

On pseudo-Riemannian manifolds with many Killing spinors

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Abstract

Let M be a pseudo-Riemannian spin manifold of dimension n and signature s and denote by N the rank of the real spinor bundle. We prove that M is locally homogeneous if it admits more than $\frac{3}{4}N$ independent Killing spinors with the same Killing number, unless $n \equiv 1 \pmod{4}$ and $s \equiv 3 \pmod{4}$. We also prove that M is locally homogeneous if it admits k_+ independent Killing spinors with Killing number λ and k_- independent Killing spinors with Killing number $-\lambda$ such that $k_+ + k_- > \frac{3}{2}N$, unless $n \equiv s \equiv 3 \pmod{4}$. Similarly, a pseudo-Riemannian manifold with more than $\frac{3}{4}N$ independent *conformal* Killing spinors is *conformally* locally homogeneous. For (positive or negative) definite metrics, the bounds $\frac{3}{4}N$ and $\frac{3}{2}N$ in the above results can be relaxed to $\frac{1}{2}N$ and N , respectively. Furthermore, we prove that a pseudo-Riemannian spin manifold with more than $\frac{3}{4}N$ parallel spinors is flat and that $\frac{1}{4}N$ parallel spinors suffice if the metric is definite. Similarly, a Riemannian spin manifold with more than $\frac{3}{8}N$ Killing spinors with the Killing number $\lambda \in \mathbb{R}$ has constant curvature $4\lambda^2$. For Lorentzian or negative definite metrics the same is true with the bound $\frac{1}{2}N$. Finally, we give a classification of (not necessarily complete) Riemannian manifolds admitting Killing spinors, which provides an inductive construction of such manifolds.

Introduction

Figuroa-O’Farrill, Meessen and Philip showed in [FMP] that M-theory backgrounds with more than 24 supersymmetries are locally homogeneous. Notice that 24 is $3/4$ of the maximal possible number of independent supersymmetries, which is 32, the dimension of the spinor module of $\text{Spin}(1, 10)$. (Notice also that $11 \equiv 3 \not\equiv 1 \pmod{4}$.) This result is obtained from a careful analysis of the Killing spinor equations of M-theory.

In this paper, inspired by the work of Figuroa-O’Farrill et al, we study Killing spinors in pseudo-Riemannian and conformal geometry for arbitrary dimensions n and signatures s . We show that conformal Killing spinors give rise to conformal Killing polyvectors and, under some simple assumptions, that Killing spinors give rise to Killing polyvectors, see Theorem 2. More precisely, in equation (1.6), we define a $\wedge^k TM$ -valued bilinear form

$$(s, t) \mapsto [s, t]_k,$$

on the spinor bundle of a pseudo-Riemannian spin manifold (M, g) , which to a pair of conformal Killing spinors s, t associates a conformal Killing polyvector field $\omega = [s, t]_k$. For $k = 1$ we obtain conformal Killing vector fields.

Using the above correspondence, we prove that the existence of more than $3/4$ of the maximal possible number N of independent Killing spinors implies local homogeneity in the pseudo-Riemannian as well as in the conformal setting, see Theorem 3 for the precise statement. For (positive or negative) definite metrics we prove that more than $\frac{1}{2}N$ Killing spinors suffice to obtain the local homogeneity. In the pseudo-Riemannian (but not in the conformal) setting, our argument requires $n \not\equiv 1 \pmod{4}$ or $s \not\equiv 3 \pmod{4}$. Allowing imaginary “Killing numbers” $\lambda I \in \text{End } S$, where $\lambda \in \mathbb{R}$ and $I^2 = -1$, see (2.1), we can prove a similar result also in the case $n \equiv 1 \pmod{4}$, $s \equiv 3 \pmod{8}$. In the remaining case, where $n \equiv 1 \pmod{4}$ and $s \equiv 7 \pmod{8}$, our method does not allow to obtain the local homogeneity from the existence of Killing spinors with the same Killing number. Instead we have to assume the existence of k_+ Killing spinors with Killing number λ and k_- Killing spinors with Killing number $-\lambda$. If $k_+ + k_- > \frac{3}{2}N$, then we prove that the pseudo-Riemannian manifold is locally homogeneous, provided that $n \not\equiv 3 \pmod{4}$ or $s \not\equiv 3 \pmod{4}$. This covers, in particular the case $n \equiv 1 \pmod{4}$. For definite metrics the assumption can be relaxed to $k_+ + k_- > N$.

Using the correspondence between Killing spinors on (M, g) and parallel spinors on the metric cone (\hat{M}, \hat{g}) over M , see Definition 3 and Theorem 6, and our recent work [ACGL] we are able to obtain more precise information for Riemannian and Lorentzian manifolds. In fact, in Theorems 4, 8 and 10 we prove:

Theorem 1

- (i) *A pseudo-Riemannian spin manifold with more than $\frac{3}{4}N$ linearly independent parallel spinors is flat. If the metric is definite, then $\frac{1}{4}N$ parallel spinors suffice.*
- (ii) *A Riemannian spin manifold with more than $\frac{3}{8}N$ Killing spinors with the Killing number $\lambda \in \mathbb{R}$ has constant curvature $4\lambda^2$.*

(iii) A pseudo-Riemannian spin manifold with a negative definite or Lorentzian metric with more than $\frac{1}{2}N$ Killing spinors with the Killing number $\lambda \in \mathbb{R}$ has constant curvature $4\lambda^2$.

Notice that a negative definite metric g of positive scalar curvature s corresponds to a positive definite metric $-g$ of negative scalar curvature $-s$. We also prove that a Riemannian spin manifold with $\frac{3}{8}N$ Killing spinors with the Killing number $\lambda \in \mathbb{R} \setminus \{0\}$ can be locally represented in the form

$$M = I \times M_1 \times M_2, \quad g = ds^2 + \cos^2(s)g_1 + \sin^2(s)g_2,$$

where (M_1, g_1) is of constant curvature 1 or of dimension ≤ 1 , (M_2, g_2) is a seven-dimensional 3-Sasakian manifold and $I \subset (0, \frac{\pi}{2})$ is an interval, see Theorem 8.

In Theorem 9, we give a local classification of Riemannian manifolds admitting a non-trivial Killing spinor, which extends Bär's classification [B] of Killing spinors on complete Riemannian manifolds.

1 From Killing spinors to Killing polyvectors

Let (M, g) be an n -dimensional pseudo-Riemannian manifold. We will always assume that M is connected.

Definition 1 A k -vector field $\omega \in \Gamma(\wedge^k TM) \cong \Gamma(\wedge^k T^*M)$ ($k \geq 1$) is called **Killing** if

$$X \lrcorner \nabla_X \omega = 0, \quad \text{for all } X \in TM.$$

It is called **conformally Killing** if there exists a $(k-1)$ -vector field $\tilde{\omega}$ such that

$$X \lrcorner \nabla_X \omega = g(X, X)\tilde{\omega}, \quad \text{for all } X \in TM. \quad (1.1)$$

Proposition 1

(i) $\omega \in \Gamma(\wedge^k TM)$ is Killing if and only if $\dot{\gamma} \lrcorner \omega$ is a parallel $(k-1)$ -vector field along γ , for every geodesic γ :

$$\nabla_{\dot{\gamma}}(\dot{\gamma} \lrcorner \omega) = 0. \quad (1.2)$$

(ii) Let (M, g) be a pseudo-Riemannian manifold with indefinite metric g . Then ω is conformally Killing if and only if $\nabla_{\dot{\gamma}}(\dot{\gamma} \lrcorner \omega) = 0$, for every null geodesic γ .

