CANONICAL RAMSEY NUMBERS FOR PARTITE HYPERGRAPHS

MATÍAS AZÓCAR CARVAJAL, GIOVANNE SANTOS, AND MATHIAS SCHACHT

Abstract. We show that canonical Ramsey numbers for partite hypergraphs grow single exponentially for any fixed uniformity.

§1 Introduction

Erdős and Rado [7] established the canonical Ramsey theorem, which generalises Ramsey's theorem [12] to an unbounded number of colours. In that work Erdős and Rado characterised all canonical colour patterns that are unavoidable in colourings of edge sets of sufficiently large hypergraphs. The canonical Ramsey number $\mathrm{ER}(K_t^{(k)})$ is the smallest integer n such that any edge colouring $\varphi\colon E(K_n^{(k)}) \longrightarrow \mathbb{N}$ of the complete k-uniform hypergraph on n vertices yields a canonical copy of $K_t^{(k)}$, i.e., a copy which exhibits one of those unavoidable colour patterns. (The precise definition of these colour patterns is not important at this point, however, we remark that for their definition the underlying vertex set is assumed to be ordered.) We consider quantitative aspects of this theorem. For the Erdős–Rado theorem discussed above it follows from the work of Erdős, Hajnal, and Rado [6, §16.4] (see also reference [5, (4.2)]), the work of Lefmann and Rödl [10], and the work of Shelah [14] that the lower and the upper bound on n grow as $(k-1)$-times iterated exponentials in polynomials of t (see also reference [13, §4]). In other words, in terms of the number of exponentiations the canonical Ramsey number and the non-canonical Ramsey number (for many colours) display the same behaviour.

We study Erdős–Rado numbers for k-partite k-uniform hypergraphs. The extremal problem for k-partite k-uniform hypergraphs is degenerate and as a result the Ramsey number grows much slower. In fact, owing to the work of Kövari, Sós, and Turán [9] and of Erdős [4] those Ramsey numbers for any fixed number of colours grow only exponential and random colourings yield a matching lower bound. Roughly speaking, we show that canonical Ramsey numbers for partite hypergraphs exhibit the same behaviour and, in fact, these extremal results will be crucial in the proof. We recall the definition of canonical colourings for partite hypergraphs.

2020 *Mathematics Subject Classification.* 05D10 (primary), 05C35, 05C65 (secondary).

Key words and phrases. Ramsey theory, canonical colourings, partite hypergraphs.

The first two authors have been supported by ANID and DAAD under ANID-PFCHA/Doctorado Acuerdo Bilateral DAAD Becas Chile/2023-62230021 and by ANID Becas/Doctorado Nacional 21221049.

Definition 1.1. For a k-partite k-uniform hypergraph $H = (V_1 \cup \ldots \cup V_k, E)$, a set $J \subseteq [k]$, and an edge $e \in E$ we write

$$e_J = e \cap \bigcup_{j \in J} V_j$$

for the restriction of e to the vertex classes indexed by J. We say a colouring $\varphi \colon E \longrightarrow \mathbb{N}$ is *J-canonical*, if for all edges e, $e' \in E$ we have

$$\varphi(e) = \varphi(e') \qquad \Longleftrightarrow \qquad e_J = e'_J\,.$$

Moreover, we say the colouring is *canonical* if it is J-canonical for some $J \subseteq [k]$ and a subhypergraph is a *canonical copy*, if the colouring φ restricted to its edges is canonical.

Since $e_\varnothing = \varnothing$ for all edges $e \in E$, we observe that $\varnothing$-canonical colourings are monochromatic. For the other extreme case we note that $[k]$-canonical colourings are injective and as usual we refer to those as *rainbow* colourings.

Similarly as above, we define $\mathrm{ER}(K^{(k)}_{t,\ldots,t})$ as the smallest integer n such that every colouring $\varphi \colon E(K^{(k)}_{n,\ldots,n}) \longrightarrow \mathbb{N}$ of the edges of the complete k-partite k-uniform hypergraph with vertex classes of size n yields a canonical copy of $K^{(k)}_{t,\ldots,t}$. It follows from the work of Rado [11] that these numbers exist and a simple probabilistic argument employing a random colouring with $t^k - 1$ colours shows

$$\mathrm{ER}(K^{(k)}_{t,\ldots,t}) \geqslant t^{(1-o(1))t^{k-1}}\,, \tag{1.1}$$

where $o(1) \longrightarrow 0$ as $t \longrightarrow \infty$. We establish a comparable upper bound, which resolves a problem raised by Dobák and Mulrenin [3].

