no $(\boldsymbol{\Sigma}^1_2, \boldsymbol{\Sigma}^1_2)$-Hausdorff gaps,*
3. *there are no $(\boldsymbol{\Pi}^1_1, \cdot)$-Hausdorff gaps,*
4. *there are no $(\boldsymbol{\Pi}^1_1, \boldsymbol{\Pi}^1_1)$-Hausdorff gaps,*
5. $\forall r\, (\aleph_1^{L[r]} < \aleph_1)$.

The significance of this proposition is related to the study of regularity properties, where assertions like "all $\boldsymbol{\Sigma}^1_2/\boldsymbol{\Pi}^1_1$ sets are regular" (in a specific sense of "regular") are equivalent to certain "transcendence statements" over L, i.e., statements in how far the actual universe V differs from L. The *non-existence* of a Hausdorff gap can be seen as a kind of regularity property, and in this context Proposition 1.4 shows that this non-existence property on the $\boldsymbol{\Sigma}^1_2/\boldsymbol{\Pi}^1_2$ level is already of the strongest possible kind, as it implies that $\aleph_1$ is

[1] In the literature, the term "Hausdorff gap" is often used to mean "well-ordered gap", but for our purposes this more general definition is more appropriate.

inaccessible in L. It is also equivalent to, for example, the assertion that all $\boldsymbol{\Pi}^1_1$ sets satisfy the perfect set property.

The proposition is proved in Section 2. In Section 3 we consider an infinite game related to Hausdorff gaps.

2 Proof of main proposition

As the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) of Proposition 1.4 are trivial, we are left with (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1).

To show the latter implication, we first recall in more detail Todorčević's proof of Theorem 1.3.

Definition 2.1. Let (A, B) be a pre-gap (not necessarily σ-directed).

1. Let C be a set. We say that A and B are *C-separated* if $C \perp B$ and for every $a \in A$ there is $c \in C$ such that $a \subseteq^* c$.
2. We say that A and B are *σ-separated* if they are C-separated by some countable C.
3. Let S be a tree on $\omega^{<\uparrow\omega}$ (the set of finite strictly increasing sequences). We call S an *(A, B)-tree* if
   (a) $\forall\sigma \in S$, $\{i \in \omega \mid \sigma^\frown\langle i\rangle \in S\}$ has infinite intersection with some $b \in B$, and
   (b) $\forall x \in [S]$, $\operatorname{ran}(x) \subseteq^* a$ for some $a \in A$.

If (A, B) is not a gap, then it is σ-separated, but the converse need not be true in general. It is, however, true whenever A is σ-directed. On the other hand, the existence of an (A, B)-tree contradicts B being σ-directed.

Lemma 2.2 (Todorčević, [5])**.** *Let (A, B) be a pre-gap. If B is σ-directed, then there is no (A, B)-tree.*

Proof. Suppose, towards contradiction, that S is an (A, B)-tree. For each $\sigma \in S$, fix some $b_\sigma \in B$ such that $\{i \mid \sigma^\frown\langle i\rangle \in S\} \cap b_\sigma$ is infinite. By σ-directedness, there is a $b \in B$ which almost contains every b_σ. In particular, for each σ, the set $\{i \mid \sigma^\frown\langle i\rangle \in S\} \cap b$ is infinite. Therefore we can inductively pick $i_0, i_1, i_2 \in b$ in such a way that $\langle i_0, i_1, i_2, \dots\rangle$ is a branch through S. Then by definition of an (A, B)-tree $\{i_0, i_1, i_2 \dots\} \subseteq^* a$ for some $a \in A$. But this means that $a \cap b$ is infinite, contradicting the orthogonality of A and B. $\square$

Todorčević's proof in fact yields the following dichotomy: if (A, B) is a pre-gap and A is analytic, then either A and B are σ-separated or there exists an (A, B)-tree. We prove a similar dichotomy for $\boldsymbol{\Sigma}^1_2$ sets, with separation by a subset of $L[r]$ replacing σ-separation.

