THE CODEGREE TURÁN DENSITY OF 3-UNIFORM TIGHT CYCLES

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Abstract. Given any $\varepsilon > 0$ we prove that every sufficiently large n-vertex 3-graph H where every pair of vertices is contained in at least $(1/3+\varepsilon)n$ edges contains a copy of C_{10}, i.e. the tight cycle on 10 vertices. In fact we obtain the same conclusion for every cycle C_ℓ with $\ell \geqslant 19$.

§1 Introduction

We consider an extremal problem for hypergraphs. A k-uniform hypergraph H is defined by a vertex set $V(H)$ and a set of edges $E(H) \subseteq V(H)^{(k)} = \{S \subseteq V(H) \colon |S| = k\}$. Throughout this note, unless specified otherwise, we refer to 3-uniform hypergraphs simply as hypergraphs. For a given hypergraph F, the *extremal number* $\mathrm{ex}(n, F)$ for n vertices is the maximum number of edges in an n-vertex hypergraph that does not contain a copy of F. The *Turán density* $\pi(F)$ is defined as

$$\lim_{n\longrightarrow\infty} \frac{\mathrm{ex}(n,F)}{\binom{n}{3}},$$

this is well-defined for every F, since the sequence $\mathrm{ex}(n, F)/\binom{n}{3} \geqslant 0$ is non-increasing. Determining the Turán densities of hypergraphs is a central problem in combinatorics. Despite considerable efforts by many researchers, Turán densities are known only for few hypergraphs. For discussion of techniques, results, and variations, see the surveys by Keevash [7], Balogh, Clemen, and Lidický [1] and Reiher [9].

Our focus here is on the variation called *codegree Turán density*, introduced by Mubayi and Zhao [8]. Given a hypergraph H and a subset $S \in V^{(2)}$, the *neighbourhood* $N_H(S)$ and *codegree* $d_H(S)$ of S are defined by

$$N_H(S) = \{v \in V(H) \colon S \cup v \in E(H)\} \qquad \text{and} \qquad d_H(S) = |N_H(S)|,$$

and when H is clear from the context we will omit it from the notation. We will omit unnecessary parenthesis and commas from the set-theoretic notation, and in particular we write $d(uv)$ instead of $d(\{u, v\})$. The minimum codegree of H among all possible sets S of size two is denoted by $\delta_2(H)$. For a hypergraph F and an integer n, the *codegree Turán number* $\mathrm{ex}_2(n, F)$ is the maximum d such that there exists an F-free hypergraph H on n vertices with $\delta_2(H) \geqslant d$; and the *codegree Turán density of* F is

$$\gamma(F) = \lim_{n\longrightarrow\infty} \frac{\mathrm{ex}_2(n,F)}{n},$$

which is also well-defined for every F [8, Proposition 1.2]. Clearly, $\gamma(F) \leqslant \pi(F)$.

Similarly, the codegree Turán density is known for only few hypergraphs (see, e.g., [1, Table 1]). In particular, a computer-assisted proof by Falgas-Ravry, Pikhurko, Vaughan, and Volec [4] determined that $\gamma(K_4^-) = 1/4$, where K_4^- is the hypergraph obtained from K_4 by removing one edge. In contrast, $\gamma(K_4)$ is not known, with Czygrinow and Nagle [2] conjecturing that $\gamma(K_4) = 1/2$.

Given two hypergraphs F, G, an *homomorphism* from F to G is a map $\varphi\colon V(F) \longrightarrow V(G)$ which preserves edges, i.e. $\varphi(e) \in E(G)$ for each $e \in E(H)$. By the phenomenon of supersaturation (see, e.g. [7, Section 2] and [8, Proposition 1.4]) it turns out that if there exists an homomorphism from F to G then $\pi(F) \leqslant \pi(G)$ and $\gamma(F) \leqslant \gamma(G)$.