Proof: An obvious calculation shows that a (conformal) Killing polyvector ω satisfies the equation (1.2) for every (null) geodesic γ . The converse statement in (i) is also clear, since every vector X is the velocity vector of a geodesic. To prove the converse statement in (ii), let $\eta \in \wedge^{k-1} T_p M$ and denote by β the symmetric bilinear form such that $\eta \lrcorner (X \lrcorner \nabla_X \omega) = \beta(X, X)$, for all $X \in T_p M$. By (1.2), we have $\beta(X, X) = 0$ for all X in the null cone of g . This shows that β is a multiple of g_p , since the null cone determines the indefinite scalar product g_p up to scale, and implies (1.1). \square

Remarks: 1) For $k = 1$ (i) reduces to the well know fact that the scalar product of a Killing vector field with the velocity vector of a geodesic is constant, which was observed by Clairaut for surfaces of revolution. In virtue of (ii), conformal Killing polyvectors give rise to conservation laws in general relativity. In particular, the function $g(\dot{\gamma}, Y)$ is constant along any null geodesic γ if Y is a conformal Killing vector field.

2) It is easy to see that an n -vector field ω on an n -dimensional manifold is conformally Killing if and only if it is parallel.

3) The equation (1.1) easily implies

$$\tilde{\omega} = \frac{1}{n} \operatorname{tr} \nabla \omega = \frac{1}{n} \sum g^{ij} e_i \lrcorner (\nabla_{e_j} \omega),$$

where e_i is any basis and (g^{ij}) is the matrix inverse to $g_{ij} = g(e_i, e_j)$.

Let (M, g) be a (strongly oriented) pseudo-Riemannian spin manifold and $S \rightarrow M$ its (real) spinor bundle.

Definition 2 A spinor field $s \in \Gamma(S)$ is called *Killing* with *Killing number* $\lambda \in \mathbb{R}$ if

$$\nabla_X s = \lambda X s, \quad \text{for all } X \in TM,$$

where Xs is the Clifford product of the vector X and the spinor s . It is called *conformally Killing* if there exists a spinor field $\tilde{s} \in \Gamma(S)$ such that

$$\nabla_X s = X \tilde{s}, \quad \text{for all } X \in TM. \quad (1.3)$$

Remarks: 1) Using the Clifford relation, $XY + YX = -2g(X, Y)$, the equation (1.3) easily implies

$$\tilde{s} = -\frac{1}{n} Ds, \quad (1.4)$$

where $Ds = \sum g^{ij} e_i \nabla_{e_j} s$ is the Dirac operator. In particular, any Killing spinor is an eigenspinor for the Dirac operator: $Ds = -n\lambda s$.

2) The Killing number is related to the scalar curvature by the formula $scal = 4n(n-1)\lambda^2$. Therefore, the scalar curvature of a pseudo-Riemannian manifold which admits a Killing spinor is constant and the Killing numbers of different Killing spinors on the same manifold coincide up to a sign. It is well known that a Riemannian manifold which admits a Killing spinor is Einstein, but this is no longer true for indefinite pseudo-Riemannian manifolds, see [Bo] and references therein.

We denote by $\gamma_v : S_p \rightarrow S_p$ the Clifford multiplication with $v \in T_p M$ and define a linear map $\gamma : \wedge^k T_p M \rightarrow \operatorname{End}(S_p)$, for all $k \geq 1$, by

$$\gamma_{v_1 \wedge \dots \wedge v_k} := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma) \gamma_{v_{\sigma_1}} \cdots \gamma_{v_{\sigma_k}},$$

where \mathfrak{S}_k is the symmetric group. For $\lambda \in \wedge^0 T_p M = \mathbb{R}$ we put $\gamma_\lambda = \lambda \mathbb{1} \in \operatorname{End}(S_p)$.

A bilinear form h on the spinor module satisfying

$$\begin{aligned} h(s, t) &= \sigma h(t, s), \\ h(\gamma_X s, t) &= \tau h(t, \gamma_X s), \end{aligned} \quad (1.5)$$

for all spinors s, t and all vectors X , is called **admissible of symmetry σ and type τ** , where $\sigma, \tau \in \{-1, +1\}$. The admissible bilinear forms on the spinor module were classified in [AC] and there always exists a *nondegenerate* admissible bilinear form. An admissible form is automatically invariant under the connected spin group and, hence, defines a parallel section of $S^* \otimes S^*$. In the following, h shall always denote a parallel nondegenerate section of $S^* \otimes S^*$ of symmetry σ and type τ . Notice that (1.5) implies

$$h(\gamma_\xi s, t) = \tau^k (-1)^{\frac{(k-1)k}{2}} h(s, \gamma_\xi t), \quad \text{for all } \xi \in \Gamma(\wedge^k TM).$$

Using the bilinear form h we define, for $k \geq 1$, a parallel section $[\cdot, \cdot]_k \in \Gamma(S^* \otimes S^* \otimes \wedge^k TM)$ by

$$g([s, t]_k, \xi) = h(\gamma_\xi s, t) \quad \forall \xi \in \Gamma(\wedge^k TM), s, t \in \Gamma(S). \quad (1.6)$$

(Here g is canonically extended to a nondegenerate symmetric bilinear form on the exterior algebra.) Such brackets occur in the classification of polyvector super-Poincaré algebras, see [AC, ACDV]. For $k = 0$ we put $[s, t]_0 = h(s, t)$.

Theorem 2 *Let s, t be conformal Killing spinors on an n -dimensional pseudo-Riemannian spin manifold (M, g) . Then $\omega = [s, t]_k \in \Gamma(\wedge^k TM)$ ($k \geq 1$) is a conformal Killing polyvector;*

$$X \lrcorner \nabla_X \omega = g(X, X) \tilde{\omega} \quad \forall X \in TM,$$

where $\tilde{\omega} \in \Gamma(\wedge^{k-1} TM)$ is given by

$$n\tilde{\omega} = (-1)^{k-1} [Ds, t]_{k-1} + \tau [s, Dt]_{k-1}. \quad (1.7)$$

Proof: Let (e_i) be a local frame and $\xi = X \wedge \eta$, where $X \in \Gamma(TM)$, $\eta \in \Gamma(\wedge^{k-1} TM)$ and $X \lrcorner \eta = 0$. We shall assume that, at a given point $p \in M$, $\nabla X|_p = \nabla e_i|_p = 0$ and $\nabla \eta|_p = 0$. Then we compute at p :

$$\begin{aligned} g(\nabla_X \omega, \xi) &= h(\gamma_\xi \nabla_X s, t) + h(\gamma_\xi s, \nabla_X t) \\ &= h(\gamma_\xi \gamma_X \tilde{s}, t) + h(\gamma_\xi s, \gamma_X \tilde{t}) \\ &= -g(X, X) \left((-1)^{k-1} h(\gamma_\eta \tilde{s}, t) + \tau h(\gamma_\eta s, \tilde{t}) \right) \\ &= -g(X, X) \left((-1)^{k-1} g([\tilde{s}, t]_{k-1}, \eta) + \tau g([s, \tilde{t}]_{k-1}, \eta) \right). \end{aligned}$$

This implies that ω is a conformal Killing polyvector and that

$$\tilde{\omega} = (-1)^k [\tilde{s}, t]_{k-1} - \tau [s, \tilde{t}]_{k-1}. \quad (1.8)$$

Expressing \tilde{s}, \tilde{t} by (1.4), we obtain (1.7). \square

Corollary 1 *Let s and t be Killing spinors with Killing numbers λ and μ , respectively, and $\omega = [s, t]_k$. Then the following is true.*

- (i) ω is a conformal Killing polyvector with $\tilde{\omega} = (\lambda(-1)^k - \mu\tau)[s, t]_{k-1}$.
- (ii) If $\mu = (-1)^k \tau \lambda$, then $\omega = [s, t]_k$ is a Killing polyvector.
- (iii) If $\lambda = \mu = 0$ then ω is parallel.