Lemma 2.3. *Let (A, B) be a pre-gap such that A is $\Sigma^1_2(r)$. Then:*

1. *either A and B are C-separated by some $C \subseteq L[r]$, or*
2. *there exists an (A, B)-tree.*

Proof. Let $A^* \subseteq \omega^{\uparrow\omega}$ be such that $x \in A^*$ iff $\mathrm{ran}(x) \in A$. Let T be a tree on $\omega \times \omega_1$, increasing in the first coordinate, such that $A^* = p[T]$ and $T \in L[r]$. Define an operation on such trees T as follows

– for $(s,h) \in T$, let
$$c_{(s,h)} := \{i > \max(\mathrm{ran}(s)) \mid \exists (s',h') \in T \text{ extending } (s,h) \text{ s.t. } i \in \mathrm{ran}(s')\}$$
– let $T' := \{(s,h) \in T \mid c_{(s,h)} \text{ has infinite intersection with some } b \in B\}$.

Now let $T_0 := T$, $T_{\alpha+1} := T'_\alpha$ and $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$ for limit λ. Note that this definition is absolute for $L[r]$ so all the trees T_α are in $L[r]$.

Let α be least such that $T_\alpha = T_{\alpha+1}$. We distinguish two cases:

– **Case 1:** $T_\alpha = \varnothing$. Let $x \in A^*$ be given. Let $f \in \omega_1^\omega$ be such that $(x,f) \in [T_0]$. Let $\gamma < \alpha$ be such that $(x,f) \in [T_\gamma] \setminus [T_{\gamma+1}]$, and let $(s,h) \subseteq (x,f)$ be such that $(s,h) \in T_\gamma \setminus T_{\gamma+1}$. Now let $c_x := c_{(s,h)}$ and note that this set is in $L[r]$ since it is constructible from T_γ and (s,h) both of which are in $L[r]$. By assumption $c_x \perp B$, and it is also clear that $\mathrm{ran}(x) \subseteq^* c_x$. It follows that the collection $C := \{c_x \mid x \in A^*\}$, with each c_x defined as above, forms a subset of $L[r]$ which separates A from B.

– **Case 2:** $T_\alpha \neq \varnothing$. In this case we will use the tree T_α to construct an (A,B)-tree S. By induction, we will construct S and to each $\sigma \in S$ associate $(s_\sigma, h_\sigma) \in T_\alpha$, satisfying the following conditions:
  – $\sigma \subseteq \tau \implies (s_\sigma, h_\sigma) \subseteq (s_\tau, h_\tau)$, and
  – $\mathrm{ran}(\sigma) \subseteq \mathrm{ran}(s_\sigma)$.

  First $\varnothing \in S$, and we associate to it $(s_\varnothing, h_\varnothing) := (\varnothing, \varnothing)$. Next, suppose $\sigma \in S$ has already been defined and $(s_\sigma, h_\sigma) \in T_\alpha$ associated to it. By assumption, $(s_\sigma, h_\sigma) \in T'_\alpha$, so $c_{(s_\sigma,h_\sigma)}$ has infinite intersection with some $b \in B$. For each $i \in c_{(s_\sigma,h_\sigma)}$ we add $\sigma^\frown\langle i\rangle$ to S. Moreover, by assumption, for each $i \in c_{(s_\sigma,h_\sigma)}$ there exists $(s',h') \in T_\alpha$ extending (s,h) such that $i \in \mathrm{ran}(s')$. Now associate precisely these (s',h') to $\sigma^\frown\langle i\rangle$, i.e., let $s_{\sigma^\frown\langle i\rangle} := s'$ and $h_{\sigma^\frown\langle i\rangle} := h'$. By induction, it follows that the condition $\mathrm{ran}(\sigma^\frown\langle i\rangle) \subseteq \mathrm{ran}(s_{\sigma^\frown\langle i\rangle})$ is satisfied.

