REPRESENTATION AND DIMENSION OF SPARSE RANDOM GRAPHS

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Abstract. Recently Guo, Patton, and Warnke [*Prague dimension of random graphs*, Combinatorica, 2023] established a conjecture of Füredi and Kantor by determining the product dimension of the binomial random graph $G(n, p)$ with constant edge probability p. We consider the sparse case when $p = p(n) \longrightarrow 0$ as $n \longrightarrow \infty$.

§1. Introduction

1.1. **Graph representations.** We consider graph representations that can be traced back to the work of Szpilrajn-Marczewski [19]. A *set representation* $\mathcal{R} = \mathcal{R}(G, T)$ of a graph $G = (V, E)$ on a set T is a collection of subsets of T labeled by vertices of G such that the edge set of G is represented by the pairs of sets with nonempty intersection, i.e.,

$$\mathcal{R} = \{S_v \subseteq T \colon v \in V\} \quad \text{such that} \quad uv \in E \iff u \neq v \text{ and } S_u \cap S_v \neq \varnothing\,.$$

Moreover, if $|T| = t$, then we say G is *t-representable*.

It is easy to see that every graph $G = (V, E)$ is representable on its edge set E by the collection $S_v = \{e \in E \colon v \in e\}$ for $v \in V$. Consequently, the *set representation number*

$$\theta_1(G) = \min\{t \in \mathbb{N} \colon G \text{ is } t\text{-representable}\}$$

is well-defined for finite graphs.* Erdős, Goodman, and Pósa [7] established the optimal general upper bound $\theta_1(G) \leqslant \lfloor n^2/4 \rfloor$ for every n-vertex graph G, which is attained by balanced complete bipartite graphs. Moreover, these authors showed $\theta_1(G)$ equals the minimum number of cliques needed to cover the edges of G.

Here it will be more convenient to consider set representations of the graph complement $\overline{G}$ of G and we set

$$\bar{\theta}_1(G) = \theta_1(\overline{G})\,.$$

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*The 1 in the subscript of θ_1 indicates that for signifying the edges of G, it suffices that the representing sets intersect in at least one element.

In other words, for a graph $G = (V, E)$ we have $\bar{\theta}_1(G) \leqslant t$ if there are independent sets $I_1, \ldots, I_t \subseteq V$ in G such that

$$E = V^{(2)} \setminus \bigcup_{i=1}^{t} I_i^{(2)}\,, \tag{1}$$

where for a set X we denote by $X^{(2)}$ the set of all 2-element subsets of X. Moreover, we have $\bar{\theta}_1(G) = t$, if in addition there is not such a family of independent sets with at most $t-1$ members. Alternatively, in this context one may think of set representations, where edges are given by disjoint pairs of sets and non-edges are represented by pairs of intersecting sets.

Alon [1] proved a general upper bound for graphs of bounded maximum degree and showed

$$\bar{\theta}_1(G) \leqslant O(\Delta^2 \log(n))$$

for all n-vertex graphs G with bounded maximum degree $\Delta(G) \leqslant \Delta$. In [6], it was shown that this upper bound is sharp up to a $O(\log \Delta)$-factor by providing, for every $\Delta \geqslant 1$, a graph G with $\Delta(G) \leqslant \Delta$ and

$$\bar{\theta}_1(G) \geqslant c\frac{\Delta^2 \log(n)}{\log(\Delta)}$$

for some universal constant $c > 0$.[†]

We study $\bar{\theta}_1$ for the binomial random graph $G(n,p)$ on n vertices where each of the $\binom{n}{2}$ edges occurs, independently, with probability p. For simplicity, we shall assume that $G(n,p)$ and all other n-vertex graphs have $[n] = \{1, \ldots, n\}$ as its vertex set. This line of research was started by Bollobás, Erdős, Spencer, and West [5]. Further work of Frieze and Reed [9] and of Guo, Patton, and Warnke [12] yields that for every fixed $p \in (0,1)$, asymptotically almost surely (a.a.s., with probability tending to 1 as $n \longrightarrow \infty$) the random graph $G \in G(n,p)$ satisfies

$$\bar{\theta}_1(G) = \Theta\left(\frac{n^2}{\log^2(n)}\right). \tag{2}$$

We investigate $\bar{\theta}_1$ for sparse random graph $G(n,p)$, i.e., when $p = p(n) \longrightarrow 0$ as $n \longrightarrow \infty$. Our first result establishes an upper bound for $\bar{\theta}_1$. We use the asymptotic short-hand notation $a(n) \ll b(n)$ for the validity of the statement $|a(n)| = o(b(n))$.

Theorem 1.1. *For every $p = p(n)$ with $\log^{5/4}(n)/\sqrt{n} \ll p(n) \leqslant 1/\log^2(n)$, the random graph $G \in G(n,p)$ a.a.s. satisfies*

$$\bar{\theta}_1(G) \leqslant 200\,\frac{p^2 n^2}{\log(n)}\,. \tag{3}$$

[†]The base of the logarithms in this paper is e.

The following matching lower bound was obtained by Guo et al. [12, Lemma 20] and, for completeness, we include the short proof in §2.4.

Proposition 1.2. *For every $p = p(n)$ with $1/n \ll p(n) \leqslant 1/\log^2(n)$, a.a.s. the random graph $G \in G(n,p)$ satisfies*

$$\bar{\theta}_1(G(n,p)) > \frac{-\log(p) \cdot p^2 n^2}{10 \log(pn) \log(n)}\,.$$

Combining Theorem 1.1 with Proposition 1.2 shows that for every $\delta > 0$ and $p = p(n)$ with $\log^{5/4}(n)/\sqrt{n} \ll p(n) \leqslant n^{-\delta}$, a.a.s. we have

$$\bar{\theta}_1(G) = \Theta\bigg(\frac{p^2 n^2}{\log(n)}\bigg)$$

for $G \in G(n,p)$. We shall prove Theorem 1.1 and Proposition 1.2 in Section 2.

