TURÁN DENSITY OF CLIQUES OF ORDER FIVE IN 3-UNIFORM HYPERGRAPHS WITH QUASIRANDOM LINKS

SÖREN BERGER, SIMÓN PIGA, CHRISTIAN REIHER, VOJTĚCH RÖDL, AND MATHIAS SCHACHT

Abstract. We show that 3-uniform hypergraphs with the property that all vertices have a quasirandom link graph with density bigger than 1/3 contain a clique on five vertices. This result is asymptotically best possible.

§1. Introduction

1.1. **Turán problems for uniformly dense hypergraphs.** We study extremal problems for 3-uniform hypergraphs and here, unless stated otherwise, a hypergraph will always be 3-uniform. Recall that given an integer n and a hypergraph F the extremal number $ex(n, F)$ is the maximum number of hyperedges that an n-vertex hypergraph can have without containing a copy of F. It is well known that the sequence $ex(n, F)/\binom{n}{3}$ converges and the limit defines the *Turán density* $\pi(F)$. Determining $\pi(F)$ is a central open problem in extremal combinatorics. In fact, even the case when F is a clique on four vertices is still unresolved and known as the 5/9-conjecture of Turán. Consequently, several interesting and still challenging variations were considered.

One such variation, suggested originally by Erdős and Sós [1], restricts the problem only to those F-free hypergraphs that are *uniformly dense* among large sets of vertices. More precisely, given a hypergraph F, Erdős and Sós asked for the supremum $d \in [0, 1]$ such that there exist arbitrarily large F-free hypergraphs $H = (V, E)$ for which every linear sized subset of the vertices induces a hypergraph of density at least d.

For hypergraphs there are several different notions of "uniform density" (see, e.g., [2, 4–6]) and we shall focus on the following notion.

Definition 1.1. *For a hypergraph $H = (V, E)$ and reals $d \in [0, 1]$, $\eta > 0$, we say that H is $(\eta, d, \wedge)$-dense if for all P, $Q \subseteq V \times V$ we have*

$$e_{\wedge}(P, Q) = \Big|\Big\{\big((x, y), (y, z)\big) \in \mathcal{K}_{\wedge}(P, Q) \colon \{x, y, z\} \in E\Big\}\Big| \geqslant d\,\big|\mathcal{K}_{\wedge}(P, Q)\big| - \eta|V|^3, \qquad (1)$$

where $\mathcal{K}_{\wedge}(P, Q) = \big\{\big((x, y), (y', z)\big) \in P \times Q \colon y = y'\big\}$.

2020 *Mathematics Subject Classification.* 05C35 (primary), 05C65, 05C80 (secondary).

Key words and phrases. hypergraphs, extremal graph theory, Turán's problem, quasirandomness.

The first and the fifth author are supported by ERC Consolidator Grant PEPCo 724903. The second author is supported by ANID/CONICYT Acuerdo Bilateral DAAD/62170017 through a Ph.D. Scholarship. The fourth author is supported by NSF grant DMS 1764385.

For a fixed hypergraph F, we define the corresponding Turán density

$$\pi_{\wedge}(F) = \sup\{d \in [0,1] \colon \text{for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists an } F\text{-free}, \\ (\eta, d, \wedge)\text{-dense hypergraph with } |V(H)| \geqslant n\}\,.$$

In [3] the last three authors obtained a general upper bound for $\pi_{\wedge}(K_t^{(3)})$, which turned out to be best possible for all $t \leqslant 16$ except for $t = 5$, 9, and 10.

Theorem 1.2. *For every integer $r \geqslant 2$ we have*

$$\pi_{\wedge}(K_{2^r}^{(3)}) \leqslant \frac{r-2}{r-1}\,.$$

Moreover, we have

$$0 = \pi_{\wedge}(K_4^{(3)}) < \frac{1}{3} \leqslant \pi_{\wedge}(K_5^{(3)}) \leqslant \frac{1}{2} = \pi_{\wedge}(K_6^{(3)}) = \dots = \pi_{\wedge}(K_8^{(3)}) \leqslant \pi_{\wedge}(K_9^{(3)}) \leqslant \pi_{\wedge}(K_{10}^{(3)}) \\ \leqslant \frac{2}{3} = \pi_{\wedge}(K_{11}^{(3)}) = \dots = \pi_{\wedge}(K_{16}^{(3)})\,. \qquad \square$$

Here we improve the bound for $K_5^{(3)}$ and show that the lower bound 1/3 is best possible.