2 Manifolds with many Killing spinors

Theorem 3 *Let (M, g) be a pseudo-Riemannian spin manifold of dimension n , signature s and with spinor bundle S of rank N .*

- (i) *If (M, g) admits $k > \frac{3}{4}N$ conformal Killing spinors, which are linearly independent at $p \in M$, then (M, g) admits n conformal Killing vector fields, which are linearly independent at $p \in M$.*
- (ii) *Assume that $n \not\equiv 1 \pmod{4}$ or $s \not\equiv 3 \pmod{4}$. If (M, g) admits $k > \frac{3}{4}N$ Killing spinors with the same Killing number, which are linearly independent at $p \in M$, then (M, g) admits n Killing vector fields, which are linearly independent at $p \in M$.*
- (iii) *Assume that $n \equiv 1 \pmod{4}$ and $s \equiv 3 \pmod{8}$. Then S admits a parallel hypercomplex structure $J_1, J_2, J_3 = J_1 J_2 \in \Gamma(\text{End } S)$, which commutes with Clifford multiplication. Let I be any complex structure on S which is a linear combination of J_1, J_2, J_3 with constant coefficients. If (M, g) admits $k > \frac{3}{4}N$ solutions $s \in \Gamma(S)$ of the equation*

$$\nabla_X s = \lambda X I s, \quad \text{for all } X \in TM, \quad (2.1)$$
with the same $\lambda \in \mathbb{R}$, which are linearly independent at $p \in M$, then (M, g) admits n Killing vector fields, which are linearly independent at $p \in M$.
- (iv) *Assume that $n \not\equiv 3 \pmod{4}$ or $s \not\equiv 3 \pmod{4}$. If (M, g) admits k_+ Killing spinors with the Killing number λ , which are independent at p , and k_- Killing spinors with the Killing number $-\lambda$, which are independent at p , such that $k_+ + k_- > \frac{3}{2}N$, then it admits n Killing vector fields, which are independent at p .*
- (v) *If g is definite, then (i)-(iv) hold under the weaker assumptions $k > \frac{1}{2}N$ and $k_+ + k_- > N$, respectively.*

Proof: S carries a parallel nondegenerate bilinear form h of symmetry σ and type τ , see (1.5). Moreover, there exists such a form of type $\tau = -1$, unless $n \equiv 1 \pmod{4}$ and $s \equiv 3 \pmod{4}$, see [AC]. (The $\text{Pin}(n)$ -invariant scalar product on the spinor module associated with a positive definite scalar product, for instance, has $\tau = -1$.) By Theorem 2, for any pair of conformal Killing spinors s, t , the vector field $[s, t]_1$ is conformal. Similarly, by Corollary 1, if s, t are Killing spinors with the same Killing number and $\tau = -1$, then $[s, t]_1$ is a Killing vector field. Therefore, to prove (i) and (ii) it suffices to show that

$$\Pi := [\cdot, \cdot]_1|_{S_0 \otimes S_0} : S_0 \otimes S_0 \rightarrow T_p M$$

is surjective if the subspace $S_0 \subset S_p$ spanned by the values of the given (conformal) Killing spinors at p has dimension $> \frac{3}{4} \dim S_p$. Suppose first that g is definite. Then we have to show that Π is surjective if $\dim S_0 > \frac{1}{2} \dim S_p$. By the definition of Π , surjectivity is equivalent to: $\nexists v \in T_p M \setminus \{0\}$ such that $\gamma_v S_0 \subset S_0^\perp$. Suppose that there exists $v \in T_p M \setminus \{0\}$ such that $\gamma_v|_{S_0} : S_0 \rightarrow S_0^\perp$. If $\dim S_0 > \frac{1}{2} \dim S_p$, then $\dim S_0^\perp < \frac{1}{2} \dim S_p < \dim S_0$ and, thus, $\ker \gamma_v \neq 0$. Since $\gamma_v^2 = -g(v, v)\mathbb{1}$, this implies $g(v, v) = 0$ and, hence, $v = 0$. This proves the surjectivity of Π , if g is definite and $\dim S_0 > \frac{1}{2} \dim S_p$. If g is indefinite, we can only conclude that v is a null vector.

Lemma 1 For any non-zero null vector v the subspace $L_v := \ker \gamma_v = \text{im } \gamma_v \subset S_p$ is h -isotropic of dimension $\frac{1}{2} \dim S_p$.

Proof: From $\gamma_v^2 = 0$ we get $\text{im } \gamma_v \subset \ker \gamma_v$. Let u be an other null vector such that $g(u, v) = 1$. Then $\text{im } \gamma_u \subset \ker \gamma_u$ and $\gamma_u \gamma_v + \gamma_v \gamma_u = -2\mathbb{1}$ implies $\ker \gamma_v \subset \text{im } \gamma_v$ and, hence, $\ker \gamma_v = \text{im } \gamma_v$. Therefore, $\dim S_p - \dim \ker \gamma_v = \dim \text{im } \gamma_v$ implies $\dim L_v = \frac{1}{2} \dim S_p$. Let us check that L_v is isotropic. For $s, t = \gamma_v t' \in L_v = \text{im } \gamma_v$, we have

$$h(s, t) = h(s, \gamma_v t') = \tau h(\gamma_v s, t') = 0,$$

since $s \in L_v = \ker \gamma_v$. □

The lemma shows that $\text{rk } \gamma_v = \frac{1}{2} \dim S_p$ for any non-zero null vector. Now we consider the bilinear form $\beta = h(\gamma_v \cdot, \cdot)$ on S_p ; $\text{rk } \beta = \text{rk } \gamma_v = \frac{1}{2} \dim S_p$. Under the assumption $\gamma_v S_0 \subset S_0^\perp$, the matrix of β with respect to a basis adapted to a direct decomposition $S_p = S_0 \oplus S_1$ is of the form

$$\begin{pmatrix} 0 & A \\ \sigma \tau A^t & B \end{pmatrix}$$

(Notice that the symmetry of β is $\sigma \tau$.) Therefore,

$$\frac{1}{2} \dim S_p = \text{rk } \beta \leq \text{rk } A + \text{rk } (\sigma \tau A^t, B) \leq 2 \dim S_1 = 2(\dim S_p - \dim S_0),$$

which implies $\dim S_0 \leq \frac{3}{4} \dim S_p$. So $\dim S_0 > \frac{3}{4} \dim S_p$ implies $\nexists v \in T_p M \setminus \{0\} : \gamma_v S_0 \subset S_0^\perp$. This shows that $\Pi : S_0 \otimes S_0 \rightarrow T_p M$ is surjective in case (i) and (ii).