1.2. **Graph dimensions.** Representing the edge set of a graph as the complement of independent sets as defined in (1) is closely related to the *product dimension* of a graph $G = (V, E)$. We recall that the product dimension $\mathrm{pdim}(G)$ is defined to be the minimum integer s such that G is an induced subgraph of the product of s cliques. Here the product $F \times F'$ of two graphs $F = (V, E)$ and $F' = (V', E')$ is given by

$$V(F \times F') = V \times V' \quad \text{and} \quad \{(u,u'),(v,v')\} \in E(F \times F') \iff uv \in E \text{ and } u'v' \in E'\,.$$

The product dimension was studied by several authors (see, e.g., [2, 6, 10, 12, 15, 17, 18] and the references therein). In particular, it was shown in [17] that it is closely related to the following notion denoted by $\mathrm{idim}(G)$, which is defined as follows: $\mathrm{idim}(G)$ is the smallest integer t such that there are t partitions $\mathcal{I}_1, \dots, \mathcal{I}_s$ of $V(G)$ so that $E(G)$ is the intersection of the edge set of the s complete multipartite graphs defined on the vertex partitions $\mathcal{I}_1, \dots, \mathcal{I}_s$. In other words, a graph $G = (V, E)$ satisfies $\mathrm{idim}(G) = s$ if there exist s partitions of V

$$\mathcal{I}_1 = (I_{1,1}, \dots, I_{1,r_1})\,, \ \dots\ , \ \mathcal{I}_s = (I_{s,1}, \dots, I_{s,r_s})$$

such that

$$E = V^{(2)} \smallsetminus \bigcup_{i=1}^{s} \bigcup_{j=1}^{r_i} I_{i,j}^{(2)} \tag{4}$$

and (4) fails for all integers smaller than s. The aforementioned connection to the product dimension from [17] states

$$\mathrm{idim}(G) \leqslant \mathrm{pdim}(G) \leqslant \mathrm{idim}(G) + 1 \tag{5}$$

for every graph G.

The definition of idim yields for every graph $G = (V, E)$ the inequality

$$\operatorname{idim}(G) \cdot \frac{n}{\alpha(G)} \cdot \binom{\alpha(G)}{2} \geqslant \binom{|V|}{2} \smallsetminus |E|\,,$$

where $\alpha(G)$ denotes the independence number of G. Owing to the fact that for $n^{-1} \ll p < 1$ the random graph $G \in G(n,p)$ a.a.s. satisfies $\alpha(G) = O(\log(pn)/p)$, we obtain the following lower bound for random graphs

$$\operatorname{idim}(G) = \Omega\left(\frac{p(1-p)n}{\log(pn)}\right). \tag{6}$$

Here we show that for a large range of p, this bound is sharp up to a $\log(pn)$-factor.

Theorem 1.3. *For every $p = p(n)$ with $\log(n)^{4/3}/n^{1/3} \ll p(n) \leqslant 1/\log^2(n)$, the random graph $G \in G(n,p)$ a.a.s. satisfies*

$$\operatorname{idim}(G) \leqslant 1000\, pn\,. \tag{7}$$

We also provide a matching lower bound by sharpening the estimate from (6).

Proposition 1.4. *For $p = p(n) \gg n^{-2}$, a.a.s. the random graph $G \in G(n,p)$ satisfies*

$$\operatorname{idim}(G) \geqslant \frac{-\log(p) \cdot pn}{5\log(n)}\,.$$

Consequently, in view of (5), for all$\delta > 0$ and $p = p(n)$ with $\log(n)^{4/3}/n^{1/3} \ll p(n) \leqslant n^{-\delta}$, a.a.s. we have

$$\operatorname{pdim}(G) = \Theta(pn)$$

for $G \in G(n,p)$. The proofs of Theorem 1.3 and Proposition 1.4 are presented Section 3.

§2. Independent covers of random graphs

The proof of Theorem 1.1 relies on a few standard estimates on the distribution of large independent sets and the distribution of its edges. We state these preparatory results in §2.1 and defer the standard proof to §2.3. The proof of Theorem 1.1 based on these facts will be given in §2.2, while §2.4 is devoted to the proof of Proposition 1.2.

2.1. **Standard facts of random graphs.** The independence number of the random graph is well understood due to the work of Matula [16], Bollobás and Erdős [3], Grimmett and McDiarmid [11], and Frieze [8] (see, e.g., [4, 13] for more details). For the proof of Theorem 1.1, it is sufficient to recall that a.a.s. the independence number of $G \in G(n,p)$ is of order $\log(n)/p$ as long as $p \gg \log(n)/n$. The following observation concerning the number of large independent sets in random graphs can be obtained by a standard second moment argument, which we include in §2.3 for completeness.

Lemma 2.1. *Let $p = p(n) \geqslant \log(n)/\sqrt{n}$, let $k = \gamma \log(n)/p$ for some $\gamma \in (0, 1/3)$, and let X_k be the random variable counting the number of independent sets of size k in the random graph $G(n,p)$. Then we have*

$$\mathbb{P}\big(|X_k - \mathbb{E}X_k| \geqslant \tfrac{1}{2}\mathbb{E}X_k\big) \leqslant \frac{4p}{1-p} \cdot \frac{k^4}{n^2}\,.$$

The following simple bounds are direct consequences of the binomial distribution and we omit the proof.

Lemma 2.2. *For all $0 < p = p(n) \leqslant 1$, the random graph $G \in G(n,p)$ a.a.s. satisfies the following for all subsets $X \subseteq V(G)$:*

(i) If $|X| \geqslant \log(n)/p$, then we have $e_G(X) \leqslant 3p|X|^2$.
(ii) If $|X| < \log(n)/p$, then we have $e_G(X) \leqslant 3|X|\log(n)$. □

2.2. **Upper bound for independent covers of random graphs.** In this section, we establish the upper bound on $\bar{\theta}_1$ based on the results from §2.1.

