Set Models

M is a set

"$M \models \varphi$" formalized (Tarski recursive definition.) in ZFC

You can say: "$\exists \varphi \, \forall M$, if $M \models \varphi \ldots$"

NB: if M set, then formal "$M \models \varphi$" is equivalent to φ^M.

Class Models

$M = \{x \mid \varphi(x)\}$ prop class

"$(M, \in) \models \varphi$"

You cannot say

"$\exists \varphi \, \forall$ class M .... "

Relativization:

For formula φ, define

$\varphi^{(M,\in)} = \ldots\ldots\ldots\ldots\ldots$

"syntactic operation"

φ^M means $(M, \in) \models \varphi$

Question: M = prop. class. When we say $ZFC \vdash$ "$M \models ZFC$" what do we mean?

∞-many theorems

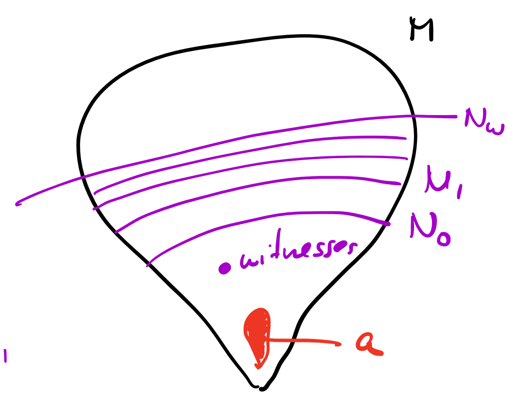
This means: for every φ from ZFC, $ZFC \vdash \varphi^M$.

Comment: Why can't we use Tarski "$M \models \varphi$" for prop. classes?

E.g: $M = \{x \mid \top\} = V$. If for any φ with Gödel number n (φ is φ_n) we could write "$M \models \varphi$" in ZFC, then $ZFC \vdash (M \models \varphi_n) \leftrightarrow \varphi$

↯ Truth predicate is undefinable!

Reflection: Down. L-Sk: $M \models \Sigma$, $a \in M$, you can find $N \subseteq M$ s.t. $a \in N$, $|N| = \max(|a|, \aleph_0)$, and $N \models \Sigma$.



"Skolem Hull"

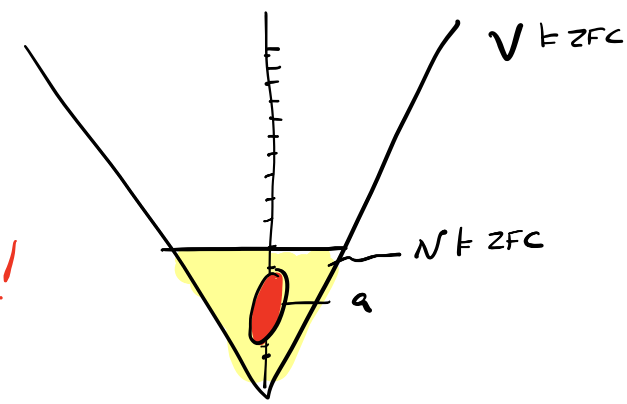
$N := H_M(a)$

In ZFC, you try to do the same:

- a = set. V = whole universe.

  Find N s.t. $a \in N$, $|N| = \max(|a|, \aleph_0)$

  and $N \models ZFC$



↯ Then $ZFC \vdash \exists M \models ZFC$

$\vdash Con(ZFC)$ ↯ Gödel!

Instead: For any finitely many axioms $ZFC^* \subseteq ZFC$,
a any set:

Refl 1: $\exists M \models ZFC^*$ with $a \subseteq M$
$|M| = \max(|a|, \aleph_0)$
(but not transitive)

Refl 2: $\exists M \models ZFC^*$ with $a \subseteq M$
M is transitive
(but M not small in size)
e.g $M = V_\alpha$

Refl 3: (Refl 1 + Mostowski Collapse)
If a was transitive, then
$\exists M \models ZFC^*$ with $a \subseteq M$
M transitive
$|M| = \max(|a|, \aleph_0)$

NB: e.g $a = \{\omega_1\}$ then if $a \in M$ and $|M| = \max(|a|, \aleph_0) = \aleph_0$
and M also transitive, $\omega_1 \in M \Rightarrow \omega_1 \subseteq M$ $\lightning$.