The proof of (iii) uses the fact that in that case there exist a unique (up to a constant factor) admissible parallel nondegenerate bilinear form h invariant under J_1, J_2 and J_3 , see [AC]. The form is of type $\tau = +1$. Using this form we obtain for two solutions s, t of (2.1) that $Y = \omega = [s, t]_1$ is a conformal Killing vector field, which satisfies (1.1) with

$$\tilde{\omega} = -h(\tilde{s}, t) - h(s, \tilde{t}) = -\lambda(h(Is, t) + h(s, It)) = 0,$$

as follows from (1.8). Therefore Y is a Killing vector field. The rest of the proof is similar to that of (i) and (ii).

To prove (iv) we first remark that the assumptions on the dimension and signature ensure the existence of an admissible parallel nondegenerate bilinear form h of type $\tau = +1$. Then we consider the subspaces $S_0(\lambda), S_0(-\lambda) \subset S_p$ spanned by the values at p of Killing spinors with Killing numbers λ and $-\lambda$, respectively. In virtue of Corollary 1, $[s, t]_1$ is a Killing vector field if s, t are Killing spinors with Killing numbers $\lambda, -\lambda$, respectively. Therefore, it suffices to show that $[S_0(\lambda), S_0(-\lambda)] = T_p M$. If this condition were not fulfilled, there would exist $0 \neq v \in T_p M$ such that $\gamma_v : S_0(\lambda) \rightarrow S_0(-\lambda)^\perp$. The assumption $\dim S_0(-\lambda)^\perp = N - k_- < k_+ = \dim S_0(\lambda)$ implies that $L_v = \ker \gamma_v \neq 0$. Then g is indefinite, v is a null vector and $L_v = \text{im } \gamma_v$ is maximally isotropic, by Lemma 1. In particular, $\text{rk } \gamma_v = N/2$. We can consider $\beta = h(\gamma_v \cdot, \cdot)$ as a linear map $S_p \rightarrow S_p^*$. From the matrix representation of β with respect to bases adapted to decompositions $S_p = S_0(\lambda) \oplus S_1$ and $S_p^* \cong S_0(-\lambda)^* \oplus S_1'$ we see that

$$\frac{1}{2} N = \text{rk } \beta \leq \min(k_-, N - k_+) + N - k_- = 2N - k_+ - k_-,$$

and, hence, $k_+ + k_- \leq \frac{3}{2}N$, which contradicts the assumption $k_+ + k_- > \frac{3}{2}N$. This proves $[S_0(\lambda), S_0(-\lambda)] = T_p M$. \square

Now we study the case where the bilinear form h has type $\tau = +1$ and the Killing spinors have the same Killing number.

Proposition 2 *Let h be a nondegenerate parallel bilinear form of symmetry σ and type $\tau = +1$ on the spinor bundle S of a pseudo-Riemannian spin manifold (M, g) and denote by $S(\lambda) \subset \Gamma(S)$ the vector space of Killing spinors with a given Killing number $\lambda \in \mathbb{R} \setminus \{0\}$. Then the image $[S(\lambda), S(\lambda)]_1 \subset \Gamma(TM)$ consists of Killing vector fields if and only if $S_0 := S(\lambda)|_p \subset S_p$ is totally isotropic for all $p \in M$ with respect to h . If S_0 is maximally isotropic at a point p then $[S(\lambda), S(\lambda)]_1 \neq 0$, hence, (M, g) admits a Killing vector field, which does not vanish at p .*

Proof: By Corollary 1, the bracket $\omega = [s, t]_1$ of $s, t \in S(\lambda)$ is a conformal Killing vector field with

$$\tilde{\omega} = -2\lambda h(s, t).$$

This shows that ω is a Killing vector field if and only if $h(s, t) = 0$. Assume now that $S_0 = S_0^\perp$ is maximally isotropic. By (1.6), $[S_0, S_0]_1 = 0$ is equivalent to $\gamma_v S_0 \subset S_0$ for all $v \in T_p M$, which is impossible since S_p is an irreducible module of the Clifford algebra $\mathcal{C}\ell(T_p M)$. \square

Remark: One can check that $[S_0, S_0]_1$ is one-dimensional for any maximally isotropic subspace S_0 of the spinor module $S_{2,3} = \mathbb{R}^4$ of $\text{Spin}(2, 3)$. For the spinor module $S_{4,5}$ of $\text{Spin}(4, 5)$ one can construct a maximally isotropic subspace S_0 such that $\dim[S_0, S_0]_1 = 4$. These examples show that in general a vector space of Killing spinors spanning a maximally isotropic subspace of S_p for all p is not sufficient to produce a transitive Lie algebra of Killing fields.

3 A multiplicative invariant

Let M be a pseudo-Riemannian spin manifold with real spinor bundle S of rank N and denote by $S(\lambda) = S(M, \lambda)$ the vector space of Killing spinors with Killing number $\lambda \in \mathbb{R}$. Then we put $k := \dim S(\lambda)$ and

$$\kappa(M, \lambda) := \frac{k}{N}, \quad \kappa(M) := \kappa(M, 0).$$

Notice that $\kappa(M) = 1$ if and only if M is flat and that $\kappa(M, \lambda) = \frac{\dim \mathbb{S}(\lambda)}{\text{rk } \mathbb{S}}$, where \mathbb{S} is the complex spinor bundle and $\mathbb{S}(\lambda) = \mathbb{S}(M, \lambda)$ the vector space of complex Killing spinors with Killing number λ . This follows from the fact that the complex spinor module $\mathbb{S}_{p,q}$ of $\mathcal{C}\ell_{p,q}$ is either the complexification of the real spinor module $S_{p,q}$ or coincides with $S_{p,q}$ endowed with a $\text{Pin}(p, q)$ -invariant complex structure, see [ACDV] Table 1. As a consequence, we have $\text{rk } \mathbb{S} = N$ or $N/2$, respectively.

Lemma 2 *Let $V = V_1 + V_2$ be an orthogonal decomposition of a complex Euclidian vector space of dimension n into subspaces of dimension n_1, n_2 respectively.*

- (i) If n_1 or n_2 is even, then the Clifford algebra $\mathcal{Cl}(V) \cong \mathcal{Cl}(V_1) \otimes \mathcal{Cl}(V_2)$ and the tensor product $S(V) = S(V_1) \otimes S(V_2)$ of irreducible $\mathcal{Cl}(V_1)$ -, $\mathcal{Cl}(V_2)$ -modules $S(V_1)$ and $S(V_2)$, respectively, is an irreducible $\mathcal{Cl}(V)$ -module.
- (ii) If n_1 and n_2 are odd, then $\mathcal{Cl}(V) \not\cong \mathcal{Cl}(V_1) \otimes \mathcal{Cl}(V_2)$ but $\mathcal{Cl}(V)$ is isomorphic to the $\mathbb{Z}/2\mathbb{Z}$ -graded tensor product $\mathcal{Cl}(V) \cong \mathcal{Cl}(V_1) \hat{\otimes} \mathcal{Cl}(V_2)$. In this case the spinor module of $\mathcal{Cl}^0(V)$ is obtained as the even part $(\Sigma \hat{\otimes} \Sigma')_0 = \Sigma_0 \hat{\otimes} \Sigma'_0 + \Sigma_1 \hat{\otimes} \Sigma'_1$ of the $\mathbb{Z}/2\mathbb{Z}$ -graded tensor product $\Sigma \otimes \Sigma'$ of irreducible $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathcal{Cl}(V_1)$ -, $\mathcal{Cl}(V_2)$ -modules Σ, Σ' , respectively. The $\mathcal{Cl}^0(V)$ -module $S(V) = (\Sigma \hat{\otimes} \Sigma')_0$ is a sum of non-equivalent irreducible semi-spinor submodules $S^\pm(V)$, which are the $\pm i$ -eigenspaces of a central element $\xi \in \mathcal{Cl}^1(V_1) \hat{\otimes} \mathcal{Cl}^1(V_2)$ of $\mathcal{Cl}^0(V)$.