Proof of Theorem 1.1. Let $p = p(n)$ be given such that $\frac{\log^{5/4} n}{\sqrt{n}} \ll p(n) \leqslant \frac{1}{\log^2(n)}$. For arbitrarily fixed $\delta > 0$, we shall show that (3) holds with probability at least $1 - \delta$ for sufficiently large n. As usual, at first we exclude some undesired events. Set

$$\gamma^{\ddagger} = \frac{10}{31}\,, \qquad k = \gamma \cdot \frac{\log(n)}{p}\,, \qquad \text{and} \qquad m = (1 - 2p - p^{3/2})n\,. \tag{8}$$

We say that two distinct vertices $x, y \in V(G)$ are *well-coverable* if the number of independent sets of size k contained in $V(G) \smallsetminus (N(x) \cup N(y) \cup \{x,y\})$ is at least

$$\frac{1}{2}(1-p)^{\binom{k}{2}}\binom{m}{k}\,.$$

Below we verify some properties to hold for $G \in G(n,p)$ with probability close to 1.

Claim 2.3. *For every $\delta > 0$ and the constants γ, k, and m from (8), the random graph $G \in G(n,p)$ satisfies the following properties with probability at least $1 - \delta/2 - o(1)$ for sufficiently large n:*

(a) both assertions of Lemma 2.2 hold,
(b) the number X_{k+2} of independent sets of size $k+2$ in G satisfies

$$\frac{1}{2}(1-p)^{\binom{k+2}{2}}\binom{n}{k+2} \leqslant X_{k+2} \leqslant \frac{3}{2}(1-p)^{\binom{k+2}{2}}\binom{n}{k+2}\,,$$

(c) and at most $5pk^4/\delta$ pairs of vertices x, y are not well-coverable.

‡The choice $\gamma = 10/31$ here is somewhat arbitrary and for an application of Lemma 2.1, we only require a constant smaller than 1/3. However, a smaller choice comes with the price that the constant 200 in (3) might need to be adjusted.

Proof of Claim 2.3. In fact, due to Lemma 2.2 a.a.s. property (*a*) is satisfied. Similarly, Lemma 2.1 implies

$$\mathbb{P}\big(\big|X_{k+2} - \mathbb{E}X_{k+2}\big| \geqslant \tfrac{1}{2}\mathbb{E}X_{k+2}\big) \leqslant \frac{4p}{1-p}\frac{(k+2)^4}{n^2} = O\bigg(\frac{\log^{1/4}(n)}{\sqrt{n}}\bigg) = o(1).$$

Concerning (*c*), we first consider the set of "non-neighbours" of a given pair of vertices x, y

$$\overline{N(x,y)} = V(G) \smallsetminus \big(N(x) \cup N(y) \cup \{x,y\}\big)\,.$$

We observe that for a fixed pair x, y of distinct vertices, Chernoff's inequality implies

$$\mathbb{P}\big(|\overline{N(x,y)}| < m\big) = \mathbb{P}\big(|N(x) \cup N(y) \cup \{x,y\}| > (2p + p^{3/2})n\big) = o(n^{-2})\,, \tag{9}$$

since

$$\frac{(p^{3/2}n)^2}{pn} = p^2 n \gg \log(n) \quad \text{and} \quad \mathbb{E}\big[|N(x) \cap N(y)|\big] = p^2(n-2) \gg \log(n)\,.$$

Moreover, Lemma 2.1 bounds the conditional probability

$$\mathbb{P}\big(x,\, y \text{ are not well-coverable} \;\big|\; |\overline{N(x,y)}| \geqslant m\big) \leqslant \frac{4p}{1-p} \cdot \frac{k^4}{m^2}\,.$$

In view of (9), we therefore have

$$\begin{aligned}\mathbb{P}\big(x,\, y \text{ are not well-coverable}\big) &\leqslant \frac{4p}{1-p} \cdot \frac{k^4}{m^2} + o\bigg(\frac{1}{n^2}\bigg)\\ &= \frac{4p}{1-p} \cdot \frac{k^4}{(1-2p-p^{3/2})^2 n^2} + o\bigg(\frac{1}{n^2}\bigg) \leqslant \frac{5pk^4}{n^2}\,.\end{aligned}$$

Consequently, Markov's inequality implies that with probability at least $1-\delta/2$, the number of not well-coverable pairs is at most

$$\frac{2}{\delta} \cdot \frac{5pk^4}{n^2} \cdot \binom{n}{2} \leqslant \frac{5pk^4}{\delta}\,,$$

i.e., property (*c*) holds with probability $1-\delta/2$. Therefore properties (*a*)–(*c*) hold with probability at least $1-\delta/2-o(1)$ and this concludes the proof of Claim 2.3. □

For the rest of the proof of Theorem 1.1, we fix a graph $G = (V, E)$ satisfying properties (*a*)–(*c*) and show that (3) holds. We set

$$t = 20\log(n)\frac{n^2}{k^2} \tag{10}$$

and we consider a random selection of independent sets $I_1, \dots, I_t$, each of size $k+2$, to show that there exists such a family of independent sets that covers all pairs of distinct vertices $\{x,y\} \notin E(G)$ that are well-coverable. In fact, for a fixed well-coverable pair $\{x,y\}$ that is not an edge and a randomly selected independent set I of size $k+2$, we have[§]