NB: True: For any fin. fragm. ZFC^* of ZFC, $ZFC \vdash$ "$\exists$ set $M \models ZFC^*$"
Not True: $ZFC \vdash$ "$\forall$ fin. fragm $ZFC^* \subseteq ZFC$, $\exists M \models ZFC^*$" ($\Rightarrow ZFC \vdash Con(ZFC)$)

Constructible Universe L (Gödel 1938)

Def: if X is a set then $Y \subseteq X$ is called "definable" if there is $\varphi(z, \ldots\ldots)$ such that $Y := \{ z \in X \mid X \models \varphi(z, a_1 \ldots a_n) \}_{a_1 \ldots a_n \in X}$

Tarski rel. because X is a set.

Def: if X is a set, then $D(X) := \{ Y \subseteq X \mid Y \text{ definable} \}$

Def:

$$L_0 = \emptyset$$
$$L_{\alpha+1} = D(L_\alpha)$$
$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \quad \text{at limits}$$
$$\mathbb{L} := \bigcup_{\alpha \in Ord} L_\alpha$$

$$V_0 = \emptyset$$
$$V_{\alpha+1} = \boxed{\mathcal{P}(V_\alpha)}$$
$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$$
$$\mathbb{V} = \bigcup_{\alpha \in Ord} V_\alpha$$

Observations:

$$L_0 = V_0 = \emptyset$$
$$L_n = V_n \quad \text{for } n \in \omega$$
$$L_\omega = V_\omega$$

$$L_{\omega+1} = ? \quad \text{vs.} \quad V_{\omega+1} = ?$$

$$|L_{\omega+1}| = |L_\omega| = \omega \qquad |V_{\omega+1}| = |\mathcal{P}(V_\omega)| = 2^{\aleph_0}$$

Lemma: $\forall \alpha \quad |L_\alpha| = |\alpha|$ (e.g. L_{ω_1} is first L-level of unctbl size)

Properties: L_α are all transitive; $\alpha < \beta \Rightarrow L_\alpha \in L_\beta$ .... etc.

Theorem: $\mathbb{L} \models ZF$

Proof: Mostly straightforward; for $\mathbb{L} \models$ Comprehension you need to use Reflection. $\square$

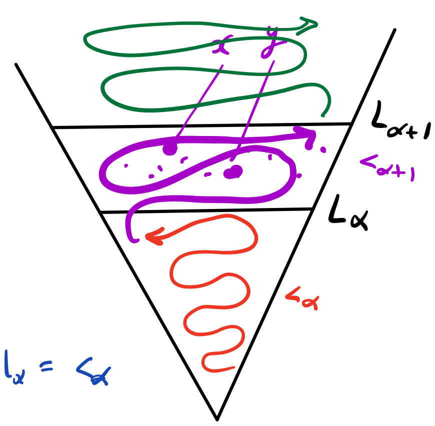
Even without assuming AC, you can define $\mathbb{L}$ and prove:

Theorem (ZF): $\mathbb{L} \models AC$

Proof: $AC \iff \forall x$ can be wellordered.

In fact, $\mathbb{L} \models$ "Global Choice" = "there is a class relation well-ordering the whole universe"

Define recursively a w.o. $<_\alpha$ of each level L_α.