Theorem 1.2. *For sufficiently large t we have $\mathrm{ER}(K^{(2)}_{t,t}) \leqslant t^{3(t+1)}$ and $\mathrm{ER}(K^{(3)}_{t,t,t}) \leqslant t^{30t^3}$. Moreover, for every $k \geqslant 4$ and t sufficiently large the following holds*

$$\mathrm{ER}(K^{(k)}_{t,\ldots,t}) \leqslant t^{t^{k^2}}\,.$$

Theorem 1.2 for $k = 2$ is optimal up to the factor 3 in the exponent and similar bounds were obtained recently by Gishboliner, Milojević, Sudakov, and Wigderson [8] and by Dobák and Mulrenin [3]. For $k = 3$ there is a more substantial gap between the lower bound (1.1) and the upper bound provided by Theorem 1.2. In view of that, it would be interesting to decide if the cubic exponent $30t^3$ in the upper bound could be improved to be quadratic.

For larger values of k the gap between the lower and the upper bound widens and in the proof of Theorem 1.2 we made no attempt for obtaining the optimal constant in front of the exponent k^2. However, our method seems fall short to obtain a factor smaller than 1/2. In particular, we leave it open if the k^2 can be improved to $o(k^2)$ or even to $O(k)$ for $k \longrightarrow \infty$, as suggested by the lower bound (1.1).

§2 Preparations

The proof of Theorem 1.2 follows the approach of Dobák and Mulrenin [3]. It is based on an unbalanced variant of Erdős' extremal result for partite hypergraphs, which we introduce in §2.1. Another key idea, often used to locate rainbow copies, is to consider bounded colourings and we introduce those in §2.2.

2.1. Extremal problem for partite hypergraphs.

The proof of Theorem 1.2 relies on the extension of the Kövari–Sós–Turán theorem [9] to hypergraphs due to Erdős [4]. For completeness we include the proof of the following variant of that result for partite hypergraphs with vertex classes of different sizes. For the inductive proof it is helpful to consider 1-uniform hypergraphs.

Proposition 2.1. *For $k \geqslant 1$ let $H = (V_1 \dot\cup \ldots \dot\cup V_k, E)$ be a k-partite k-uniform hypergraph of density $d = \frac{|E|}{|V_1|\cdots|V_k|}$. If for some positive integers $t_1, \ldots, t_k$ we have*

$$\left(\frac{d}{4^{k-1}}\right)^{\prod_{j<i} t_j} |V_i| \geqslant 2t_i \tag{2.1}$$

for every $i \in [k]$, then the number $K^{(k)}_{t_1,\ldots,t_k}(H)$ of complete k-partite k-uniform hypergraphs in H with vertex classes $U_j \subseteq V_j$ and $|U_j| = t_j$ for every $j \in [k]$ satisfies

$$K^{(k)}_{t_1,\ldots,t_k}(H) > \left(\frac{d}{2^{2k-1}}\right)^{\prod_{j\in[k]} t_j} \prod_{j\in[k]} \binom{|V_j|}{t_j}. \tag{2.2}$$

Proof. The proof is by induction on k. For $k = 1$ assumption (2.1) reduces to $d|V_1| \geqslant 2t_1$ and yields $\binom{d|V_1|}{t_1} > (d/2)^{t_1}\binom{|V_1|}{t_1}$. Consequently, conclusion (2.2) holds.

For the inductive step consider a k-uniform hypergraph $H = (V_1 \dot\cup \ldots \dot\cup V_k, E)$ of density d. For every vertex $v \in V_k$ we denote by $H(v)$ the $(k-1)$-uniform hypergraph on vertex classes $V_1, \ldots, V_{k-1}$ defined by the link of v and we set

$$K(v) = K^{(k-1)}_{t_1,\ldots,t_{k-1}}(H(v)),$$

i.e., $K(v)$ is the number of complete $(k-1)$-partite $(k-1)$-uniform hypergraphs in $H(v)$ with vertex classes $U_j \subseteq V_j$ and $|U_j| = t_j$ for every $j \in [k-1]$. Let $V_k^\star \subseteq V_k$ be the set of those vertices v with