$$\begin{aligned}
\mathbb{P}_I\big(\{x,y\} \subseteq I\big) &\geqslant \frac{\frac{1}{2}(1-p)^{\binom{k}{2}}\binom{m}{k}}{\frac{3}{2}(1-p)^{\binom{k+2}{2}}\binom{n}{k+2}} = \frac{1}{3}\cdot\frac{1}{(1-p)^{2k+1}}\cdot\frac{m\cdots(m-k+1)}{n\cdots(n-k-1)}\cdot(k+2)(k+1) \\
&\geqslant \frac{1}{3}\cdot\frac{1}{(1-p)^{2k+1}}\cdot\left(\frac{m-k}{n-k}\right)^k\cdot\frac{k^2}{n^2} \\
&= \frac{1}{3}\cdot\frac{1}{(1-p)^{2k+1}}\cdot\left(1-2p-p^{3/2}-\frac{2pk+p^{3/2}k}{n-k}\right)^k\cdot\frac{k^2}{n^2} \\
&\geqslant \frac{1}{3}\cdot\frac{(1-2p-p^{3/2}-4pk/n)^k}{(1-p)^{2k+1}}\cdot\frac{k^2}{n^2} \geqslant \frac{1}{3}\cdot\left(\frac{1-2p-p^{3/2}-4pk/n}{1-2p+p^2}\right)^k\cdot\frac{k^2}{n^2} \\
&\geqslant \frac{k^2}{3n^2}\cdot\left(1-\frac{8pk}{n}-2p^{3/2}\right)^k \geqslant \frac{k^2}{3n^2}\cdot\big(1-2.01p^{3/2}\big)^k \\
&\geqslant \frac{k^2}{3n^2}\cdot\big(1-2.01p^{3/2}k\big) \geqslant (1-2.01\gamma)\frac{k^2}{3n^2} \geqslant \frac{k^2}{10n^2}\,, \qquad (11)
\end{aligned}$$

where we used $p \leqslant 1/\log^2(n)$ to bound $p^{3/2}k \leqslant \gamma$ in the second to last inequality and the choice of γ for the last inequality. Consequently, for a well-coverable pair $\{x,y\}$, we have

$$\mathbb{P}_I\big(\{x,y\} \nsubseteq I_1 \cup \cdots \cup I_t\big) \leqslant \left(1-\frac{k^2}{10n^2}\right)^t \leqslant \exp\left(-\frac{tk^2}{10n^2}\right) \overset{(10)}{=} \frac{1}{n^2}\,.$$

Hence, we conclude that there exists a family of t independent sets $I_1,\dots,I_t$ (each of size $k+2$) which cover all well-coverable pairs $\{x,y\} \in V^{(2)} \smallsetminus E$. It remains to deal with the pairs that are not well-coverable.

Let $B=(V,E_B)$ be the graph of those pairs, i.e.,

$$E_B = \big\{\{x,y\} \in V^{(2)} \smallsetminus E\colon x,\ y \text{ are not a well-coverable pair}\big\}\,.$$

Owing to *(c)*, we have

$$\sum_{v\in V} d_B(v) = 2\,|E_B| \leqslant \frac{10pk^4}{\delta}\,. \qquad (12)$$

For a fixed vertex $v \in V$, suppose G induced on the neighbourhood of v in B is s-colourable and let $J_1,\dots,J_s$ be independent sets defining such a colouring. Then the sets $J_1 \cup \{v\},\dots J_s \cup \{v\}$ are independent and cover all edges of B incident to v. Applying this argument to every vertex v implies

$$\bar{\theta}_1(G) \leqslant t + \sum_{v\in V} \chi\big(G[N_B(v)]\big)\,. \qquad (13)$$

[§]Here we add the subscript I in $\mathbb{P}_I$ to stress that we consider the probability space over all independent sets of size $k+2$ in G.

We employ property *(a)* to bound $\chi(G[N_B(v)])$. If $d_B(v) < \log(n)/p$, then part *(ii)* of Lemma 2.2 implies that $G[N_B(v)]$ is $6\log(n)$-degenerate and, therefore, in this case

$$\chi\big(G[N_B(v)]\big) \leqslant 6\,\log(n) + 1 \tag{14}$$

holds. Similarly, if $d_B(v) \geqslant \log(n)/p$, then combining parts *(i)* and *(ii)* of Lemma 2.2 tells us that $G[N_B(v)]$ is $(6pd_B(v) + 1)$-colourable, i.e.,

$$\chi\big(G[N_B(v)]\big) \leqslant 6\,pd_B(v) + 1\,. \tag{15}$$

Therefore, combining (14) and (15) with (13) yields

$$\bar{\theta}_1(G) \leqslant t + \sum_{v\in V}\big(6\max\{\log(n)\,, pd_B(v)\} + 1\big) \overset{(12)}{\leqslant} t + 60\frac{p^2k^4}{\delta} + 6n\log(n) + n\,.$$

Since $p \gg \log^{5/4}(n)/\sqrt{n}$, our choice of k in (8) yields

$$60\frac{p^2k^4}{\delta} + 6n\log(n) + n \ll \frac{p^2n^2}{\log(n)}\,.$$

Consequently, our choice of t in (10) leads to

$$\bar{\theta}_1(G) \leqslant 20\log(n)\frac{n^2}{k^2} + o\bigg(\frac{p^2n^2}{\log(n)}\bigg) \overset{(8)}{=} \bigg(\frac{31^2}{5} + o(1)\bigg)\frac{p^2n^2}{\log(n)} \leqslant 200\frac{p^2n^2}{\log(n)}$$

and this concludes the proof of Theorem 1.1. □

2.3. **Independent sets in random graphs.** The proof of Lemma 2.1 relies on the second moment method and follows from standard estimates on the variance of the number of independent sets in random graphs (see, e.g., Krivelevich et al. [14]).

Proof of Lemma 2.1. Let $p = p(n) \geqslant \log(n)/\sqrt{n}$, let $k = \gamma\log(n)/p$ for some $\gamma \in (0, 1/3)$, and let X_k be the random variable counting the number of independent sets of size k in $G(n,p)$. Since $\gamma < 1$ and $p \gg n^{-1/2}$, we have

$$k\log(n/k) - pk^2/2 \geqslant \Omega\big(\log^2(n)\big)$$

and, consequently,

$$\mathbb{E}X_k = (1-p)^{\binom{k}{2}}\binom{n}{k} \geqslant \exp\big(-pk^2/2 + k\log(n/k)\big) \geqslant \exp\Big(\Omega\big(\log^2(n)\big)\Big) \gg \frac{n^2}{pk^4}\,. \tag{16}$$