  Now we have a tree S on $\omega^{\uparrow\omega}$. By definition, for every $\sigma \in S$ the set of its successors $c_{(s_\sigma,h_\sigma)}$ has infinite intersection with some $b \in B$. Now let $x \in [S]$. By construction, $\bigcup\{(s_\sigma,h_\sigma) \mid \sigma \subseteq x\}$ forms an infinite branch through T_α, whose projection $a := \bigcup\{s_\sigma \mid \sigma \subseteq x\}$ is a member of $p[T_\alpha] \subseteq p[T_0] = A^*$. Since by assumption $\mathrm{ran}(\sigma) \subseteq \mathrm{ran}(s_\sigma)$ holds for all $\sigma \subseteq x$, it follows that $\mathrm{ran}(x) \subseteq \mathrm{ran}(a)$. This proves that S is an (A,B)-tree. □

Corollary 2.4. *If $\forall r\, (\aleph_1^{L[r]} < \aleph_1)$ then there are no $(\boldsymbol{\Sigma}^1_2, \cdot)$-Hausdorff gaps.*

Proof. Let (A,B) be a pre-gap such that A and B are σ-directed and A is $\Sigma^1_2(r)$. By Lemma 2.2, the second alternative of Lemma 2.3 is impossible, hence there is a $C \subseteq L[r]$ which separates A from B. Since the reals of $L[r]$ are countable, C is countable, so A and B are σ-separated. Since A is also σ-directed, (A,B) cannot be a gap. □

This completes the proof of the implication (5) $\Rightarrow$ (1) of our main proposition.

To show the implication (4) $\Rightarrow$ (5), assume, towards contradiction, that $\aleph_1^{L[r]} = \aleph_1$ for some r. We need to construct a Hausdorff gap (A, B) in $L[r]$, such that both A and B have $\boldsymbol{\Pi}^1_1$ definitions, and moreover, such that (A, B) remains a gap in V. It turns out that we can use Hausdorff's original construction of the (ω_1, ω_1)-gap. It is easy to see that this remains a gap in any larger model as long as $\aleph_1$ is preserved (the gap is said to be *indestructible*), so what remains to be checked is that in $L[r]$, Hausdorff's construction can be carried out in such a way that both A and B are $\boldsymbol{\Pi}^1_1$ sets. We refer the reader to [3, Section 4.3], where such a construction is worked out in detail. With this the proof of Proposition 1.4 is complete.

Remark 2.5. A remark is due concerning the proof of (4) $\Rightarrow$ (5). First of all, note that, if we only wanted A and B to have $\boldsymbol{\Sigma}^1_2$-definitions, this would be easy, since we could just use the $\boldsymbol{\Sigma}^1_2$-good wellorder of the reals of $L[r]$. To obtain the stronger result that A and B have $\boldsymbol{\Pi}^1_1$-definitions, the proof in [3, Section 4.3] used a method due to Arnold Miller [4]. We chose not to include the proof here because

1. it is rather long, and involves ideas not directly related to the contents of this note, and
2. Stevo Todorčević (private communication) informed us that it can be proved in a simpler way.

3 Haudorff gaps and determinacy

We consider the effect of infinite games on Hausdorff gaps. The best result we could hope for in this setting is for AD (the axiom of determinacy) to imply that there are no Hausdorff gaps. Unfortunately, we are only able to prove this from the stronger assumption of $\mathsf{AD}_{\mathbb{R}}$ (the axiom of real determinacy).