$$e(H(v)) \geqslant \frac{|E|}{2|V_k|}.$$

In particular, $\sum_{v\in V_k^\star} e(H(v)) \geqslant |E|/2$ and, more importantly, for every $v \in V_k^\star$ the link hypergraph $H(v)$ satisfies the assumption (2.1) with k replaced by $k-1$. Consequently, the inductive hypothesis applied to $H(v)$ for every $v \in V_k^\star$ yields

$$\sum_{v\in V_k^\star} K(v) > \sum_{v\in V_k^\star} \left(\frac{e(H(v))}{2^{2k-3}|V_1|\cdots|V_{k-1}|}\right)^{\prod_{j\in[k-1]} t_j} \prod_{j\in[k-1]} \binom{|V_j|}{t_j}$$

and Jensen's inequality with weights $1/|V_k|$ for every $v \in V_k^\star$ tells us

$$\begin{aligned}\sum_{v\in V_k^\star} K(v) &> |V_k|\cdot\left(\frac{|E|/2}{2^{2k-3}|V_1|\cdots|V_{k-1}||V_k|}\right)^{\prod_{j\in[k-1]}t_j}\prod_{j\in[k-1]}\binom{|V_j|}{t_j}\\ &= |V_k|\cdot\left(\frac{d}{4^{k-1}}\right)^{\prod_{j\in[k-1]}t_j}\prod_{j\in[k-1]}\binom{|V_j|}{t_j}. \end{aligned}\tag{2.3}$$

Finally, we arrive at the desired lower bound on $K^{(k)}_{t_1,\dots,t_k}(H)$ by double counting and another application of Jensen's inequality, i.e.,

$$\begin{aligned} K^{(k)}_{t_1,\dots,t_k}(H) &\geqslant \prod_{j\in[k-1]}\binom{|V_j|}{t_j}\cdot\binom{\sum_{v\in V_k}K(v)/\prod_{j\in[k-1]}\binom{|V_j|}{t_j}}{t_k}\\ &\overset{(2.3)}{>} \prod_{j\in[k-1]}\binom{|V_j|}{t_j}\cdot\binom{\left(\frac{d}{4^{k-1}}\right)^{\prod_{j\in[k-1]}t_j}\cdot|V_k|}{t_k}\\ &\overset{(2.1)}{>} \prod_{j\in[k-1]}\binom{|V_j|}{t_j}\cdot\left(\frac{1}{2}\left(\frac{d}{4^{k-1}}\right)^{\prod_{j\in[k-1]}t_j}\right)^{t_k}\binom{|V_k|}{t_k}\\ &\geqslant \left(\frac{d}{2^{2k-1}}\right)^{\prod_{j\in[k]}t_j}\prod_{j\in[k]}\binom{|V_j|}{t_j}, \end{aligned}$$

and this concludes the proof of Proposition 2.1. □

2.2. **Bounded colourings.** For a k-partite k-uniform hypergraph $H = (V_1 \cup \ldots \cup V_k, E)$ and $J \subseteq [k]$ we write V_J for the set of $|J|$-element vertex sets intersecting the vertex classes indexed by J, i.e.,

$$V_J = \big\{\{v_{i_1},\dots,v_{i_{|J|}}\}\colon v_{i_j}\in V_j \text{ for all } j\in J\big\}\,.$$

Clearly, we have $|V_\varnothing| = 1$ and $|V_J| = \prod_{j\in J}|V_j|$.

Roughly speaking, the proof of Theorem 1.2 pivots on a classification of the colourings given by the following definition.

Definition 2.2. We say a colouring $\varphi\colon E \longrightarrow \mathbb{N}$ of the edge set of a k-partite k-uniform hypergraph $H = (V_1 \cup \ldots \cup V_k, E)$ is (δ, J)-*bounded* for some $\delta > 0$ and some set $J \subsetneq [k]$ if all but at most $\delta|V_J|$ of the $|J|$-tuples $S \in V_J$ satisfy

$$\big|\{e\in E\colon e_J = S \text{ and } \varphi(e)=\ell\}\big| \leqslant \delta\big|V_{[k]\smallsetminus J}\big|$$

for every colour $\ell \in \mathbb{N}$. For $j = 0, \dots, k-1$ we say the colouring φ is (δ, j)-*bounded*, if it is (δ, J)-*bounded* for every j-element set $J \in [k]^{(j)}$.

Moreover, we say φ is $\boldsymbol{\delta}$-*bounded* for some $\boldsymbol{\delta} = (\delta_0, \dots, \delta_{k-1}) \in (0, 1]^k$ if φ is (δ, j)-bounded for every $j = 0, \dots, k-1$.