Below we shall establish

$$\frac{\mathrm{Var}(X_k)}{(\mathbb{E}X_k)^2} \leqslant \frac{p}{1-p}\cdot\frac{k^4}{n^2}\,,$$

which yields Lemma 2.1 by Chebyshev's inequality. Viewing X_k as a sum of $\binom{n}{k}$ indicator random variables yields

$$\begin{aligned}\mathrm{Var}(X_k) &\leqslant \mathbb{E}X_k + \sum_{j=2}^{k-1}\binom{n}{k}\binom{k}{j}\binom{n-k}{k-j}\left((1-p)^{2\binom{k}{2}-\binom{j}{2}} - (1-p)^{2\binom{k}{2}}\right)\\ &= \mathbb{E}X_k + (\mathbb{E}X_k)^2\sum_{j=2}^{k-1}\frac{\binom{n-k}{k-j}\binom{k}{j}}{\binom{n}{k}}\left(\frac{1}{(1-p)^{\binom{j}{2}}}-1\right).\end{aligned}$$

Optimising the right hand side of the above inequality, we claim that the following function

$$g(j) = \frac{\binom{k}{j}\binom{n-k}{k-j}}{\binom{n}{k}}\cdot\frac{1-(1-p)^{\binom{j}{2}}}{(1-p)^{\binom{j}{2}}}$$

attains a maximum at $j = 2$. In fact, for $j = 2, \ldots, k-2$, we consider the quotient

$$\frac{g(j+1)}{g(j)} = \frac{(k-j)^2}{(j+1)(n-2k+j+1)}\cdot\frac{1}{(1-p)^j}\cdot\frac{1-(1-p)^{\binom{j+1}{2}}}{1-(1-p)^{\binom{j}{2}}}$$

First, we note that we can bound the last quotient from above by 3. In fact, this inequality amounts to verifying

$$(1-p)^{\binom{j}{2}}\cdot\left(3-(1-p)^j\right)\leqslant 2\,.$$

This can be checked for $j = 2$. For general j, it can be shown by induction that the left-hand side is non-increasing with j. Consequently,

$$\begin{aligned}\frac{g(j+1)}{g(j)} &\leqslant \frac{2k^2}{n}\cdot\frac{(1-p)^{-j}}{j+1}\cdot 3\\ &\leqslant \frac{6k^2}{n}\cdot\frac{\exp(j(p+p^2))}{j+1}\\ &= \frac{6\gamma^2\log^2(n)}{p^2 n}\cdot\frac{\exp(j(p+p^2))}{j+1}\,.\end{aligned}$$

Next, we consider two cases which depend on the value of p to further bound $g(j+1)/g(j)$. If $\log(n)/n^{1/2} \leqslant p < 1/n^{1/3}$, then the first factor is at most $6\gamma^2$. Since $x \longmapsto \exp(cx)/(x+1)$ is a convex function on $\mathbb{R}_{>0}$, the second factor is maximised either for $j = 2$ or $j = k-2$. For $j = 2$, the second factor is bounded by $1/2$ in this range of p. For $j = k-2 \leqslant \gamma\log(n)/p$, the second factor can be bounded by

$$\frac{\exp((k-2)(p+p^2))}{k-1}\leqslant\frac{2p\exp((1+p)\gamma\log(n))}{\gamma\log(n)} = \frac{2pn^{(1+p)\gamma}}{\gamma\log(n)}\longrightarrow 0$$

since $\gamma < 1/3$. Consequently, in this range of p, we have

$$\frac{g(j+1)}{g(j)}\leqslant 6\gamma^2\cdot\frac{1}{2}<\frac{1}{2} \tag{17}$$

for sufficiently large n.

If $p \geqslant n^{-1/3}$ and $j = 2$, then the first factor can be bounded by $6\gamma^2 \log^2(n)/n^{1/3}$ and the second factor is at most $\exp(4)/3$. For $j = k-2$, we arrive at

$$\frac{g(k-1)}{g(k-2)} \leqslant \frac{6\gamma^2 \log^2(n)}{p^2 n} \cdot \frac{2pn^{(1+p)\gamma}}{\gamma \log(n)} = \frac{12\gamma \log(n) \cdot n^{(1+p)\gamma}}{pn} \longrightarrow 0$$

for $n \longrightarrow 0$. Hence, for p in this range, the bound in (17) remains valid. Therefore, we have

$$\frac{\mathrm{Var}(X_k)}{(\mathbb{E}X_k)^2} \leqslant \frac{1}{\mathbb{E}X_k} + \sum_{j=2}^{k-1} g(j) \overset{(17)}{\leqslant} \frac{1}{\mathbb{E}X_k} + 2g(2) \leqslant \frac{1}{\mathbb{E}X_k} + \frac{k^2(k-1)^2}{n(n-1)} \cdot \frac{p}{1-p} \overset{(16)}{\leqslant} \frac{p}{1-p} \cdot \frac{k^4}{n^2}$$

which concludes the proof of Lemma 2.1 by Chebyshev's inequality. □

2.4. **Lower bound for independent covers of random graphs.** In this section, we provide a lower bound for $\bar{\theta}_1$ in random graphs and establish Proposition 1.2. In the proof, we shall use the following simple lemma, which will also be useful in Section 3.

Lemma 2.4. *Let $p = p(n) \gg n^{-2}$ and suppose $(\mathcal{F}_n)_{n \in \mathbb{N}}$ is a sequence of families of n-vertex graphs such that for the random graph $G \in G(n,p)$, we have $\mathbb{P}(G \in \mathcal{F}_n) = 1 - o(1)$. Then for every sufficiently large n, there exists some integer $m = m(n,p)$ such that*

$$\frac{1}{2} p \binom{n}{2} \leqslant m \leqslant \frac{3}{2} p \binom{n}{2} \qquad \text{and} \qquad \big|\mathcal{F}_{n,m}\big| = \big|\{F \in \mathcal{F}_n \colon e(F) = m\}\big| \geqslant \frac{1}{2}\binom{\binom{n}{2}}{m}.$$