Definition 3.1. Given a pre-gap (A, B), we define the *Hausdorff game*, denoted by $G_{\mathrm{H}}(A, B)$. It is played as follows:

$$\begin{array}{l|llll} \mathrm{I}: & c_0 & (s_1, c_1) & (s_2, c_2) & \dots \\ \hline \mathrm{II}: & i_0 & i_1 & i_2 & \dots \end{array}$$

where $s_n \in \omega^{<\omega}$, $c_n \in [\omega]^\omega$ and $i_n \in \omega$. The conditions for player I are that

1. $\min(s_n) > \max(s_{n-1})$ for all $n \geq 1$,
2. $\min(c_n) > \max(s_n)$,
3. $c_n \not\perp B$ for all n, and
4. $i_n \in \mathrm{ran}(s_{n+1})$ for all n.

Conditions for player II are that

5. $i_n \in c_n$ for all n.

If all five conditions are satisfied, let $s^* := s_1{}^\frown s_2{}^\frown \dots$ be an infinite increasing sequence formed by the play of the game. Player I wins iff $\mathrm{ran}(s^*) \subseteq^* a$ for some $a \in A$.

Remark 3.2. In the original version of this paper, the winning condition for Player I was simply "$\mathrm{ran}(s^*) \in A$" (which felt somewhat more intuitive) and only the "$\Rightarrow$" directions of Theorem 3.3 was proved. The referee suggested that the implication of that theorem be upgraded to an equivalence. This, however, required the small (but very natural) change in the definition of the game.

Theorem 3.3. *Suppose (A, B) is a pre-gap and $G_{\mathrm{H}}(A, B)$ is the corresponding Hausdorff game. Then:*

1. *Player I has a winning strategy in $G_{\mathrm{H}}(A, B)$ if and only if there exists an (A, B)-tree.*
2. *Player II has a winning strategy in $G_{\mathrm{H}}(A, B)$ if and only if A and B are σ-separated.*

Proof.

1. First, suppose S is an (A, B)-tree, and let us informally describe the strategy for I. The first move c_0 is defined as $\{n \mid \langle n \rangle \in S\}$. After Player II picks $i_0 \in c_0$, I responds by playing $s_1 := \langle i_0 \rangle$ and $c_1 := \{n \mid \langle i_0, n \rangle \in S\}$. Player II then picks $i_1 \in c_1$, whereupon I responds by playing $s_2 := \langle i_1 \rangle$ and $c_2 := \{n \mid \langle i_0, i_1, n \rangle \in S\}$. So it goes on, i.e., at each stage k, Player II picks $i_k \in c_k$ and I responds by $s_{k+1} := \langle i_k \rangle$ and $c_{k+1} := \{n \mid s_0{}^\frown \dots {}^\frown s_{k+1}{}^\frown \langle n \rangle \in S\}$ (note that by induction $s_0{}^\frown \dots {}^\frown s_k \in S$). Clearly, this describes a legitimate strategy for I which results in a play where s^* is a branch through S, so $\mathrm{ran}(s^*) \subseteq^* a$ for some $a \in A$ and I wins the game.

For the converse direction, let σ be a winning strategy for player I, and let T_σ be the tree of partial positions according to σ. If $p \in T_\sigma$ is a position of the form $p = \langle c_0, i_0, (s_1, c_1), i_1, \dots, (s_n, c_n) \rangle$, we use the notation $p^* := s_1{}^\frown \dots {}^\frown s_n$.

Now we use T_σ to inductively construct the tree S. To each $s \in S$ we associate a $p_s \in T_\sigma$ (of odd length), such that

1. $s \subseteq t$ iff $p_s \subseteq p_t$, and
2. $\mathrm{ran}(s) \subseteq \mathrm{ran}(p_s^*)$.

First $\varnothing \in S$ and $p_\varnothing = \varnothing$. Suppose $s \in S$ and p_s are already defined and $\mathrm{ran}(s) \subseteq \mathrm{ran}(p_s^*)$ holds. Assume $p_s = \langle \dots, (s_n, c_n) \rangle$. For every $i_n \in c_n$, let (s_{n+1}, c_{n+1}) be the response of the strategy σ to $p_s{}^\frown \langle i_n \rangle$. Let $s^\frown \langle i_n \rangle$ be in S and associate to it $p_{s^\frown \langle i_n \rangle} := p_s{}^\frown \langle i_n \rangle^\frown \langle (s_{n+1}, c_{n+1}) \rangle$. Since for each $i_n \in c_n$ we know that $i_n \in \mathrm{ran}(s_{n+1})$, it follows that $\mathrm{ran}(s^\frown \langle i_n \rangle) \subseteq \mathrm{ran}(p^*_{s^\frown \langle i_n \rangle})$, completing the induction step.