Note that being $(\delta,\varnothing)$-bounded simply means that no colour appears more than

$$\delta|V_{[k]}| = \delta|V_1|\cdots|V_k|$$

times. Moreover, it is easy to check that $(\frac{\delta^2}{1+\delta}, j+1)$-boundedness implies (δ, j)-boundedness.

It is well known, that $\boldsymbol{\delta}$-bounded colourings for sufficiently small choices of $\delta_0,\dots,\delta_{k-1}$ yield large rainbow subhypergraphs. In fact, being (δ, J)-bounded implies that the number of those obstructions to a rainbow coloured subhypergraph consisting of two edges e, e' of the same colour with $e\cap e'\in V_J$ is at most

$$\delta_{|J|}|V_J||V_{[k]\smallsetminus J}|^2\,. \tag{2.4}$$

This was already exploited in the work of Babai [2], of Lefmann and Rödl [10] and of Alon, Jiang, Miller, and Pritikin [1] in similar context. The next proposition is based on the same observation.

Proposition 2.3. *For $k\geqslant 2$ and $\boldsymbol{\delta}=(\delta_0,\dots,\delta_{k-1})\in(0,1]^k$ let $\varphi\colon E(K^{(k)}_{V_1,\dots,V_k})\longrightarrow\mathbb{N}$ be a $\boldsymbol{\delta}$-bounded colouring with of the complete k-partite k-uniform hypergraph $K^{(k)}_{V_1,\dots,V_k}$ with vertex partition $V_1\dot\cup\dots\dot\cup V_k$. If for some positive integer $t\leqslant\frac{1}{2}\min\big\{|V_1|,\dots,|V_k|\big\}$ we have*

$$\delta_j\leqslant\frac{1}{2^{3k-j}\cdot t^{2k-j-1}} \tag{2.5}$$

for every $j=0,\dots,k-1$, then φ yields a rainbow copy of the complete k-partite k-uniform subhypergraph $K^{(k)}_{t,\dots,t}$ with vertex classes of size t.

Proof. For every $i\in[k]$ we choose a random subset $W_i\subseteq V_i$ of size $2t$ and these k choices are carried out independently. For a subset $J\subsetneq[k]$ let X_J be the random variable of the number of pairs of edges $\{e,e'\}$ present in the induced subhypergraph $H[W_1,\dots,W_k]$ with

$$e\cap e'\in W_J \quad\text{and}\quad \varphi(e)=\varphi(e')\,.$$

Clearly, we have

$$\mathbb{E}X_J\overset{(2.4)}{\leqslant}\prod_{j\in J}\frac{2t}{|V_j|}\cdot\prod_{j\notin J}\frac{2t\cdot(2t-1)}{|V_j|\cdot(|V_j|-1)}\cdot\delta_{|J|}|V_J||V_{[k]\smallsetminus J}|^2\leqslant\delta_{|J|}\cdot(2t)^{2k-|J|}\overset{(2.5)}{<}\frac{t}{2^k}$$

and, hence, $\mathbb{E}\big[\sum_{J\subsetneq[k]}X_J\big]<t$. Consequently, there exists a choice of sets $W_i\subseteq V_i$ which, after removing one vertex for every instance counted by X_J for every $J\subsetneq[k]$, induces a rainbow copy of $K^{(k)}_{t,\dots,t}$. □

We shall use a variant of Proposition 2.3, where we move away from complete partite hypergraphs. Instead we start with a partite hypergraph of density d and we are interested in large rainbow subhypergraphs of similar density.

Proposition 2.4. *For $k \geqslant 2$ let $H = (V_1 \cupdot \dots \cupdot V_k, E)$ be a k-partite k-uniform hypergraph of density $d = \frac{|E|}{|V_1|\cdots|V_k|} > 0$ and for $\boldsymbol{\delta} = (\delta_0, \dots, \delta_{k-1}) \in (0,1]^k$ let $\varphi\colon E \longrightarrow \mathbb{N}$ be a $\boldsymbol{\delta}$-bounded colouring. If for some integer m we have*

$$\frac{2^{k+3}}{d} \leqslant m \leqslant \min\big\{|V_1|, \dots, |V_k|\big\} \quad \text{and} \quad \delta_j < \frac{1}{2^{k+1} \cdot m^{2k-j}} \tag{2.6}$$

for every $j = 0, \dots, k-1$, then H contains a rainbow subhypergraph of density at least $d/2$ with vertex classes $U_j \subseteq V_j$ and $|U_j| = m$ for every $j \in [k]$.

