Reflection Theorem

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Löwenheim-Skolem Theorem

Theorem (Löwenheim-Skolem Theorem, in ZFC − Foundation)

Let $\mathfrak{M} = (M, I)$ be an $\mathcal{L}$-model and $N_0 \subseteq M$. Then there exists a set $N \subseteq M$ such that $N_0 \subseteq N$, $|N| \leq \max(|N_0|, |\mathcal{L}|)$ and $\mathfrak{M} \restriction N \preceq \mathfrak{M}$.

Lemma (Tarski-Vaught Criterion)

Let $\mathfrak{M}$ and $\mathfrak{N}$ be models such that $\mathfrak{N} \subseteq \mathfrak{M}$. Then the following are equivalent:

- $\mathfrak{N} \preceq \mathfrak{M}$.
- *For all $\exists y \varphi(\vec{x}, y) \in \mathcal{L}$ and $\vec{a} \in N$, $\mathfrak{M} \models \exists y \varphi[\vec{a}]$ implies $\mathfrak{M} \models \varphi[\vec{a}, b]$ for some $b \in N$.*

From set models to proper class models...

As it is already known, we cannot apply LST-theorem directly to V. More precisely, we cannot generalize LST-theorem like this:

"Let N_0 be a set. Then there exists a set $N \supseteq N_0$ such that $N \preceq V$."

It is meaningless to say $N \preceq V$, since we are not allowed to quantify over formulas. However, for any list $\varphi_0, \cdots, \varphi_{n-1}$ of finitely many formulas, we can write down a sentence like:

$$\exists M(\bigwedge_{i<n} M \preceq_{\varphi_i} V),$$

where $M \preceq_{\varphi_i} V$ means

$$(M, \in) \models \varphi_i[a_0, \cdots, a_{n-1}] \text{ if and only if } (V, \in) \models \varphi_i[a_0, \cdots, a_{n-1}]$$

for every $a_0, \cdots, a_{n-1} \in M$.

Reflection Principle

Theorem (Reflection Principle)

(i) Let $\varphi(x_1, \cdots, x_n)$ *be a formula. For each set* M_0*, there is a set* M *such that* $M_0 \subseteq M$ *and*

$$\varphi^M(x_1, \cdots, x_n) \leftrightarrow \varphi(x_1, \cdots, x_n).$$

for every $x_1, \cdots, x_n \in M$*. (We say that* M *reflects* φ*.)*
(ii) Moreover, there is a transitive set $M \supseteq M_0$ *that reflects* φ*; moreover, there is a limit ordinal* α *such that* $M_0 \subseteq V_\alpha$ *and* V_α *reflects* φ*.*
(iii) Assuming the Axiom of Choice, there is a set M *such that* $M_0 \subseteq M$*,* M *reflects* φ *and* $|M| \leq \max(|M_0|, \aleph_0)$*. In particular, there is a countable* M *that reflects* φ*.*

Reflection Principle

Before proving the reflection principle, we prove a ‘class version’ of Tarski-Vaught criterion:

Lemma

Let $\Phi = \{\varphi_i : i < n\}$ be a subformula-closed set of formulas. Let A, B be classes with $\varnothing \neq A \subseteq B$. Then the following are equivalent:

(1) $\bigwedge_{i<n} A \preceq_{\varphi_i} B$.

(2) *For all existential formulas* $\varphi_i = \exists y \varphi_j(\vec{x}, y) \in \Phi$, $\forall \vec{a} \in A(\varphi_i^B(\vec{a}) \to \exists b \in A \varphi_j^B(\vec{a}, b))$.

Reflection Principle

It suffices to show the following lemma:

Lemma

Let $\Phi = \{\varphi_i : i < n\}$ be a finite set of formulas. For each set M_0, there exists a (transitive) set M such that $M_0 \subseteq M$ and,

$$(\dagger) \text{ for all } \vec{x} \in M \text{ and } \varphi \in \Phi, \exists y \varphi(\vec{x}, y) \text{ implies } \exists y \in M \varphi(\vec{x}, y).$$

Assuming AC, there exists a set N such that ($\dagger$) holds for N and $|N| \leq \max(|M_0|, \aleph_0)$.

Reflection Principle

Now it is easy to see the following theorem holds:

Theorem (Reflection Principle)

(i) Let $\varphi(x_1, \cdots, x_n)$ be a formula. For each set M_0, there is a set M such that $M_0 \subseteq M$ and

$$\varphi^M(x_1, \cdots, x_n) \leftrightarrow \varphi(x_1, \cdots, x_n).$$

for every $x_1, \cdots, x_n \in M$. (We say that M reflects φ.)
(ii) Moreover, there is a transitive set $M \supseteq M_0$ that reflects φ; moreover, there is a limit ordinal α such that $M_0 \subseteq V_\alpha$ and V_α reflects φ.
(iii) Assuming the Axiom of Choice, there is a set M such that $M_0 \subseteq M$, M reflects φ and $|M| \leq \max(|M_0|, \aleph_0)$. In particular, there is a countable M that reflects φ.

Reflection Theorem

Theorem (Reflection Theorem)

Let $\Phi = \{\varphi_i : i < n\}$ *be a finite set of formulas. Assume that* B *is a non-empty class and* $\langle A(\alpha) : \alpha \in \mathrm{Ord}\rangle$ *is a transfinite sequence such that:*

(i) $\alpha < \beta$ *implies* $A(\alpha) \subseteq A(\beta)$,

(ii) *if* α *is limit, then* $A(\alpha) = \bigcup_{\beta<\alpha} A(\beta)$, *and*

(iii) $B = \bigcup_{\alpha\in\mathrm{Ord}} A(\alpha)$.

Then $\forall\alpha\exists\beta > \alpha(A(\beta) \neq \varnothing \wedge \bigwedge_{i<n} A(\beta) \preceq_{\varphi_i} B \wedge \beta$ *is limit*).

Some Corollaries

Corollary

Let Λ *be a finite set of axioms of* ZF. *Then*

(1) $\mathrm{ZF} \vdash \exists\alpha \in \mathrm{Ord}(V_\alpha \models \Lambda \cup (\mathrm{ZF} - \mathrm{Replacement}))$,

(2) $\mathrm{ZFC} \vdash \exists\alpha \in \mathrm{Ord}(V_\alpha \models \Lambda \cup (\mathrm{ZFC} - \mathrm{Replacement}))$, *and*

(3) $\mathrm{ZFC} \vdash \exists M(M \models \Lambda \cup (\mathrm{ZF} - \mathrm{Replacement}) \wedge |M| = \aleph_0 \wedge M$ *is transitive*$)$.

Theorem

If $\Gamma \supseteq \mathrm{ZF}$ *is consistent, then* Γ *is not finitely axiomatizable.*

Thanks!