Proof. We set $m_- = \frac{1}{2}p\binom{n}{2}$ and $m_+ = \frac{3}{2}p\binom{n}{2}$. Since $p \gg n^{-2}$ a.a.s., the random graph $G \in G(n,p)$ satisfies $m_- \leqslant e(G) \leqslant m_+$. Combined with the assumption on $\mathcal{F}_n$, this tells us

$$\sum_{m=m_-}^{m_+} \mathbb{P}\big(G \in \mathcal{F}_n \mid e(G) = m\big) \cdot \mathbb{P}\big(e(G) = m\big) = \mathbb{P}\big(G \in \mathcal{F}_n \text{ and } m_- \leqslant e(G) \leqslant m_+\big)$$
$$= 1 - o(1)\,.$$

Consequently, there is some $m \in [m_-, m_+]$ such that

$$\mathbb{P}\big(G \in \mathcal{F}_n \mid e(G) = m\big) \geqslant \frac{1}{2}\,.$$

Therefore,

$$\mathbb{P}\big(G \in \mathcal{F}_{n,m}\big) = \mathbb{P}\big(G \in \mathcal{F}_n \text{ and } e(G) = m\big)$$
$$\geqslant \frac{1}{2} \cdot \mathbb{P}\big(e(G) = m\big) = \frac{1}{2} \cdot \binom{\binom{n}{2}}{m} p^m (1-p)^{\binom{n}{2}-m}\,.$$

and the desired lower bound $|\mathcal{F}_{n,m}| \geqslant \frac{1}{2}\binom{\binom{n}{2}}{m}$ follows. □

We conclude this section with the proof of the lower bound on $\bar{\theta}_1$ for random graphs, which follows the lines of Guo et al. [12, Lemma 20].

Proof of Proposition 1.2. For $1/n \ll p \leqslant 1/\log^2(n)$, Frieze [8] (see, e.g., [13, Theorem 7.4]) showed that a.a.s. for $G \in G(n,p)$, we have

$$\alpha(G) = (2+o(1))\frac{\log(pn)}{p}\,.$$

Setting

$$\alpha = 2.1\frac{\log(pn)}{p} \qquad \text{and} \qquad t = \frac{-\log(p)\cdot p^2n^2}{10\log(pn)\log(n)}\,,$$

consider the family of n-vertex graphs

$$\mathcal{F}_n(\alpha,t) = \{F\colon V(F)=n,\ \alpha(F)\leqslant\alpha,\ \text{and } \bar{\theta}_1(F)\leqslant t\}\,.$$

This definition yields the following upper bound on the number of graphs in $\mathcal{F}_n(\alpha,t)$

$$\big|\mathcal{F}_n(\alpha,t)\big| \leqslant \left(\sum_{i=0}^{\alpha}\binom{n}{i}\right)^t \leqslant \binom{n}{\alpha+1}^t \leqslant \left(\frac{en}{\alpha+1}\right)^{(\alpha+1)t} < \exp\big((\alpha+1)t\log(n)\big)\,. \tag{18}$$

We suppose by contradiction that a.a.s. the random graph $G \in G(n,p)$ satisfies $\bar{\theta}_1(G)\leqslant t$ and consequently a.a.s. $G\in\mathcal{F}_n(\alpha,t)$. Owing to Lemma 2.4, there exists some $m \geqslant \frac{1}{2}p\binom{n}{2}$ such that

$$\begin{aligned}\big|\mathcal{F}_{n,m}(\alpha,t)\big| = \big|\{F\in\mathcal{F}_n(\alpha,t)\colon e(F)=m\}\big| &\geqslant \frac{1}{2}\binom{\binom{n}{2}}{m} \geqslant \frac{1}{2}\binom{\binom{n}{2}}{\frac{1}{2}p\binom{n}{2}}\\ &\geqslant \frac{1}{2}\left(\frac{2}{p}\right)^{\frac{1}{2}p\binom{n}{2}} \geqslant \exp\left(\frac{-\log(p)}{4.5}pn^2\right)\,.\end{aligned} \tag{19}$$

Since $|\mathcal{F}_n(\alpha,t)| \geqslant |\mathcal{F}_{n,m}(\alpha,t)|$, comparing (19) with (18) yields

$$t \geqslant \frac{-\log(p)\cdot pn^2}{4.5(\alpha+1)\log(n)} > \frac{-\log(p)\cdot p^2n^2}{10\log(pn)\log(n)}\,,$$

which contradicts to our choice of t and concludes the proof of the proposition. □

§3. Product dimension of random graphs

In this section, we prove the bounds on the product dimension for sparse random graphs. In view of (5), it suffices to bound idim and, in this section, we only focus on that parameter. The proofs are based similar ideas as the proofs in Section 2 concerning the bounds on $\bar{\theta}_1$.

3.1. **Upper bound for the product dimension of random graphs.** In this section, we establish the upper bound on idim for random graphs. This is based on similar ideas as the proof of Theorem 1.1. However, for finding partitions of independent subsets, the argument needs some technical adjustments.

Proof of Theorem 1.3. Let $p = p(n)$ be given such that $\frac{\log^{4/3}(n)}{n^{1/3}} \ll p(n) \leqslant \frac{1}{\log^2(n)}$. For arbitrarily fixed $\delta > 0$, we shall show that (7) holds with probability at least $1-\delta$ for sufficiently large n. As in the proof of Theorem 1.1, we fix the constants

$$\gamma = \frac{10}{31}\,, \qquad k = \gamma \cdot \frac{\log(n)}{p} \qquad \text{and} \qquad m = (1-2p-p^{3/2})n\,. \tag{20}$$

Again, we say that two distinct vertices x, $y \in V(G)$ are *well-coverable* if the number of independent sets of size k contained in $V(G) \smallsetminus (N(x) \cup N(y) \cup \{x,y\})$ is at least

$$\frac{1}{2}(1-p)^{\binom{k}{2}}\binom{m}{k}\,.$$

Since the choice in (20) is identical with the choice in (8) and since the range of p in Theorem 1.3 is a subset of the range in Theorem 1.1, we can appeal to Claim 2.3. Consequently, we know that with probability at least $1-\delta/2-o(1)$, the random graph $G \in G(n,p)$ satisfies:

(A) both assertions of Lemma 2.2 hold,

(B) the number X_{k+2} of independent sets of size $k+2$ in G satisfies

$$\frac{1}{2}(1-p)^{\binom{k+2}{2}}\binom{n}{k+2} \leqslant X_{k+2} \leqslant \frac{3}{2}(1-p)^{\binom{k+2}{2}}\binom{n}{k+2}\,,$$

(C) and at most $5pk^4/\delta$ pairs of vertices x, y are not well-coverable.