Now it is clear that the tree S has exactly the sets "c_n" from the definition of the Hausdorff game, as the branching-points, and each such set c_n has infinite intersection with some $b \in B$, by assumption. Moreover, if x is a branch through S, then by construction $z := \bigcup\{p_s \mid s \subseteq x\}$ forms a branch through T_σ satisfying $\operatorname{ran}(x) \subseteq \operatorname{ran}(z^*)$. Since z is an infinite play of the game according to the winning strategy σ, it follows that $\operatorname{ran}(z^*) \subseteq^* a$ for some $a \in A$, hence also $\operatorname{ran}(x) \subseteq^* a$, and so S is an (A, B)-tree.

2. First, assume that A and B are σ-separated by some set $\{a_n \mid n < \omega\}$. All Player II has to do is make sure that, for every k, $i_k \notin \bigcup_{n \leq k} a_n$. But since $a_n \cap b$ is finite for all n and all $b \in B$, whereas the sets c_k played by Player I have infinite intersection with some $b \in B$, it is clear that, at each stage k, Player II is indeed able to pick $i_k \in \left(c_k \setminus \bigcup_{n \leq k} a_n\right)$. Now consider any result of such a play s^*. Clearly $\{i_k \mid k < \omega\} \subseteq \operatorname{ran}(s^*)$ holds. Moreover, for every n, there are at most n elements in $\{i_k \mid k < \omega\} \cap a_n$, implying that $\{i_k \mid k < \omega\}$ is orthogonal to all a_n, and hence to A. But then $\operatorname{ran}(s^*)$ is also orthogonal to A, so, in particular, Player I cannot win this game.

It remains to prove the converse direction. Let τ be a winning strategy for player II, and let T_τ be the tree of partial plays according to τ. Our method will be similar to the proof of the standard Banach-Mazur theorem, but the problem is that the tree T_τ has uncountable branching. Therefore we first thin it out to another tree $\tilde{T}_\tau$, as follows: for every node of even length $p = \langle \dots, (s_n, c_n), i_n \rangle \in T_\tau$, fix s and i and consider the collection $\operatorname{Succ}_{T_\tau}(p, s, i) := \{(s, c) \mid p^\frown\langle(s, c)\rangle^\frown\langle i \rangle \in T_\tau\}$. In other words, this is the collection of all valid moves by player I following position p, such that the first component of the move is s, and such that II's next move according to τ is i. If this collection is non-empty, throw away all members of $\operatorname{Succ}_{T_\tau}(p, s, i)$, and their generated subtrees, except for one, so that $\operatorname{Succ}_{T_\tau}(p, s, i)$ becomes a singleton. Do this for every $s \in \omega^{<\omega}$ and every $i \in \omega$, and inductively form the new tree $\tilde{T}_\tau$—this is also going to be a tree of positions according to τ, but it will be only countably branching. Now we can use a Banach-Mazur-style argument on the tree $\tilde{T}_\tau$.

For every $p \in \tilde{T}_\tau$ and $x \in \omega^{\uparrow\omega}$, where $p = \langle \dots (s_n, c_n), i_n \rangle$, we say that p *is compatible with* x if $p^* \subseteq x$ and $i_n \in \operatorname{ran}(x)$. We say that p *rejects* x if it is compatible with x and maximally so with respect to $\tilde{T}_\tau$, i.e., if for every (s, c) such that $p^\frown\langle(s, c)\rangle \in \tilde{T}_\tau$ and $p^{*\frown} s \subseteq x$, $i := \tau(p^\frown\langle(s, c)\rangle) \notin \operatorname{ran}(x)$.