- Suppose $(L_\alpha, <_\alpha)$ is w.o.
- For $L_{\alpha+1}$: take $x, y \in L_{\alpha+1}$:
  - If $x, y \in L_\alpha$, then $x <_{\alpha+1} y :\iff x <_\alpha y$ $\Big\}$ $<_{\alpha+1} \restriction L_\alpha = <_\alpha$
  - If $x \in L_\alpha$ and $y \in L_{\alpha+1} \setminus L_\alpha$: $x <_{\alpha+1} y$ (new sets come after old sets)

• If $x, y \in L_{\alpha+1} \setminus L_\alpha$: then

$x <_{\alpha+1} y \iff$ the least φ and $<_\alpha^{lex}$-least $a_1 .. a_n \in L_\alpha$ defining x

are $<_\alpha^{lex}$-less than

the least ψ and $<_\alpha^{lex}$-least $b_1 .. b_k \in L_\alpha$ defining y.

$$x = \{ z \mid L_\alpha \models \varphi(z, a_1 .. a_n) \}$$

$$y = \{ z \mid L_\alpha \models \psi(z, b_1 ... b_k) \}$$

$$\left. \begin{array}{l} \langle \ulcorner \varphi \urcorner, a_1 .. a_n \rangle \\ \langle \ulcorner \psi \urcorner, b_1 ... b_k \rangle \end{array} \right\} \text{which appears first?}$$

Let's move on to CH:

Theorem: "L_α-definition" is absolute for transitive $M \models ZFC^*$

Proof: "being definable", $D(x)$ and ordinals are all absolute. $\square$

(check.)

This means: really "$\alpha \longmapsto L_\alpha$" is absolute.

Therefore: $\alpha \in M$ then $L_\alpha \in M$

Theorem: "$\mathbb{L}$ is the minimal transitive model of ZFC"

Proof: If $M \models ZFC$, M proper class, M transitive

Then $Ord \subseteq M$. So $L_\alpha \in M$ $\forall \alpha$.

So $L_\alpha \subseteq M$ $\forall \alpha$

So $\mathbb{L} \subseteq M$

Def: Axiom of Constructibility: "$\forall x \ \exists \alpha \ (x \in L_\alpha)$"

Typically: "$V = \mathbb{L}$"

Note: $\mathbb{L} \models (V = \mathbb{L})$

Proof: $\forall x \in L \ \exists \alpha \ x \in L_\alpha$

$\Rightarrow \forall x \in L \ \exists \alpha \ (x \in L_\alpha)^{\mathbb{L}}$

$\Rightarrow (\forall x \ \exists \alpha \ (x \in L_\alpha))^L$

$\Rightarrow L \models (V = L)$ $\square$

Theorem: If M trans. prop. class $M \models ZFC +$ "$V = L$"

Then $M = L$.

Proof: $L \subseteq M$ always.

If $M \models (V=L)$

$\therefore M \models (\forall x \ \exists \alpha \ x \in L_\alpha)$

$\therefore \forall x \in M \ \exists \alpha \in M \ (x \in L_\alpha)^M$

$\therefore \quad \forall x \in M \quad \exists \alpha \quad (x \in L_\alpha)$

$\therefore \quad M \subseteq L$ $\square$

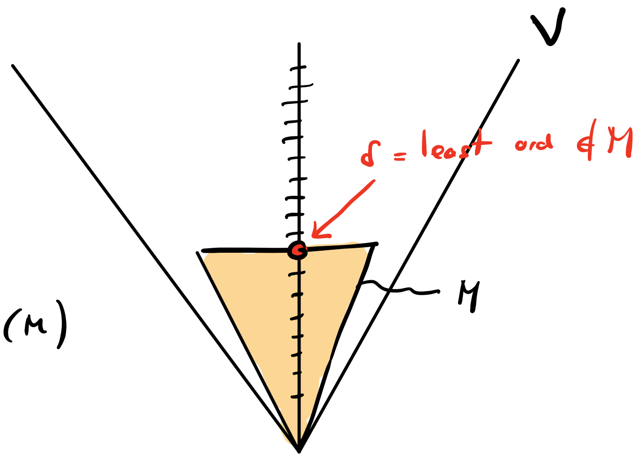
We also need set-versions of the above

Def: Let M be a transitive set model $M \models ZFC^*$.

Then the height of M is

$$o(M) = \mathrm{Ord} \cap M = \text{least ord. } \delta \notin M.$$

($M \models$ "δ is the class Ord")



Theorem: M trans. set $M \models ZFC^*$, $\delta = o(M)$

then $L_\delta \subseteq M$.