For the proof here, we will require also the following additional property:

(D) Every $x \in V$ is contained in at most $\frac{3}{2}(1-p)^{\binom{k+1}{2}}\binom{m_1}{k+1}$ independent sets of size $k+2$ for $m_1 = (1-p)n + 2\sqrt{pn\log(n)}$.

Again, it follows from Lemma 2.1 that $G \in G(n,p)$ satisfies (D) with probability at least $1-o(1)$. In fact, Chernoff's inequality tells us that for a fixed vertex $x \in V$, we have

$$\mathbb{P}\big(|V \smallsetminus (N(x) \cup \{x\})| > m_1\big) = o(n^{-1})\,.$$

Applying Lemma 2.1 with $k+1$ to the random graph induced on $V \smallsetminus (N(x) \cup \{x\})$ yields

$$\mathbb{P}\Big((D) \text{ fails for } x \,\Big|\, |V \smallsetminus (N(x) \cup \{x\})| \leqslant m_1\Big) \leqslant \frac{4p}{1-p} \cdot \frac{(k+1)^4}{m_1^2}\,.$$

Consequently, $\mathbb{P}((D) \text{ fails for } x) = O(pk^4/n^2) + o(1/n)$ and Markov's inequality implies that property (D) holds for all $x \in V$ since

$$n \cdot \frac{pk^4}{n^2} \overset{(20)}{=} \gamma^4 \frac{\log^4(n)}{p^3 n} = o(1)$$

by the assumption that $p \gg \log^{4/3}(n)/n^{1/3}$.

Summarising the discussion above, we showed that properties (A)–(D) hold with probability at least $1-\delta/2-o(1)\geqslant 1-\delta$. For the rest of the proof, we fix a graph G satisfying properties (A)–(D) and verify (7).

For that, we fix

$$r=\frac{n}{8k}\qquad\text{and}\qquad s=320\cdot\frac{n\log(n)}{k}\,. \tag{21}$$

We consider a collection of $s\cdot r$ random independent sets $I_{1,1},\dots,I_{1,r},I_{2,1},\dots,I_{s,r}$, each of size $k+2$ and each chosen uniformly at random from the set of all such independent sets in $G=(V,E)$. In order to define partitions $\mathcal{J}_1,\dots,\mathcal{J}_s$ for $\sigma=1,\dots,s$ and $\varrho=1,\dots,r$, we set

$$J_{\sigma,\varrho}=I_{\sigma,\varrho}\smallsetminus\bigcup_{\varrho'=1}^{\varrho-1}I_{\sigma,\varrho'}\qquad\text{and}\qquad\mathcal{J}_\sigma=(J_{\sigma,1},\dots,J_{\sigma,r})\,. \tag{22}$$

Note that we may extend $\mathcal{J}_\sigma$ by adding trivial partition classes of single vertices to "complete" them to a partition of all of V. However, these trivial classes have no bearing in the proof and, without loss of generality, we may assume that they are not necessary.

We claim that with positive probability, the collection $\mathcal{J}_1,\dots,\mathcal{J}_s$ covers all well-coverable non-edges of G, i.e., for the graph $B=(V,E_B)$ defined by

$$E_B=\big\{\{x,y\}\in V^{(2)}\smallsetminus E\colon x,\,y\text{ are not a well-coverable pair}\big\}\,,$$

we have

$$E\cup E_B\supseteq V^{(2)}\smallsetminus\bigcup_{\sigma=1}^{s}\bigcup_{\varrho=1}^{r}J^{(2)}_{\sigma,\varrho}\,. \tag{23}$$

Let q_{xy} be the probability that a fixed, well-coverable, non-adjacent pair x, y is contained in a randomly chosen independent set of size $k+2$. Then the same calculation as in (11) shows that q_{xy} can be bounded from below by

$$q_{xy}=\mathbb{P}_I(\{x,y\}\subseteq I)\geqslant\frac{k^2}{10n^2}\,.$$

Similarly, we let q_x be the probability that a fixed vertex is contained in a randomly chosen independent set. Employing properties (B) and (D), we can upper bound q_x by

$$\begin{aligned}
q_x=\mathbb{P}_I(x\in I)&\leqslant\frac{\frac32(1-p)^{\binom{k+1}{2}}\binom{m_1}{k+1}}{\frac12(1-p)^{\binom{k+2}{2}}\binom{n}{k+2}}=3\cdot\frac{\binom{(1-p)n+2\sqrt{pn\log(n)}}{k+1}}{(1-p)^{k+1}\binom{n}{k+2}}\\
&\leqslant 3\cdot\frac{(1-p+2\sqrt{\log(n)p/n})^{k+1}\binom{n}{k+1}}{(1-p)^{k+1}\binom{n}{k+2}}\\
&\leqslant 3\cdot\frac{(1-p)^{k+1}(1+4\sqrt{\log(n)p/n})^{k+1}\cdot(k+2)}{(1-p)^{k+1}\cdot(n-k-1)}\\
&\leqslant 3\cdot\exp\big(4(k+1)\sqrt{\log(n)p/n}\big)\cdot\frac{k+2}{n-k-1}\leqslant\frac{4k}{n}\,.
\end{aligned}$$