Theorem 1.3 (Main result)**.** *We have that*

$$\pi_{\wedge}(K_5^{(3)}) = \frac{1}{3}\,.$$

Theorem 1.3 has a consequence for hypergraphs with quasirandom links. For a hypergraph H and a vertex x, define the *link graph of* x, $L_H(x)$ by the graph with vertex set V and edges $\{yz \in V^{(2)} \colon xyz \in E(H)\}$. Recall that for given $d \in [0,1]$ and $\delta > 0$ we say that a graph $G = (V, E)$ is (δ, d)-quasirandom if for every subset of vertices $X \subseteq V$ the number of edges $e(X)$ inside X satisfies

$$\left|e(X) - d\tbinom{|X|}{2}\right| \leqslant \delta |V|^2\,.$$

One can check that if all the vertices of a hypergraph H have a (δ, d)-quasirandom link graph, then H is $(f(\delta), d, \wedge)$-dense, where $f(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$. In fact, such hypergraphs would even satisfy in addition a matching upper bound for $e_{\wedge}(P, Q)$ in (1) and, hence, having quasirandom links is a stronger property. However, the lower bound construction for $\pi_{\wedge}(K_5^{(3)})$ (see below) has quasirandom links with density 1/3 and, therefore, Theorem 1.3 yields an asymptotically optimal result for such hypergraphs.

Example 1.4. *For a map $\psi \colon V^{(2)} \longrightarrow \mathbb{Z}/3\mathbb{Z}$ we define the hypergraph $H_\psi = (V, E)$ by*

$$xyz \in E \qquad \Longleftrightarrow \qquad \psi(xy) + \psi(xz) + \psi(zy) \equiv 1 \pmod 3\,. \tag{2}$$

Observe that for any set of five different vertices $U = \{u_1, u_2, u_3, u_4, u_5\}$ the following equality follows by double counting

$$\sum_{u_i u_j u_k \in U^{(3)}} \big(\psi(u_i u_j) + \psi(u_i u_k) + \psi(u_j u_k)\big) = 3 \sum_{u_i u_j \in U^{(2)}} \psi(u_i u_j)\,.$$

Since the second sum is zero modulo 3 at least one of the ten triplets in the first sum fails to satisfy (2). Consequently, H_ψ is $K_5^{(3)}$-free for every map ψ.

Moreover, if ψ is chosen uniformly at random, then it is not difficult to show that for every fixed $\delta > 0$ with high probability the hypergraph H_ψ has the property that all link graphs are $(\delta, 1/3)$-quasirandom.

Summarising the discussion above we arrive at the following corollary, which in light of Example 1.4 is asymptotically best possible.

Corollary 1.5. *For every $\varepsilon > 0$ there exist $\delta > 0$ and an integer n_0 such that every hypergraph on at least n_0 vertices with all link graphs being $(\delta, 1/3 + \varepsilon)$-quasirandom contains a copy of $K_5^{(3)}$.*

The proof of Theorem 1.3 is based on the regularity method for hypergraphs and in the next section we recall the relevant concepts. We then sketch the main ideas of the proof in Section 3. Roughly speaking, after an application of the hypergraph regularity method, the proof splits into two parts (see Propositions 3.1 and 3.2). The proof of Proposition 3.1 will appear in the full version of the article and the proof Proposition 3.2 is presented in Section 3.

§2. Hypergraph regularity and reduced hypergraphs

Given a large hypergraph $H = (V, E)$, the regularity lemma for hypergraphs provides a vertex partition $V_1, V_2, \dots, V_t$ together with partitions $\mathcal{P}^{ij}$ of the edges of the complete bipartite graphs between all $\binom{t}{2}$ pairs of classes V_i, V_j. Each class $P^{ij} \in \mathcal{P}^{ij}$ is ε-regular in the sense of Szemerédi's regularity lemma for graphs. Moreover, the hypergraph H is "regular" among most *triads*, i.e., among most of the tripartite graphs

$$P^{ijk}_{\alpha\beta\gamma} = P^{ij}_{\alpha} \cup P^{ik}_{\beta} \cup P^{jk}_{\gamma}$$

with $P^{ij}_\alpha \in \mathcal{P}^{ij}$, $P^{ik}_\beta \in \mathcal{P}^{ik}$, and $P^{jk}_\gamma \in \mathcal{P}^{jk}$. Roughly speaking, here "regular" means, that the hyperedges of H match the same proportion of triangles for every tripartite subgraph of such a triad.