Let $\bar{A} := \{x \mid x \subseteq^* a \text{ for some } a \in A\}$. It is clear that for every $x \in \bar{A}$ there is a $p \in \tilde{T}_\tau$ which rejects x—otherwise we could inductively find an infinite branch z through $\tilde{T}_\tau$ such that $z^* = x$, which would imply that $x \notin \bar{A}$ since τ is winning for Player II. For each $p \in \tilde{T}_\tau$ let $K_p := \{x \mid p \text{ rejects } x\}$. Then $A \subseteq \bar{A} \subseteq \bigcup_{p \in \tilde{T}_\tau} K_p$ and $\tilde{T}_\tau$ is countable, so the result will follow if we can prove that each K_p is σ-separated from B.

For this, fix some $p = \langle \dots (s_n, c_n), i_n \rangle$, and for every $s \in \omega^{<\omega}$ such that $i_n \in \operatorname{ran}(s)$ and $\min(s) > \max(p^*)$, consider the set

$$a_s := \bigcup\{\operatorname{ran}(x) \mid x \in K_p \text{ and } p^{*\frown} s \subseteq x\}.$$

We claim that the collection $\{a_s \mid i_n \in \operatorname{ran}(s) \text{ and } \min(s) > \max(p^*)\}$ σ-separates K_p from B. First, clearly if $x \in K_p$ then there exists some s, satisfying the conditions, such that $p^* {}^\frown s \subseteq x$, so that $\operatorname{ran}(x) \subseteq a_s$. Secondly, suppose $K_p \not\perp B$. Then there is some s such that a_s has infinite intersection with some $b \in B$. Let $a'_s := a_s \setminus \max(s)$. According to the rules of the game, player I is then allowed to play the move "(s, a'_s)" after position p. The only problem is that $p^\frown\langle(s, a'_s)\rangle$ might not be in $\tilde{T}_\tau$. However, by construction there is some c such that $i := \tau(p^\frown\langle(s,c)\rangle) = \tau(p^\frown\langle(s, a'_s)\rangle)$ and $p^\frown\langle(s,c)\rangle \in \tilde{T}_\tau$. But then we must have $i \in a'_s$, so by definition there is some $x \in K_p$ such that $p^* {}^\frown s \subseteq x$ and $i \in \operatorname{ran}(x)$. But then $p^\frown\langle(s,c)\rangle^\frown\langle i\rangle$ is still compatible with x and hence p does not reject x, contradicting $x \in K_p$.

So we must have $a_s \perp B$ for all s, and this completes the proof. □

Corollary 3.4. $\mathsf{AD}_\mathbb{R}$ *implies that every pre-gap* (A, B) *is either* σ*-separated or there exists an* (A, B)*-tree. Hence, there is no Hausdorff gap.*

Note that from the way we have defined the game $G_\mathrm{H}(A, B)$, it looks as though both A and B were parameters in the definition. However, a closer look reveals that B is only relevant for the condition that requires Player I to play c_n such that $c_n \not\perp B$ at every move. As this is a "closed" condition on the play of the game, the complexity of B is irrelevant for the game's determinacy. Hence, we obtain the following stronger corollary:

Corollary 3.5. *Let* $\boldsymbol{\Gamma}$ *be a projective pointclass and assume that* $\mathsf{AD}_\mathbb{R}$ *holds for* $\boldsymbol{\Gamma}$*. Then there are no* $(\boldsymbol{\Gamma}, \cdot)$*-Hausdorff gaps.*

In our definition of the game, it was essential for Player I to be able to make real number moves. Therefore, the following is still open:

Question 3.6. *Can* $\mathsf{AD}_\mathbb{R}$ *be replaced by* AD *in the above results?*

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