Corollary 2 Under the assumptions of Lemma 2 the following is true.

- (i) If n_1 or n_2 is even, then as a $\text{Spin}(V_1) \times \text{Spin}(V_2)$ -module the spinor module $S(V)$ of $\text{Spin}(V)$ is isomorphic to the tensor product $S(V) \cong S(V_1) \otimes S(V_2)$.
- (ii) If n_1 and n_2 are odd, then as a $\text{Spin}(V_1) \times \text{Spin}(V_2)$ -module, $S(V) \cong 2S(V_1) \otimes S(V_2)$.

Corollary 3 Let $M = M_1 \times M_2$ be the product of two pseudo-Riemannian spin manifolds. Then $\kappa(M) = \kappa(M_1)\kappa(M_2)$. In particular, $\kappa(M) = \kappa(M_1)$ if and only if M_2 is flat.

Proof: Since $\kappa(M) = \frac{\dim \mathbb{S}(M,0)}{\text{rk } \mathbb{S}}$, the statement of Corollary 3 is obtained from Corollary 2, using that parallel spinors correspond to invariants of the holonomy group under the spinor representation and that the holonomy group of M is the product of the holonomy groups of the factors M_1, M_2 . In fact, $\mathbb{S}(M, 0) \cong \mathbb{S}(M_1, 0) \otimes \mathbb{S}(M_2, 0)$ if n_1 and n_2 are even and $\mathbb{S}(M, 0) \cong 2\mathbb{S}(M_1, 0) \otimes \mathbb{S}(M_2, 0)$ if n_1 and n_2 are odd. \square

Remark: The invariant $\kappa(M, \lambda)$ for $\lambda \neq 0$ is not multiplicative. For instance, $\kappa(S^2, \frac{1}{2}) = 1$ but $\kappa(S^2 \times S^2, \frac{1}{2}) = 0$.

4 Manifolds with many parallel spinors

Theorem 4 Let (M, g) be a pseudo-Riemannian spin manifold.

- (i) If $\kappa(M) > \frac{3}{4}$, then (M, g) is flat.
- (ii) If the metric g is definite and $\kappa(M) > \frac{1}{4}$, then (M, g) is flat. A complete simply connected Riemannian spin manifold (M, g) with $\kappa(M) = \frac{1}{4}$ is the product of a flat manifold and a manifold with holonomy group $\text{SU}(2)$.

Proof: (i) follows from Theorem 3 (i) and (v), since the conformal Killing vector fields $[s, t]_1$ are parallel if s, t are parallel spinors, see Corollary 1. Next we prove (ii). It follows from Wang's classification of parallel spinors on manifolds

with connected irreducible holonomy group [W] that a locally irreducible Riemannian manifold (M, g) has $\kappa(M) \leq \frac{1}{4}$ and $\kappa(M) = \frac{1}{4}$ implies that M has holonomy algebra $\mathfrak{h} = \mathfrak{su}(2)$. Applying the (local) de Rham decomposition and Corollary 3, we conclude that a Riemannian manifold with $\kappa(M) > \frac{1}{4}$ is flat and that a complete simply connected Riemannian manifold with $\kappa(M) = \frac{1}{4}$ is the product $M = M_0 \times M_1$ of a flat manifold M_0 and an irreducible manifold M_1 with $\kappa(M_1) = \frac{1}{4}$ and holonomy group $SU(2)$. \square

Theorem 5 *Let (\hat{M}, \hat{g}) be the Lorentzian cone over a pseudo-Riemannian manifold (M, g) with either negative definite metric or of signature $(+, \dots, +, -)$. If $\kappa(\hat{M}) > \frac{1}{2}$, then \hat{M} is flat and M has constant curvature 1.*

Proof: If (\hat{M}, \hat{g}) is not flat, we can decompose it locally as a product of indecomposable pseudo-Riemannian manifolds. By Corollary 3, there exists an indecomposable factor M_1 of dimension > 1 with $\kappa(M_1) > \frac{1}{2}$. It cannot be Riemannian, by the previous theorem. Hence it is a Lorentzian indecomposable manifold. By [ACGL] Theorem 4.1, $M_1 = \hat{N}_1$ is (locally) a cone over a pseudo-Riemannian manifold N_1 . Moreover, by [ACGL] Theorem 9.1, the local holonomy algebra $\hat{\mathfrak{h}}$ of M_1 contains the subalgebra $\mathfrak{e} := p \wedge E$, where $T_x M_1 = V = \mathbb{R}p + \mathbb{R}q + E$, p, q are isotropic vectors with $\hat{g}(p, q) = 1$ and E is the positive definite orthogonal complement of $\text{span}\{p, q\}$. The Clifford algebra has the decomposition $\mathcal{C}\ell(V) = \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell(E)$. The Clifford algebra $\mathcal{C}\ell_{1,1}$ is the full matrix algebra of real 2×2 matrices and is generated by

$$\gamma_p = \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma_q = -\sqrt{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with respect to the standard basis (e_1, e_2) of \mathbb{R}^2 . If S_E is an irreducible $\mathcal{C}\ell(E)$ -module then $S_V = \mathbb{R}^2 \otimes S_E$ is an irreducible $\mathcal{C}\ell(V)$ -module. Under the isomorphism $\mathcal{C}\ell(V) \cong \mathcal{C}\ell_{1,1} \otimes \mathcal{C}\ell(E)$ a vector $v = f \oplus e \in \mathbb{R}^{1,1} \oplus E$ is mapped to $f \otimes 1 + \nu \otimes e$, where $\nu = \frac{1}{2}(p+q)(p-q)$ is the volume element in $\mathcal{C}\ell_{1,1}$, which satisfies $\nu^2 = 1$, $\nu e_1 = e_1$, $\nu e_2 = -e_2$.

Lemma 3 *The space of \mathfrak{e} -invariant spinors is given by*

$$S_V^{\mathfrak{e}} = e_1 \otimes S_E \subset S_V = \mathbb{R}^2 \otimes S_E.$$

Proof: A spinor $s = e_1 \otimes s_1 + e_2 \otimes s_2 \in S_V$ is invariant under $\mathfrak{e} \subset \hat{\mathfrak{h}}$ if and only if

$$0 = \gamma_{p \wedge e} s = -\sqrt{2} e_1 \otimes \gamma_e s_2,$$

for all $e \in E$, which is equivalent to $s_2 = 0$. \square

The lemma shows that $\dim S_V^{\hat{\mathfrak{h}}} \leq \dim S_V^{\mathfrak{e}} = \frac{1}{2} \dim S_V$ and, hence, $\kappa(M_1) \leq \frac{1}{2}$, which contradicts the assumption. \square

5 Cones \hat{M} over pseudo-Riemannian manifolds M and relation between Killing spinors on M and parallel spinors on \hat{M}

Definition 3 Let (M, g) be a pseudo-Riemannian manifold of signature (p, q) . The manifold $\hat{M} = \mathbb{R}^+ \times M$ endowed with the pseudo-Riemannian metric $\hat{g} = dr^2 + r^2g$ of signature $(p+1, q)$ is called the cone over (M, g) .