The proof of Proposition 2.4 parallels the proof of Proposition 2.3. Roughly speaking, we show that a randomly chosen subhypergraph inherits the density of H. However, for technical reasons we will refrain from removing vertices from the randomly chosen vertex sets and instead we require smaller values of δ_j in the assumption (2.6) leading to no expected obstructions for the rainbow subhypergraph.

Proof. For every $i \in [k]$ we choose a random subset $W_i \subseteq V_i$ of size m and these k choices are carried out independently. Let $H_{\mathcal{W}} = H[W_1, \dots, W_k]$ be the random subhypergraph induced on the chosen vertex sets. Again for every $J \subsetneq [k]$ we consider the random variable X_J counting the pairs of edges $\{e, e'\}$ spanned in $H_{\mathcal{W}}$ with

$$e \cap e' \in W_J \quad \text{and} \quad \varphi(e) = \varphi(e')\,.$$

Appealing to the $(\delta_{|J|}, J)$-boundedness of φ we have

$$\mathbb{E}X_J \overset{(2.4)}{\leqslant} \prod_{j \in J} \frac{m}{|V_j|} \cdot \prod_{j \notin J} \frac{m \cdot (m-1)}{|V_j| \cdot (|V_j| - 1)} \cdot \delta_{|J|} |V_J| |V_{[k] \smallsetminus J}|^2 < \delta_{|J|} \cdot m^{2k-|J|} \overset{(2.6)}{=} \frac{1}{2^{k+1}}$$

and, owing to Markov's inequality, we arrive at

$$\mathbb{P}\Big(\sum_{J \subsetneq [k]} X_J > 1\Big) < \frac{1}{2}\,. \tag{2.7}$$

For the edge density of $H_{\mathcal{W}}$ we consider the random variable $Y = \sum_{e \in E} \mathbb{1}_e = E(H_{\mathcal{W}})$. Clearly,

$$\mathbb{E}Y = \prod_{i=1}^{k} \frac{m}{|V_i|} \cdot |E| = dm^k$$

and we can bound the variance by

$$\operatorname{Var} Y \leqslant \mathbb{E}Y + \sum_{\varnothing \neq J \subsetneq [k]} \sum_{e \in E} \sum_{\substack{f \in E \\ e \cap f \in V_J}} \mathbb{E}[\mathbb{1}_e \mathbb{1}_f] \leqslant \mathbb{E}Y + \sum_{\varnothing \neq J \subsetneq [k]} m^{k-|J|} \cdot \mathbb{E}Y < 2^k m^{k-1} \mathbb{E}Y\,.$$

Consequently, Chebyshev's tells us

$$\mathbb{P}\Big(Y < \frac{1}{2}\mathbb{E}Y\Big) \leqslant \frac{\operatorname{Var} Y}{(\frac{1}{2}\mathbb{E}Y)^2} \leqslant \frac{2^{k+2} m^{k-1}}{dm^k} \leqslant \frac{2^{k+2}}{dm} \overset{(2.6)}{<} \frac{1}{2}\,. \tag{2.8}$$

Owing to the estimates (2.7) and (2.8), there exist subsets $U_i \subseteq V_i$ for every $i \in [k]$ each of size m such that no two edges of $H[U_1, \dots, U_k]$ have the same colour under φ and we have $e(H[U_1, \dots, U_k]) \geqslant dm^k/2$. □

§3 Proof the main result

In this section we deduce Theorem 1.2 from Propositions 2.1 – 2.4.

Proof of Theorem 1.2. We first address the general case giving a slightly weaker bound for $k = 2$ and 3. For a fixed integer k, let t is sufficiently large, set $c = 1/2^{2k+1}$. We fix auxiliary constants

$$\delta_j = \left(\frac{c}{t^k}\right)^{(2kt^k)^{k-1-j}} \qquad \text{and} \qquad m_j = \left(\frac{1}{\delta_j}\right)^{t^k} = \left(\frac{t^k}{c}\right)^{(2k)^{k-1-j}t^{k(k-j)}}$$

for $j = 2, \dots, k-1$ and we set

$$\delta_1 = \delta_0 = \left(\frac{c}{t^k}\right)^{(2kt^k)^{k-2}}$$

and $\boldsymbol{\delta} = (\delta_0, \dots, \delta_{k-1})$. Moreover, by assumption of Theorem 1.2 we have

$$n = t^{t^{k^2}}.$$

The choices above yield the following order of the involved constants

$$2 \leqslant k < \frac{1}{c} < t < \frac{1}{\delta_{k-1}} < m_{k-1} < \dots < \frac{1}{\delta_2} < m_2 < \frac{1}{\delta_1} = \frac{1}{\delta_0} < n\,.$$