We are interested in studying codegree Turán densities of tight cycles. Given an integer $\ell \geqslant 3$, a *tight cycle* C_ℓ is a hypergraph with vertex set $\{v_1, \dots, v_\ell\}$ and edge set $\{v_i v_{i+1} v_{i+2} \colon i \in \mathbb{Z}/\ell\mathbb{Z}\}$. Whenever ℓ is divisible by 3 we have that C_ℓ is 3-partite, which implies that $\gamma(C_\ell) = \pi(C_\ell) = 0$ (see [3]), so the interesting cases concern ℓ not divisible by 3 only. Recently, Kamčev, Letzter, and Pokrovsky [6] determined $\pi(C_\ell)$ for those values of ℓ, as long as ℓ is sufficiently large.

The following lower bound construction shows that $\gamma(C_\ell) \geqslant 1/3$ for ℓ not divisible by 3.

Example 1.1. Let $n \in 3\mathbb{N}$ and let $H = (V, E)$ be a hypergraph where $V = V_1 \dot\cup V_2 \dot\cup V_3$ with $|V_i| = n/3$ and

$$E = \{uvw \in V^{(3)} \colon u \in V_i, v \in V_j, w \in V_k \text{ and } i + j + k \equiv 1 \bmod 3\}\,.$$

It is easy to check that $\delta_2(H) \geqslant n/3 - 1$. Let C_ℓ be a cycle in H, we will show that ℓ is divisible by 3. For each $v \in V(C_\ell)$ let $c(v) = j$ if $v \in V_j$ and set $\Omega = \sum_{e \in E(C_\ell)} \sum_{v \in e} c(v)$. By construction, we have that $\sum_{v \in e} c(v) \equiv 1 \bmod 3$ for each $e \in E(C_\ell)$, therefore $\Omega \equiv \ell \bmod 3$. Moreover, since every vertex of C_ℓ is contained exactly in 3 edges, we also have that $\Omega \equiv 0 \bmod 3$. Hence, $\ell \equiv 0 \bmod 3$.

The previously known best upper bound for codegree Turán densities of tight cycles is due to Balogh, Clemen, and Lidický [1]. One of their results yields $\gamma(C_\ell) \leqslant 0.3993$ for every $\ell \geqslant 5$ except $\ell = 7$.

In this note we establish an upper bound matching Example 1.1 for almost every ℓ not divisible by three.

Theorem 1.2. *For $\ell \in \{10, 13, 16\}$ and for every $\ell \geqslant 19$ not divisible by 3, $\gamma(C_\ell) = 1/3$.*

We can use homomorphisms to obtain codegree Turán densities for longer cycles using shorter ones. Indeed, there is an homomorphism from $C_{\ell+3}$ to C_ℓ (by wrapping around the last three vertices), and therefore $\gamma(C_{\ell+3}) \leqslant \gamma(C_\ell)$. Moreover, for any $t \geqslant 2$, there is an homomorphism from $C_{t\ell}$ to C_ℓ (by transversing C_ℓ t times), so $\gamma(C_{t\ell}) \leqslant \gamma(C_\ell)$ also holds. Combining these two observations, it is easy to see that we only need to prove $\gamma(C_{10}) \leqslant 1/3$.

§2 Proof of Theorem 1.2

Given $\varepsilon > 0$ let $n_0 \in \mathbb{N}$ be sufficiently large and let H be a hypergraph on $n \geqslant n_0$ vertices with $\delta_2(H) \geqslant (1/3+\varepsilon)n$. It suffices to show that H contains an homomorphic image of C_{10}. For a contradiction, suppose not. We separate the rest of the proof in a series of claims.

Claim 1. *Every edge of H is contained in a copy of K_4^-.*

Proof. Let $e = xyz \in E(H)$ and note that $d(xy)+d(xz)+d(yz) \geqslant (1+3\varepsilon)n$. Hence, there is a vertex $v \in V \setminus \{x,y,z\}$ such that v is in two neighbourhoods $N(xy), N(xz), N(yz)$. Suppose $v \in N(xy) \cap N(xz)$. Then the edges $\{xyz, xyv, xzv\}$ form a copy of K_4^-. ■

We say the only vertex of degree 3 in a K_4^- is the *apex* of K_4^-. We say that a pair of distinct vertices $u, v \in V(H)$, is an *apex pair* if there is a copy of K_4^- containing u and v, where either u or v is the apex. Similarly, we say they are a *base pair* if there is a copy of K_4^- containing u and v where neither of them is the apex.