Important structural properties of a hypergraph H after an application of the hypergraph regularity lemma can be captured by the *reduced hypergraph*, which can be viewed as a generalisation of the reduced graph in the context of Szemerédi's regularity lemma for graphs. Given a set of indices I of size t and pairwise disjoint, non-empty sets of vertices $\mathcal{P}^{ij}$ for every pair of indices $ij \in I^{(2)}$, let for every triplet of distinct indices $ijk \in I^{(3)}$ a tripartite hypergraph $\mathcal{A}^{ijk}$ with vertex classes $\mathcal{P}^{ij}$, $\mathcal{P}^{ik}$, and $\mathcal{P}^{jk}$ be given. We consider the disjoint union of all those hyperedges and, hence, we obtain a $\binom{|I|}{2}$-partite hypergraph $\mathcal{A}$ with

$$V(\mathcal{A}) = \dot{\bigcup_{ij \in I^{(2)}}} \mathcal{P}^{ij} \qquad \text{and} \qquad E(\mathcal{A}) = \dot{\bigcup_{ijk \in I^{(3)}}} E(\mathcal{A}^{ijk})\,.$$

We say $\mathcal{A}$ is a *reduced* hypergraph with *index set I*, *vertex classes $\mathcal{P}^{ij}$*, and *constituents $\mathcal{A}^{ijk}$*.

An application of the hypergraph regularity lemma to a given hypergraph H naturally defines a reduced hypergraph $\mathcal{A}$ in which the vertices $P^{ij} \in \mathcal{P}^{ij}$ represent a set of pairs between the vertex classes V_i and V_j. Moreover, a hyperedge $P^{ij}_{\alpha}P^{ik}_{\beta}P^{jk}_{\gamma}$ in the reduced hypergraph signifies that H is regular and dense on the triad $P^{ijk}_{\alpha\beta\gamma}$.

As mentioned above the properties of the hypergraph H are often transferred to the reduced hypergraph. We consider $\wedge$-dense and $K_5^{(3)}$-free hypergraphs H and below we discuss the corresponding properties for the reduced hypergraph $\mathcal{A}$ after an appropriate application of the hypergraph regularity lemma.

Roughly speaking, the $\wedge$-density condition translates into a minimal codegree condition for almost all pairs of vertices from different vertex classes in almost all constituents of the reduced graphs. However, one can always move to a large reduced hypergraph in which *all* pairs of vertices have large codegree (see [2] for details). This inspires the following definition of $(d, \wedge)$-density for reduced hypergraphs. For a given real number $d \in [0,1]$, we say that a reduced hypergraph $\mathcal{A}$ is $(d, \wedge)$-dense, if for every three distinct indices $ijk \in I^{(3)}$ and vertices $P^{ij} \in \mathcal{P}^{ij}$ and $P^{ik} \in \mathcal{P}^{ik}$ we have

$$d\big(P^{ij}, P^{ik}\big) = \big|\{P^{jk} \in \mathcal{P}^{jk} \colon P^{ij}P^{ik}P^{jk} \in E(\mathcal{A}^{ijk})\}\big| \geqslant d\,|\mathcal{P}^{jk}|\,.$$

As discussed above (see [2] for details), an appropriate application of the hypergraph regularity lemma to a $(\eta, d+\varepsilon, \wedge)$-dense hypergraph H allows a reduction to a $(d+\varepsilon/2, \wedge)$-dense reduced hypergraphs.

Furthermore, we say a reduced hypergraph $\mathcal{A}$ with index set I *supports a* $K_5^{(3)}$ if there is a 5-element subset $J \subseteq I$ and vertices $P^{ij} \in \mathcal{P}^{ij}$ for every $ij \in J^{(2)}$ such that all ten triples $P^{ij}P^{ik}P^{jk}$ with $ijk \in J^{(3)}$ are hyperedges present in $\mathcal{A}$. Note that, if the reduced hypergraph $\mathcal{A}$ defined from a hypergraph H through an appropriate application of the regularity lemma supports a $K_5^{(3)}$, then the embedding/counting lemma yields a $K_5^{(3)} \subseteq H$.

The discussion above reduces the proof of Theorem 1.3 to the following statement for $\wedge$-dense reduced hypergraphs.