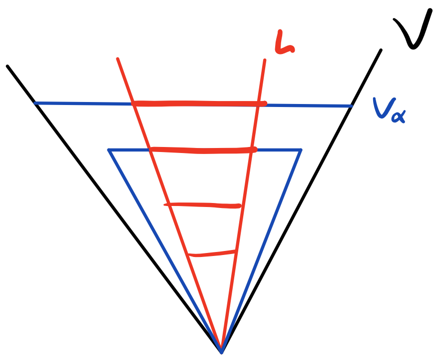
Theorem: M trans. set $M \models ZFC^* + (V=L)$, $\delta = o(M)$

then $L_\delta = M$.

Proofs exactly the same as above, but "$\forall \alpha < \delta$"

L is so minimal that the only possible

sub models of $ZFC^* + V=L$, are L_δ themselves!



Summary:

$$M \models ZFC^* \implies \begin{array}{ll} \mathbb{L} \subseteq M & (M \text{ class}) \\ L_\delta \subseteq M & (M \text{ set}) \end{array}$$

$$M \models ZFC^* + V=L \implies \begin{array}{ll} \mathbb{L} = M & (M \text{ class}) \\ L_\delta = M & (M \text{ set}) \end{array}$$

Theorem: $\mathbb{L} \models CH$

Proof: Let $x \subseteq \omega$, $x \in L$. We'll show that $x \in L_{\omega_1}$.

This is enough, because:

$$P(\omega) \subseteq L_{\omega_1}$$

$$\implies 2^{\aleph_0} = |P(\omega)| \leq |L_{\omega_1}| = \omega_1$$

Let $x \in \omega$. Let $a = \omega \cup \{x\}$ (to make it transitive).

By Reflection 3 there is M which is

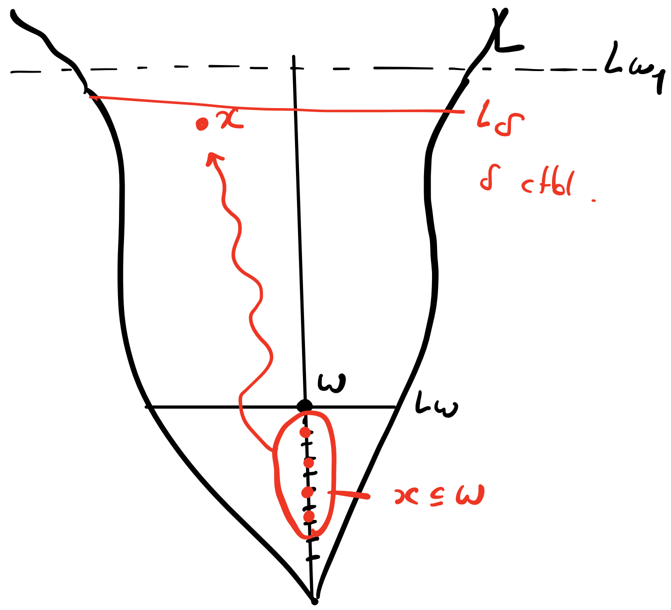
- $|M| = \max(|a|, \aleph_0) = \aleph_0$
- M transitive
- $a \subseteq M$ (because a transitive)

  $\implies x \in M$
- $M \models ZFC^* + V = L$

But we saw that $M = L_\delta$, with $\delta = o(M)$.

Since M ctbl, $o(M) < \omega_1$

So $M = L_\delta \subseteq L_{\omega_1}$. $\quad x \in M \subseteq L_{\omega_1}$. $\boxtimes$



"Condensation Lemma"

Same works for any $x \subseteq \kappa \implies x \in L_{\kappa^+}$

$$\therefore \quad \underset{\underset{2^\kappa}{\shortparallel}}{|P(\kappa)|} \leq |L_{\kappa^+}| = \kappa^+$$

So: $\quad \mathbb{L} \models GCH$

$$\boxed{Con(ZF) \rightarrow Con(ZFC + GCH)}$$