Besides this monotonicity we shall employ a few more relations between these constants. In fact, the proof relies on Propositions 2.1 – 2.4 and their applications will be justified by the tailored inequalities (3.1)–(3.5) below. These estimates are mainly based on the facts that for $j \geqslant 2$ we have $m_j = \delta_j^{-t^k}$, that $\delta_{j-1} = \delta_j^{2kt^k}$, and that we assume that t is sufficiently large as a function of k.

We shall apply Proposition 2.1 in three different ways and for those applications we rely on the following three sets of inequalities

$$\left(\frac{\delta_0}{4^{k-1}}\right)^{t^{k-1}} n \geqslant 2t\,, \tag{3.1}$$

$$\frac{\delta_1}{4^{k-1}} \cdot \sqrt{\delta_1 n} \geqslant 2t \qquad \text{and} \qquad \left(\frac{\delta_1}{4^{k-1}}\right)^{t^{k-1}} n \geqslant 2t\,, \tag{3.2}$$

and for $j = 2, \dots, k-1$

$$\left(\frac{\delta_j^2/2}{4^{k-1}}\right)^{t^{j-1}} m_j \geqslant 2t \qquad \text{and} \qquad \left(\frac{\delta_j^2/2}{4^{k-1}}\right)^{t^{k-1}} n \geqslant 2t\,. \tag{3.3}$$

Similarly, preparing for an application of Proposition 2.3, we observe

$$\delta_{k-1} = \frac{1}{2^{2k+1} \cdot t^k} \quad \text{and} \quad \delta_j \leqslant \delta_{k-2} \leqslant \frac{1}{2^{3k-j} \cdot t^{2k-j-1}} \quad \text{for } j = 0, \dots, k-2\,. \tag{3.4}$$

Finally, for an intended application of Proposition 2.4, we note that for all $0 \leqslant j < j_\star \leqslant k-1$ and $j_\star \geqslant 2$ we have

$$\delta_{j_\star} m_{j_\star} > 2^{j_\star+3} \qquad \text{and} \qquad \frac{\delta_j}{\delta_{j_\star}} < \frac{1}{2^{j_\star+1} m_{j_\star}^{2j_\star-j}} \,. \tag{3.5}$$

This concludes the discussion of the constants.

We shall show that every colouring $\varphi\colon E(K^{(k)}_{n,\dots,n}) \longrightarrow \mathbb{N}$ yields a canonical copy of $K^{(k)}_{t,\dots,t}$. Let $V_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} V_k$ be the vertex partition of $K^{(k)}_{n,\dots,n}$. Given a colouring φ we consider several cases depending on the 'boundedness properties' of φ.

Note that, if φ is indeed $\boldsymbol{\delta}$-bounded, then in view of (3.4) our choice of $\boldsymbol{\delta}$ yields a rainbow coloured copy of $K^{(k)}_{t,\dots,t}$ by Proposition 2.3. Consequently, there exists a minimal index $j_\star \in \{0,\dots,k-1\}$ such that φ is not $(\delta_{j_\star}, j_\star)$-bounded and let $J_\star \in [k]^{(j_\star)}$ be a set that witnesses this and without loss of generality we may assume $J_\star = \{1,\dots,j_\star\}$.

In case $j_\star = 0$, then one of the colours appears at least $\delta_0 n^k$ times. In view of inequality (3.1), Proposition 2.1 applied with

$$d = \delta_0\,, \qquad n_1 = \dots = n_k = n\,, \qquad \text{and} \qquad t_1 = \dots = t_k = t\,,$$

yields a monochromatic copy of $K^{(k)}_{t,\dots,t}$ in this case.

In case $j_\star = 1$, then there is a set $U \subseteq V_1$ of size at least $\delta_1 n$ such that every $u \in U$ is contained in at least $\delta_1 n^{k-1}$ edges of the same colour and we denote this colour by $\ell(u)$.