Claim 2. *Every pair of distinct vertices is either an apex pair or a base pair, but not both.*

Proof. Observe that Claim 1 together with the minimum codegree condition imply that every pair of vertices is contained in a copy of K_4^-. In particular, every pair is an apex pair or a base pair.

Suppose that the pair uv is simultaneously an apex pair and a base pair. Consequently, we can assume that there are K and K', copies of K_4^-, both containing the vertices u and v and such that v is the apex of K and neither u nor v is the apex of K'. Let $V(K) = \{u,v,x,y\}$ and $V(K') = \{u,v,a,b\}$ be the vertex sets of K and K' respectively, where a is the apex of K'. Observe that the ordering $(\boldsymbol{v}, u, \boldsymbol{a}, b, v, \boldsymbol{a}, u, \boldsymbol{v}, x, y)$ forms an homomorphic copy of C_{10}, where we marked the apexes for clarity. ■

We define an auxiliary directed graph D with on the vertex set $V(H)$ with arcs given by

$$E(D) = \left\{(u,v) \in V(H)^2 \colon uv \text{ is an apex pair with apex } v\right\}.$$

Claim 3. *D does not contain a directed cycle of length 2.*

Proof. Suppose $(a,x),(x,a) \in E(D)$. Then there are K and K', copies of K_4^-, both containing the vertices a and x, and such that a is the apex of K and x is apex of K'. Let $V(K) = \{a,x,b,c\}$ and $V(K') = \{a,x,y,z\}$ be the vertex sets of K and K' respectively. Observe that the ordering $(\boldsymbol{x}, \boldsymbol{a}, y, \boldsymbol{x}, z, \boldsymbol{a}, \boldsymbol{x}, b, \boldsymbol{a}, c)$ forms an isomorphic copy of C_{10}, where we marked the apexes for clarity. ■

Let $B = \{uv \in V(H)^{(2)} \colon uv \text{ is a base pair}\}$ and note that due to Claims 2 and 3 for every pair $uv \in V(H)^{(2)}$ exactly one of the following alternatives hold:

(*i*) $(u,v) \in E(D)$,
(*ii*) $(v,u) \in E(D)$, or
(*iii*) $uv \in B$.

The following claim shows that the edges of B and the arcs of D are strongly related.

Claim 4. *For every $v \in V(H)$ we have:*

(*a*) *If $d_B(v) > 0$, then $d_D^+(v) \geqslant (1/3+\varepsilon)n$.*
(*b*) *If $d_D^+(v) > 0$, then $d_B(v) \geqslant (1/3+\varepsilon)n$.*
(*c*) *If $d_D^-(v) > 0$, then $d_D^-(v) \geqslant (1/3+\varepsilon)n$.*

Proof. Since the proofs are all analogous, we only show (*a*). Let u be such that $uv \in B$ and let $w \in N(uv)$ chosen arbitrarily. Due to Claim 1 there is a K_4^- containing the edge uvw. Observe that neither u nor v can be the apex of such K_4^-, otherwise, uv would be an apex pair, contradicting Claim 2. Therefore w is the apex, and $(v,w) \in E(D)$. Hence $N(uv) \subseteq N_D^+(v)$, meaning that $|N_D^+(v)| \geqslant (1/3+\varepsilon)n$. ■

Finally, if there is an vertex $v^\star \in V(H)$ with $d_B(v^\star) > 0$, $d_D^+(v^\star) > 0$, and $d_D^-(v^\star) > 0$, then Claim 4 yields a contradiction with Claim 2 or Claim 3, since there would be a pair for which two of the three alternatives (*i*), (*ii*), (*iii*) hold. We shall find such vertex $v^\star$.

First, suppose there are two distinct vertices u, v with $d_D^+(u) = d_D^+(v) = 0$. Then $uv \in B$, due to Claim 2, and in particular $d_B(u), d_B(v) > 0$. However, Claim 4 yields a contradiction, since this implies $d_D^+(u), d_D^+(v) > 0$. Hence, there is at most one vertex having zero out-degree in D.