Recall that a spin structure (in the strong sense) on (M, g) is a $\text{Spin}_0(p, q)$ -equivariant two-fold covering $P_{\text{Spin}_0(p, q)}(M) \rightarrow P_{SO_0(p, q)}(M)$ of the principal bundle of strongly oriented orthonormal frames. Let us denote by $P_{\text{Spin}_0(p+1, q)}(M) \supset P_{\text{Spin}_0(p, q)}(M)$ and $P_{SO_0(p+1, q)}(M) \supset P_{SO_0(p, q)}(M)$ the $\text{Spin}_0(p+1, q)$ - and $SO_0(p+1, q)$ -principal bundles obtained by enlarging the structure groups. Then $P_{\text{Spin}_0(p, q)}(M) \rightarrow P_{SO_0(p, q)}(M)$ extends naturally to a $\text{Spin}_0(p+1, q)$ -equivariant two-fold covering

$$\Theta : P_{\text{Spin}_0(p+1, q)}(M) \rightarrow P_{SO_0(p+1, q)}(M).$$

Using the isometric inclusion $M \cong \{1\} \times M \subset \hat{M} = \mathbb{R}^+ \times M$, we can identify $P_{SO_0(p+1, q)}(M)$ with the restriction $P_{SO_0(p+1, q)}(\hat{M})|_M$ of the bundle of strongly oriented orthonormal frames of \hat{M} . In particular, the frame $(e_1, \dots, e_n) \in P_{SO_0(p, q)}(M) \subset P_{SO_0(p+1, q)}(M)$ is mapped to the frame $(\partial_r, e_1, \dots, e_n) \in P_{SO_0(p+1, q)}(\hat{M})$ under this identification. Similarly, we identify $P_{SO_0(p+1, q)}(\hat{M})$ with the pullback of $P_{SO_0(p+1, q)}(M)$ via the projection $\pi : \hat{M} \rightarrow M$. Under this identification $(\partial_r, e_1, \dots, e_n) \in P_{SO_0(p+1, q)}(\hat{M})_{(r, x)}$ is mapped to $(re_1, \dots, re_n) \in P_{SO_0(p, q)}(M)_x \subset P_{SO_0(p+1, q)}(M)_x$ for all $x \in M$. Then

$$P_{\text{Spin}_0(p+1, q)}(\hat{M}) := \pi^* P_{\text{Spin}_0(p+1, q)}(M) \rightarrow \pi^* P_{SO_0(p+1, q)}(M) = P_{SO_0(p+1, q)}(\hat{M})$$

defines a spin structure on \hat{M} .

Lemma 4 Let (\hat{M}, \hat{g}) be the cone over a pseudo-Riemannian spin manifold (M, g) of signature (p, q) .

(i) If $s = p - q \equiv 0, 2, 4, 5$ or $6 \pmod{8}$, then the spinor bundle \hat{S} of \hat{M} is related to the spinor bundle S of M by a canonical isomorphism

$$\hat{S}|_M \cong S.$$

(ii) If $s = p - q \equiv 1, 3$ or $7 \pmod{8}$, then the semi-spinor bundles \hat{S}^\pm of \hat{M} are related to the spinor bundle of M by canonical isomorphisms

$$\hat{S}^\pm|_M \cong S.$$

(iii) If $n = \dim M = p + q$ is even, then the complex spinor bundles $\mathbb{S}, \hat{\mathbb{S}}$ of M and \hat{M} , respectively, are related by a canonical isomorphism

$$\hat{\mathbb{S}}|_M \cong \mathbb{S}.$$

(iv) If n is odd, then the complex semi-spinor bundles $\hat{\mathbb{S}}^\pm$ of \hat{M} are related to the spinor bundle \mathbb{S} of M by canonical isomorphisms

$$\hat{\mathbb{S}}^\pm|_M \cong \mathbb{S}.$$

Proof: Let (e_0, \dots, e_n) be an orthonormal basis of $\mathbb{R}^{p+1,q}$. Recall that by definition

$$\text{Spin}(p, q) \subset \text{Spin}(p+1, q) \subset \mathcal{C}\ell_{p+1,q}^0 = \langle e_i e_j | i, j = 0, \dots, n \rangle.$$

The even part $\mathcal{C}\ell_{p+1,q}^0$ of the Clifford algebra $\mathcal{C}\ell_{p+1,q}$ is mapped isomorphically onto $\mathcal{C}\ell_{p,q}$ by

$$\begin{aligned} e_i e_j &\mapsto e_i e_j, \\ e_i e_0 &\mapsto e_i, \quad i, j = 1, \dots, n. \end{aligned}$$

Using this isomorphism $\mathcal{C}\ell_{p+1,q}^0 \cong \mathcal{C}\ell_{p,q}$ the spinor module $S_{p,q}$ of $\text{Spin}(p, q)$ can be extended to an irreducible $\text{Spin}(p+1, q)$ -module. In fact, $S_{p,q}$ is the restriction of an irreducible $\mathcal{C}\ell_{p,q}$ -module to $\text{Spin}(p, q) \subset \mathcal{C}\ell_{p,q}$. Restricting this $\mathcal{C}\ell_{p,q}$ -module to $\text{Spin}(p+1, q) \subset \mathcal{C}\ell_{p+1,q}^0 \cong \mathcal{C}\ell_{p,q}$ gives the desired $\text{Spin}(p+1, q)$ -module. The $\text{Spin}(p+1, q)$ -module $S_{p,q}$ is equivalent to the spinor module $S_{p+1,q}$ if the spinor module $S_{p+1,q}$ is irreducible, which is the case if $s = p - q \equiv 0, 2, 4, 5$ or $6 \pmod{8}$, see [AC] Prop. 1.3. Otherwise it is equivalent to one of the semi-spinor modules $S_{p+1,q}^\pm$. The semi-spinor modules $S_{p+1,q}^+$ and $S_{p+1,q}^-$ are always equivalent as $\text{Spin}(p, q)$ -modules (and even as $\text{Spin}(p+1, q)$ -modules if $s \equiv 1 \pmod{8}$). This implies (i) and (ii). The proof of (iii) and (iv) is similar. \square

Notice that for $s = p - q \equiv 5 \pmod{8}$ the spinor module $S_{p+1,q}$ is irreducible and admits a $\text{Spin}(p+1, q)$ -invariant complex structure, see [AC] Prop. 1.3. Its complexification is isomorphic to the complex spinor module of $\text{Spin}(p+1, q)$ (see [ACDV] Table 1), which is a sum of two semi-spinor modules.

For $\Sigma \in \{S, \mathbb{S}, \hat{S}, \hat{\mathbb{S}}, \hat{S}^\pm, \hat{\mathbb{S}}^\pm\}$, let us denote by $\Sigma(\lambda)$ the vector space of Killing spinors $s \in \Gamma(\Sigma)$ with Killing number $\lambda \in \mathbb{R}$.