The box principle yields a subset $U_\star \subseteq U$ of size at least $\sqrt{\delta_1 n}$ such that either all colours $\ell(u)$ for $u \in U_\star$ are equal or they are all distinct. We consider the k-partite k-uniform hypergraph $H_\star$ with vertex partition

$$U_\star \mathbin{\dot\cup} V_2 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} V_k$$

and

$$E(H_\star) = \bigcup_{u \in U_\star} \big\{ e \in V_{[k]} \colon u \in e \text{ and } \varphi(e) = \ell(u) \big\}\,.$$

Since every vertex $u \in U_\star$ has degree at least $\delta_1 n^{k-1}$, the hypergraph $H_\star$ has density at least δ_1. Again we apply Proposition 2.1 to $H_\star$, this time with

$$d = \delta_1\,, \quad n_1 = |U_\star| \geqslant \sqrt{\delta_1 n}\,, \quad n_2 = \dots = n_k = n\,, \quad \text{and} \quad t_1 = \dots = t_k = t\,,$$

which is justified by (3.2) and we obtain either a monochromatic or an $\{1\}$-canonical copy of $K^{(k)}_{t,\dots,t}$. This concludes the proof for the cases $j_\star \leqslant 1$.

It is left to consider the case $J_\star = \{1,\dots,j_\star\}$ for some $j_\star = 2,\dots,k-1$. Let $U_{J_\star} \subseteq V_{J_\star}$ be a set of size at least $\delta_{j_\star} n^{j_\star}$ such that every $j_\star$-tuple $S \in U_{J_\star}$ extends to at least $\delta_{j_\star} n^{k-j_\star}$ edges of the same colour and we denote this colour by $\ell(S)$. We consider the $j_\star$-partite $j_\star$-uniform hypergraph G with vertex partition

$$V_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} V_{j_\star} \quad \text{and} \quad E(G) = U_{J_\star}\,.$$

Moreover, we define a colouring $\varphi_G \colon E(G) \longrightarrow \mathbb{N}$ through

$$\varphi_G(S) = \ell(S)$$

for all $S \in E(G)$. Owing to the minimal choice of $j_\star$, a moment of thought reveals that the colouring φ_G is $(\delta_j/\delta_{j_\star}, j)$-bounded for every $j = 0, \dots, j_\star - 1$. In other words, the colouring φ_G of the $j_\star$-partite $j_\star$-uniform hypergraph G is $(\delta_0/\delta_{j_\star}, \dots, \delta_{j_\star-1}/\delta_{j_\star})$-bounded. In view of the estimates (3.5), we may apply Proposition 2.4 to G with

$$k = j_\star\,, \quad d = \delta_{j_\star}\,, \quad \big(\delta_0/\delta_{j_\star}, \dots, \delta_{j_\star-1}/\delta_{j_\star}\big)\,, \quad \text{and} \quad m_{j_\star}\,.$$

This way we obtain a rainbow $j_\star$-uniform subhypergraph $G_\star$ of density at least $\delta_{j_\star}/2$ with vertex classes $U_j \subseteq V_j$ and $|U_j| = m$ for every $j \in [j_\star]$.

Finally, we consider the natural k-uniform extension $H_\star$ of $G_\star$ on the vertex partition

$$U_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} U_{j_\star} \mathbin{\dot\cup} V_{j_\star+1} \mathbin{\dot\cup} \dots \mathbin{\dot\cup} V_k$$

with

$$E(H_\star) = \bigcup_{S \in E(G_\star)} \big\{e \in V_{[k]} \colon S \subseteq e \text{ and } \varphi(e) = \ell(S)\big\}$$

and note that the colouring φ restricted to $H_\star$ is $J_\star$-canonical. Moreover, since every $j_\star$-tuple $S \in E(G_\star)$ extends to at least $\delta_{j_\star} n^{k-j_\star}$ distinct k-tuples of colour $\ell(S)$ under φ, the k-uniform hypergraph $H_\star$ has density at least $\delta_{j_\star}^2/2$. Another application of Proposition 2.1 to $H_\star$ with

$$d = \frac{\delta_{j_\star}^2}{2}\,, \quad n_1 = \dots = n_{j_\star} = m\,, \quad n_{j_\star+1} = \dots = n_k = n\,, \quad \text{and} \quad t_1 = \dots = t_k = t\,,$$

which is justified by the estimates (3.3), yields a $J_\star$-canonical copy of $K^{(k)}_{t,\dots,t}$. This concludes the proof of Theorem 1.2 for $k \geqslant 4$ and it is left to discuss the better bounds on n for the cases $k = 2$ and $k = 3$.