For a fixed $\sigma \in [s]$, we want to bound the probability that a fixed, well-coverable, non-adjacent pair x, y is covered by a partition $\mathcal{J}_\sigma$, i.e., $\{x,y\} \subseteq J_{\sigma,\varrho}$ for some $\varrho \in [r]$. Note that this event happens if there is some $\varrho \in [r]$ such that

$$\{x,y\} \subseteq I_{\sigma,\varrho} \qquad \text{and} \qquad \{x,y\} \cap I_{\sigma,\varrho'} = \varnothing \quad \text{for all } \varrho' < \varrho.$$

Consequently, we have

$$\mathbb{P}_I\big(\{x,y\} \subseteq J_{\sigma,\varrho}\big) = q_{xy} \cdot \mathbb{P}_I\big(x \notin I \text{ and } y \notin I\big)^{\varrho-1} = q_{xy} \cdot \big(1 - q_x - q_y + q_{xy}\big)^{\varrho-1}$$

and

$$\begin{aligned}\mathbb{P}_I\big(\{x,y\} \text{ is covered by } \mathcal{J}_\sigma\big) &= q_{xy} \cdot \sum_{\varrho=1}^{r} \mathbb{P}_I\big(x \notin I \text{ and } y \notin I\big)^{\varrho-1} \geqslant \frac{k^2}{10n^2} \cdot \sum_{\varrho=0}^{r-1} \Big(1 - \frac{8k}{n}\Big)^{\varrho} \\ &= \frac{k^2}{10n^2} \cdot \frac{1 - \big(1 - 8k/n\big)^r}{8k/n} \geqslant \frac{k}{80n}\big(1 - \exp(-8kr/n)\big) \\ &\overset{(21)}{=} \frac{k}{80n}\big(1 - 1/\mathrm{e}\big) \geqslant \frac{k}{160n}\,.\end{aligned}$$

It follows that

$$\mathbb{P}\big(\{x,y\} \text{ is not covered by any } \mathcal{J}_\sigma \text{ for } \sigma \in [s]\big) \leqslant \Big(1 - \frac{k}{160n}\Big)^s < \exp\Big(-\frac{sk}{160n}\Big) \overset{(21)}{=} \frac{1}{n^2}\,.$$

Therefore, it follows that there is a collection of independent sets $I_{\sigma,\varrho}$ for $\sigma \in [s]$ and $\varrho \in [r]$ such that the corresponding partitions $\mathcal{J}_1, \dots, \mathcal{J}_s$ defined in (22) satisfy (23), i.e., they cover all well-coverable, non-adjacent pairs of vertices.

It remains to deal with the pairs that are not well-coverable. First, we consider the vertices $v \in V$ with $d_B(v) \geqslant \log(n)/p$. For those vertices, we again consider colourings of the induced subgraph $G[N_B(v)]$. In fact, we turn every colour class C of $G[N_B(v)]$ into a partition consisting of the class $C \cup \{v\}$ and trivial classes for every vertex from $V \smallsetminus (C \cup \{v\})$. Following the lines of the proof of Theorem 1.1, by appealing to (A), we can cover all edges of B incident to v with

$$\chi\big(G[N_B(v)]\big) \leqslant 6pd_B(v) + 1 \leqslant 7pd_B(v)$$

partitions. Let $B' \subseteq B$ be the subgraph of those pairs which are not yet covered. By definition $\Delta(B') < \log(n)/p$, and so by Vizing's theorem, we infer that the edges of B' can be covered with $1 + \log(n)/p$ matchings. Again, we can extend every such matching to a partition of V by adding trivial classes and conclude

$$\mathrm{idim}(G) \leqslant s + \sum\big\{7pd_B(v)\colon d_B(v) > \log(n)/p\big\} + \frac{\log(n)}{p} + 1\,.$$

In view of (C), we have $\sum_{v\in V} d_B(v) \leqslant 10pk^4/\delta$, which implies the bound

$$\sum \big\{7pd_B(v)\colon d_B(v) > \log(n)/p\big\} \leqslant \frac{70p^2k^4}{\delta}\,.$$

Moreover, our assumption that $p \gg \frac{\log^{4/3}(n)}{n^{1/3}}$ and our choice of k in (20) guarantee

$$\frac{70p^2k^4}{\delta} \ll pn \qquad \text{and} \qquad \frac{\log(n)}{p} + 1 \ll pn$$

and we arrive at

$$\operatorname{idim}(G) \leqslant s + o(pn) \overset{(21)}{=} 320 \cdot \frac{n\log(n)}{k} + o(pn) \overset{(20)}{\leqslant} 1000\,pn\,,$$

which concludes the proof of Theorem 1.3. □

3.2. **Lower bound for the product dimension of random graphs.** The proof of Proposition 1.4 follows the lines of the proof of Proposition 1.2.

Proof of Proposition 1.4. Let $p = p(n) \gg n^{-2}$ and set

$$s = \frac{-\log(p)\cdot pn}{5\log(n)}\,. \tag{24}$$

Let $\mathcal{F}_n(s)$ be the family of all n-vertex graphs F with $\operatorname{idim}(F) \leqslant s$. Obviously, we can bound the size of this family by

$$\big|\mathcal{F}_n(s)\big| \leqslant n^{sn} = \exp\big(sn\cdot\log(n)\big)\,.$$

On the other hand, assuming by contradiction that a.a.s. $\operatorname{idim}(G) \leqslant s$ holds for $G \in G(n,p)$, Lemma 2.4 yields

$$\big|\mathcal{F}_n(s)\big| \geqslant \frac{1}{2}\binom{\binom{n}{2}}{\frac{1}{2}p\binom{n}{2}} \geqslant \frac{1}{2}\left(\frac{2}{p}\right)^{\frac{1}{2}p\binom{n}{2}} > \exp\big(\tfrac{1}{5}\log(1/p)\cdot pn^2\big)\,.$$

Comparing the lower and the upper bound on $|\mathcal{F}_n(s)|$ implies

$$sn\cdot\log(n) \geqslant \log\big(|\mathcal{F}_n(s)|\big) > \frac{1}{5}\log(1/p)\cdot pn^2\,,$$

which contradicts the choice of s in (24) and concludes the proof of Proposition 1.4. □

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