Notice that if $\lambda \neq 0$ one can always normalise the metric such that $\lambda = \pm \frac{1}{2}$ (as for a space of constant curvature 1). Now let $\Sigma = S$ or \mathbb{S} . Multiplication by the volume element $\nu = e_1 \cdots e_n \in \mathcal{C}\ell(TM)$ maps $\Sigma(\lambda)$ to $\Sigma((-1)^{n+1}\lambda)$. In particular, it defines isomorphisms $\Sigma(\lambda) \cong \Sigma(-\lambda)$, if n is even. For odd dimensional manifolds, however, the vector spaces $\Sigma(\lambda)$ and $\Sigma(-\lambda)$ have in general different dimensions.

Using Lemma 4, the following theorem can be proven as for Riemannian manifolds, see Bär [B].

Theorem 6 *Let (\hat{M}, \hat{g}) be the cone over a pseudo-Riemannian spin manifold (M, g) of signature (p, q) .*

(i) *The restriction $\Gamma(\hat{S}) \ni s \mapsto s|_M \in \Gamma(S)$ defines isomorphisms*

$$\hat{S}(0) \rightarrow S\left(\frac{1}{2}\right) \cong S\left(-\frac{1}{2}\right),$$

if $s = p - q \equiv 0, 2, 4, 5$ or $6 \pmod{8}$ and

$$\hat{S}^\pm(0) \rightarrow S\left(\pm\frac{\epsilon}{2}\right),$$

for some $\epsilon \in \{1, -1\}$, if $s = p - q \equiv 1, 3$ or $7 \pmod{8}$.

(ii) The restriction $\Gamma(\hat{\mathbb{S}}) \ni s \mapsto s|_M \in \Gamma(\mathbb{S})$ defines isomorphisms

$$\hat{\mathbb{S}}(0) \rightarrow \mathbb{S}\left(\frac{1}{2}\right) \cong \mathbb{S}\left(-\frac{1}{2}\right),$$

if $n = \dim M$ is even and

$$\hat{\mathbb{S}}^\pm(0) \rightarrow \mathbb{S}\left(\pm\frac{\epsilon}{2}\right),$$

for some $\epsilon \in \{1, -1\}$, if n is odd.

6 Riemannian manifolds with many Killing spinors

Theorem 7 *Let (M, g) be a simply connected Riemannian spin manifold.*

(i) *Assume that one of the following conditions is satisfied:*

a) *(M, g) is complete and not of constant curvature 1.*

b) *The holonomy algebra of M is different from $\mathfrak{so}(n)$, where $n = \dim M$.*

Then the holonomy algebra $\hat{\mathfrak{h}}$ of the cone (\hat{M}, \hat{g}) is irreducible.

(ii) *If $\hat{\mathfrak{h}}$ is irreducible, then (\hat{M}, \hat{g}) admits a parallel spinor if and only if $\hat{\mathfrak{h}}$ belongs to the following list: $\mathfrak{su}(m)$ ($m \geq 3, k = 2$), $\mathfrak{sp}(m)$ ($m \geq 2, k = m + 1$), $\mathfrak{spin}(7)$ ($k = 1$) or \mathfrak{g}_2 ($k = 1$), where k in brackets indicates the number of linearly independent parallel complex spinors. The projection of the space of parallel complex spinors onto the space of parallel complex semi-spinors is zero for one of the two semi-spinor bundles, unless $n + 1 \not\equiv 0 \pmod{4}$.*

Proof: The irreducibility of the holonomy algebra follows from Gallot's theorem [G] under the assumption a) and from [ACGL] Theorem 4.1 under the assumption b). The remaining statements follow from Wang's classification of parallel spinors on manifolds with connected irreducible holonomy group [W] and the observation that there is no cone with holonomy group $SU(2) = Sp(1)$. \square

Theorem 8 *Let (M, g) be a Riemannian spin manifold which is not of constant positive curvature $4\lambda^2$.*

(i) *Then $\kappa(M, \lambda) \leq \frac{3}{8}$.*

- (ii) Assume that for every $p \in M$ we have $\kappa(U, \lambda) = \frac{3}{8}$ for every sufficiently small open neighborhood $U \subset M$ of p . Then either (M, g) is locally isometric to seven-dimensional 3-Sasakian manifold or there exists a dense open subset $M' \subset M$ such that every point of M' has a neighborhood isometric to a Riemannian manifold of the form

$$(I \times M_1 \times M_2, ds^2 + \cos^2(s)g_1 + \sin^2(s)g_2),$$

where (M_1, g_1) is of constant curvature 1 or of dimension ≤ 1 and (M_2, g_2) is a seven-dimensional 3-Sasakian manifold.

Proof: We use the correspondence between Killing spinors on M and parallel spinors on the cone \hat{M} . If $\kappa(M, \lambda) > \frac{3}{8}$, then, according to Theorem 6, $\kappa(\hat{M}) > \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$. This is impossible if the holonomy algebra \hat{h} of \hat{M} is irreducible, due to Theorem 7 (ii). The maximal value $\kappa(\hat{M}) = \frac{3}{16}$ is, in fact, attained for the holonomy algebra $\hat{h} = \mathfrak{sp}(2)$ (since there is no cone with holonomy $\mathfrak{su}(2)$). The cone \hat{M} has local holonomy $\mathfrak{sp}(2)$ if and only if the seven-dimensional manifold M is locally 3-Sasakian. This proves (i) and (ii) if \hat{h} is irreducible. In the reducible case, the claims (i) and (ii) now follow from [ACGL] Theorem 4.1 using Corollary 3. \square

Recall [B] that the holonomy algebra \hat{h} of the cone (\hat{M}, \hat{g}) over a simply connected Riemannian manifold (M, g) belongs to the list of irreducible linear Lie algebras described in Theorem 7 (ii) if and only if (M, g) is Einstein-Sasaki, 3-Sasakian, strictly nearly parallel G_2 or strictly nearly Kähler, respectively. We will call these geometric structures on (M, g) Bär geometries.

Theorem 9 *Let (M, g) be an n -dimensional Riemannian spin manifold which admits a nontrivial Killing spinor with Killing constant $\lambda \in \mathbb{R}$.*

- (i) *If $\lambda = 0$, then (M, g) is locally a product $M = M_0 \times M_1 \times \cdots \times M_r$ of a flat Riemannian manifold M_0 with an arbitrary number of Riemannian manifolds M_i with irreducible holonomy group from the following list: $SU(m)$, $Sp(m)$, $Spin(7)$ or G_2 .*
- (ii) *If $\lambda \neq 0$, then (M, g) has holonomy $\mathfrak{h} = \mathfrak{so}(n)$. Moreover, if the cone \hat{M} is locally irreducible, then (M, g) carries locally one of the Bär geometries and if \hat{M} is locally reducible, then, on a dense open subset, (M, g) can be locally represented in the form*

$$M = I \times M_1 \times M_2, \quad g = ds^2 + \cos^2(s)g_1 + \sin^2(s)g_2, \quad (6.1)$$

where $I \subset (0, \frac{\pi}{2})$ is an interval and (M_1, g_1) and (M_2, g_2) are Riemannian manifolds which either admit a nontrivial Killing spinor with Killing constant $\pm\lambda$ or which are of dimension ≤ 1 .