For the case $k = 2$ one can check that the same proof works for $n = t^{3(t+1)}$ for sufficiently large t with $\delta_1 = \delta_0 = 2^{-6}t^{-3}$. In fact, for graphs the proof is somewhat simpler, since the case $j_\star \geqslant 2$ does not arise.

Similarly, for $k = 3$ and $n = t^{30t^3}$ one can check that the choices

$$\delta_2 = \frac{1}{2^7 t^3}\,, \qquad m_2 = t^{7t}\,, \qquad \text{and} \qquad \delta_1 = \delta_0 = \frac{1}{2^{10} \cdot t^{29t}}$$

satisfy inequalities (3.1)–(3.5) for sufficiently large t and, consequently, the proof presented yields the claimed bound in this case. □

References

[1] N. Alon, T. Jiang, Z. Miller, and D. Pritikin, *Properly colored subgraphs and rainbow subgraphs in edge-colorings with local constraints*, Random Structures Algorithms **23** (2003), no. 4, 409–433, DOI 10.1002/rsa.10102. MR2016871 ↑2.2

[2] L. Babai, *An anti-Ramsey theorem*, Graphs Combin. **1** (1985), no. 1, 23–28, DOI 10.1007/BF02582925. MR796179 ↑2.2

[3] D. Dobák and E. Mulrenin, *Sharp exponents for bipartite Erdős-Rado numbers*, available at arXiv:2410.08982. ↑1, 1, 2

[4] P. Erdős, *On extremal problems of graphs and generalized graphs*, Israel J. Math. **2** (1964), 183–190, DOI 10.1007/BF02759942. MR183654 ↑1, 2.1

[5] P. Erdős and A. Hajnal, *Ramsey-type theorems*, Discrete Appl. Math. **25** (1989), no. 1-2, 37–52, DOI 10.1016/0166-218X(89)90045-0. Combinatorics and complexity (Chicago, 1987). MR1031262 ↑1

[6] P. Erdős, A. Hajnal, and R. Rado, *Partition relations for cardinal numbers*, Acta Math. Acad. Sci. Hungar. **16** (1965), 93–196, DOI 10.1007/BF01886396. MR202613 ↑1

[7] P. Erdős and R. Rado, *A combinatorial theorem*, J. London Math. Soc. **25** (1950), 249–255, DOI 10.1112/jlms/s1-25.4.249. MR37886 ↑1

[8] L. Gishboliner, A. Milojević, B. Sudakov, and Y. Wigderson, *Canonical Ramsey numbers of sparse graphs*, available at arXiv:2410.08644. ↑1

[9] T. Kövari, V. T. Sós, and P. Turán, *On a problem of K. Zarankiewicz*, Colloq. Math. **3** (1954), 50–57, DOI 10.4064/cm-3-1-50-57. MR65617 ↑1, 2.1

[10] H. Lefmann and V. Rödl, *On Erdős–Rado numbers*, Combinatorica **15** (1995), no. 1, 85–104, DOI 10.1007/BF01294461. MR1325273 ↑1, 2.2

[11] R. Rado, *Direct decomposition of partitions*, J. London Math. Soc. **29** (1954), 71–83, DOI 10.1112/jlms/s1-29.1.71. MR65616 ↑1

[12] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. (2) **30** (1930), no. 4, 264–286, DOI 10.1112/plms/s2-30.1.264. MR1576401 ↑1

[13] Chr. Reiher, V. Rödl, M. Sales, K. Sames, and M. Schacht, *On quantitative aspects of a canonisation theorem for edge-orderings*, J. Lond. Math. Soc. (2) **106** (2022), no. 3, 2773–2803, DOI 10.1112/jlms.12648. MR4498567 ↑1

[14] S. Shelah, *Finite canonization*, Comment. Math. Univ. Carolin. **37** (1996), no. 3, 445–456. MR1426909 ↑1

Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany
Email address: `matias.azocar.carvajal@uni-hamburg.de`

Departamento de Ingeniería Matemática, Universidad de Chile, Santiago, Chile
Email address: `gsantos@dim.uchile.cl`

Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany
Email address: `schacht@math.uni-hamburg.de`