Secondly, take two disjoint edges e_1 and e_2 and note that Claim 1 implies that there are vertices $v_1 \in e_1$ and $v_2 \in e_2$ with $d_D^-(v_1), d_D^-(v_2) > 0$. One of them, say v_1, has positive out-degree as well, i.e. $d_D^+(v_1) > 0$. Since Claim 4 yields $d_B(v_1) > 0$ we are done by taking $v^\star = v_1$. □

§3 Concluding remarks

It would be interesting to settle the remaining values of $\gamma(C_\ell)$. The case $\ell = 4$ is equivalent to the determination of $\gamma(K_4)$ and, as mentioned in the introduction, Czygrinow and Nagle [2] conjectured $\gamma(K_4) = 1/2$. It seems plausible that Example 1.1 is optimal for all other values of ℓ not divisible by three. In other words, that $\gamma(C_\ell) = 1/3$ for every $\ell \geqslant 5$ not divisible by three. Note that by our previous remarks, for this result it would suffice to show $\gamma(C_5) \leqslant 1/3$ and $\gamma(C_7) \leqslant 1/3$.

Determining whether Example 1.1 is optimal for $\mathrm{ex}(n, C_\ell)$ for $\ell \geqslant 5$ not divisible by three is a natural question. We believe a more careful analysis of the proof of Theorem 1.2 yields a constant $c \in \mathbb{N}$ such that

$$\mathrm{ex}_2(n, C_{10}) \leqslant \frac{n}{3} + c\,,$$

for sufficiently large n and finding the optimal constant c remains open.

Finally, for k-uniform hypergraphs with $k \geqslant 4$, the problem of determining $\gamma\big(C_\ell^{(k)}\big)$ in general remains open. For general lower-bound constructions see [5, Section 10].

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References

[1] J. Balogh, F. C. Clemen, and B. Lidický, "Hypergraph Turán problems in ℓ_2-norm," *Surveys in combinatorics 2022*, ser. London Math. Soc. Lecture Note Ser. Vol. 481, Cambridge Univ. Press, Cambridge, 2022, pp. 21–63 (↑ 1, 2).

[2] A. Czygrinow and B. Nagle, "A note on codegree problems for hypergraphs," *Bull. Inst. Combin. Appl.*, vol. 32, pp. 63–69, 2001 (↑ 2, 4).

[3] P. Erdős, "On extremal problems of graphs and generalized graphs," *Israel J. Math.*, vol. 2, pp. 183–190, 1964, DOI: 10.1007/BF02759942 (↑ 2).

[4] V. Falgas-Ravry, O. Pikhurko, E. Vaughan, and J. Volec, "The codegree threshold of K_4^-," *J. Lond. Math. Soc. (2)*, vol. 107, no. 5, pp. 1660–1691, 2023, DOI: 10.1112/jlms.12722 (↑ 2).

[5] J. Han, A. Lo, and N. Sanhueza-Matamala, "Covering and tiling hypergraphs with tight cycles," *Combin. Probab. Comput.*, vol. 30, no. 2, pp. 288–329, 2021, DOI: 10.1017/S0963548320000449 (↑ 4).

[6] N. Kamčev, S. Letzter, and A. Pokrovskiy, "The Turán density of tight cycles in three-uniform hypergraphs," *Int. Math. Res. Not. IMRN*, no. 6, pp. 4804–4841, 2024, DOI: 10.1093/imrn/rnad177 (↑ 2).

[7] P. Keevash, "Hypergraph Turán problems," *Surveys in combinatorics 2011*, ser. London Math. Soc. Lecture Note Ser. Vol. 392, Cambridge Univ. Press, Cambridge, 2011, pp. 83–139 (↑ 1, 2).

[8] D. Mubayi and Y. Zhao, "Co-degree density of hypergraphs," *J. Combin. Theory Ser. A*, vol. 114, no. 6, pp. 1118–1132, 2007, DOI: 10.1016/j.jcta.2006.11.006 (↑ 1, 2).

[9] Chr. Reiher, "Extremal problems in uniformly dense hypergraphs," *European J. Combin.*, vol. 88, pp. 103117, 22, 2020, DOI: 10.1016/j.ejc.2020.103117 (↑ 1).

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