Remark: Notice that the Theorem 9 (ii) gives an inductive decomposition of a manifold with a nontrivial Killing spinor with $\lambda \neq 0$ in terms of an arbitrary number of manifolds (M_i, g_i) , which carry each one of the Bär geometries or are of dimension ≤ 1 . We remark also that a manifold (M, g) of constant curvature 1 can be locally decomposed as (6.1) with

manifolds (M_1, g_1) and (M_2, g_2) which are either of constant curvature 1 or of dimension ≤ 1 .

Proof: (i) is an immediate consequence of Wang's classification of irreducible connected holonomy groups preserving a non-trivial spinor [W].

In the case (ii), it follows from Bär's classification [B] that the cone (\hat{M}, \hat{g}) over (M, g) is locally irreducible if and only if (M, g) carries locally one of the Bär geometries. We check, in this case, that M is locally irreducible. We consider the modified covariant derivative $\tilde{\nabla}_X := \nabla_X - \lambda\gamma_X$, $X \in T_xM$, on the complex spinor bundle over M , where ∇ stands for the Levi-Civita connection. Assume that $M = M_1 \times M_2$ is a Riemannian product. Then we compute the curvature of $\tilde{\nabla}$ at $x \in M$:

$$\tilde{R}(X_1, X_2)_x = \lambda[\gamma_{X_1}, \gamma_{X_2}] = 2\lambda\gamma_{X_1}\gamma_{X_2},$$

for $X_i \in TM$ tangent to M_i and such that $\nabla X_i|_x = 0$, $i = 1, 2$. This implies that the local holonomy $\tilde{\mathfrak{h}}$ algebra of $\tilde{\nabla}$ contains $\mathfrak{spin}(n)$, because the holonomy algebra at x contains all curvature operators $\tilde{R}(X_1, X_2)_x$ and the Clifford products X_1X_2 generate $\mathfrak{spin}(n)$ (as a Lie algebra). Since, by [B], $\tilde{\mathfrak{h}}$ can be identified with the local holonomy algebra $\hat{\mathfrak{h}}$ of the Levi-Civita connection of the cone \hat{M} , we can conclude that $\hat{\mathfrak{h}}$ contains the subalgebra $\mathfrak{so}(n) \subset \mathfrak{so}(n+1)$. One can easily check that this is not possible for $\hat{\mathfrak{h}}$ belonging to the list of irreducible holonomy algebras of Riemannian cones admitting a parallel spinor, see Theorem 7 (ii). This shows that (M, g) is locally irreducible if (\hat{M}, \hat{g}) is locally irreducible. In particular, \mathfrak{h} belongs to Berger's list of irreducible holonomy algebras, excluding the Ricci-flat holonomies (but so far including the holonomies of irreducible symmetric spaces). In dimension $n = 7$ this already implies $\mathfrak{h} = \mathfrak{so}(7)$. In dimension $n = 6$ this implies $\mathfrak{h} = \mathfrak{so}(6)$, using that a strict nearly Kähler manifold cannot be Kähler. In the remaining cases (M, g) is locally Sasaki-Einstein (or even 3-Sasakian). The curvature tensor of such a manifold satisfies

$$R(\xi, X)Y = \xi g(X, Y) - Xg(\xi, Y)$$

for all vector fields X, Y on M , where ξ is the Sasaki vector field. This identity immediately implies that $\mathfrak{h} = \mathfrak{so}(n)$, since \mathfrak{h} contains all curvature operators and their brackets.

If the cone (\hat{M}, \hat{g}) is locally reducible, then it follows from [ACGL] Theorem 4.1 that $\mathfrak{h} = \mathfrak{so}(n)$ and that, on a dense open subset of M , (M, g) is locally isometric to (6.1). \square

7 Pseudo-Riemannian manifolds with Lorentzian cone, which admit many Killing spinors

Theorem 10 *Let (M, g) be spin with either a negative definite metric or a metric of Lorentzian signature $(+, \dots, +, -)$. If (M, g) is not of positive constant curvature $4\lambda^2$, then $\kappa(M, \lambda) \leq \frac{1}{2}$.*

Proof: The spinor module S_V of $\text{Spin}(V)$, $V = T_x\hat{M}$, is either irreducible or it splits as

$$S_V = S_V^+ \oplus S_V^-, \quad S_V^\pm = e_1 \otimes S_E^\pm + e_2 \otimes S_E^\mp,$$

where S_E^\pm are the semi-spinor modules of $\text{Spin}(E)$ and we use the notation of Lemma 3. As in the proof of Theorem 5, we can assume that the cone (\hat{M}, \hat{g}) is indecomposable. If the cone is Riemannian, we have $\kappa(M, \lambda) \leq \frac{3}{8}$, by Theorem 8. Therefore, we can assume that it is Lorentzian. In that case the holonomy algebra contains $\mathfrak{e} = p \wedge E$, by [ACGL] Theorem 9.1. Then $(S_V)^\mathfrak{e} = e_1 \otimes S_E$ if S_V is irreducible and $(S_V)^\mathfrak{e} = e_1 \otimes S_E^\pm$ otherwise. This shows that $\dim \hat{S}(0) = \frac{1}{2} \text{rk} \hat{S}$, which implies $\dim S(\lambda) \leq \frac{1}{2} N$, by Theorem 6. \square

Remark that a pseudo-Riemannian manifold (M, g) of dimension n which admits a Killing spinor with (real) Killing number $\lambda \in \mathbb{R} \setminus \{0\}$ has positive scalar curvature $s = 4n(n-1)\lambda^2$. If g is negative definite of scalar curvature $s > 0$, then the Riemannian metric $-g$ has negative scalar curvature $-s$. This allows to treat also Riemannian manifolds with negative scalar curvature.

References

- [AC] D.V. Alekseevsky and V. Cortés, *Classification of N -(super)-extended Poincaré algebras and bilinear invariants of the spinor representation of $\text{Spin}(p, q)$* , Comm. Math. Phys. **183** (1997), no. 3, 477-510.
- [ACDV] D.V. Alekseevsky, V. Cortés, C. Devchand and A. Van Proeyen, *Polyvector super-Poincaré algebras*, Comm. Math. Phys. **253** (2005), no. 2, 385-422.
- [ACGL] D.V. Alekseevsky, V. Cortés, A.S. Galaev and T. Leistner, *Cones over pseudo-Riemannian manifolds and their holonomy*, to appear in Journal für die reine und angewandte Mathematik, arXiv:0707.3063.
- [B] C. Bär, *Real Killing spinors and holonomy*, Comm. Math. Phys. **154** (1993), no. 3, 509-521.
- [Bo] C. Bohle, *Killing and Twistor spinors on Lorentzian manifolds*, Diplomarbeit, Freie Universität Berlin, 1998.
- [FMP] J. Figueroa-O'Farrill, P. Meessen, S. Philip, *Supersymmetry and homogeneity of M -theory backgrounds*, Classical Quantum Gravity **22** (2005), no. 1, 207-226.
- [G] S. Gallot, *Équations différentielles caractéristiques de la sphère*, Ann. Sci. École Norm. Sup. (4) **12** (1979), no. 2, 235-267.
- [W] M.Y. Wang, *Parallel spinors and parallel forms*, Ann. Global Anal. Geom. **7** (1989), no. 1, 59-68.