Proposition 2.1. *For every $\varepsilon > 0$ every sufficiently large $\big(\frac{1}{3}+\varepsilon, \wedge\big)$-dense reduced hypergraph $\mathcal{A}$ supports a $K_5^{(3)}$.*

§3. Proof of Proposition 2.1

Our proof of Proposition 2.1 is divided in two main parts. First we reduce the problem to the case in which the reduced hypergraph $\mathcal{A}$ on some index set I can be *bicoloured*. By this we mean that there is a colouring $\varphi\colon V(\mathcal{A}) \longrightarrow \{red, blue\}$ of the vertices such that for every $ij \in I^{(2)}$ we have

$$\varphi^{-1}(red) \cap \mathcal{P}^{ij} \neq \varnothing \qquad \text{and} \qquad \varphi^{-1}(blue) \cap \mathcal{P}^{ij} \neq \varnothing \tag{3}$$

and there are no hyperedges in $\mathcal{A}$ with all three vertices of the same colour. Given such a colouring φ, we define the *minimum monochromatic codegree density of $\mathcal{A}$ and φ* by

$$\delta_2(\mathcal{A},\varphi) = \min_{ijk\in I^{(3)}} \min\Big\{\frac{d(P^{ij},P^{ik})}{|\mathcal{P}^{jk}|}\colon P^{ij}\in\mathcal{P}^{ij},\ P^{ik}\in\mathcal{P}^{ik},\ \text{and}\ \varphi(P^{ij})=\varphi(P^{ik})\Big\}.$$

The following proposition reduces Proposition 2.1 to bicoloured reduced hypergraphs.

Proposition 3.1. *Given $\varepsilon>0$ there exists $\delta>0$ such that for every $t\in\mathbb{N}$ the following holds. If $\mathcal{A}$ is a sufficiently large reduced $(\frac{1}{3}+\varepsilon,\boldsymbol{\wedge})$-dense hypergraph, then one of the following statements holds*

(*i*) *$\mathcal{A}$ supports a $K_5^{(3)}$ or*
(*ii*) *there exists a bicoloured reduced hypergraph $\mathcal{A}_*$ with $\delta_2(\mathcal{A}_*,\varphi)\geqslant\frac{1}{3}+\delta$ and with an index set of size at least t, which does not support a $K_5^{(3)}$.*

For the proof of Proposition 3.1 we mainly analyse holes in the reduced hypergraph $\mathcal{A}$ that do not support a $K_5^{(3)}$. It turns out that essentially the whole vertex set can be covered by two almost disjoint holes, which then can be used to define an appropriate colouring φ (and the details will appear in the full version of the article). The next proposition completes the proof of Proposition 2.1 by contradicting alternative (*ii*) of Proposition 3.1.

Proposition 3.2. *For every $\varepsilon>0$ every sufficiently large bicoloured reduced hypergraph $\mathcal{A}$ with $\delta_2(\mathcal{A},\varphi)\geqslant\frac{1}{3}+\varepsilon$ supports a $K_5^{(3)}$.*

Proof. Given $\varepsilon>0$ we fix a sufficiently small auxiliary constant ξ with $0<\xi\ll\varepsilon$ such that $\frac{1/6-\varepsilon}{\xi}$ equals to some integer s. Moreover, let I be a sufficiently large index set such that its cardinality satisfies the partition relation $|I|\longrightarrow(5)^2_s$, i.e., it is at least as large as the s-colour Ramsey number for the graph clique K_5. Let $\mathcal{A}$ be a bicoloured reduced hypergraph with index set I and vertex classes $\mathcal{P}^{ij}$ for $ij\in I^{(2)}$ and let $\varphi\colon V(\mathcal{A})\longrightarrow\{\mathit{red},\mathit{blue}\}$ satisfy $\delta_2(\mathcal{A},\varphi)\geqslant 1/3+\varepsilon$.

In the proof we shall use the following notation. For two vertices P, $P'\in V(\mathcal{A})$ and a subset $\mathcal{U}\subseteq V(\mathcal{A})$ we denote by $N_{\mathcal{U}}(P,P')$ the neighbourhood restricted to U. Similarly, for two subsets $\mathcal{U},\mathcal{U}'\subseteq V(\mathcal{A})$ we write $N_{\mathcal{U}\times\mathcal{U}'}(P)$ for the set of ordered pairs $(U,U')\in\mathcal{U}\times\mathcal{U}'$ such that PUU' is a hyperedge in $\mathcal{A}$.

For every $ij\in I^{(2)}$ set

$$\mathcal{R}^{ij}=\varphi^{-1}(\mathit{red})\cap\mathcal{P}^{ij}\qquad\text{and}\qquad \varrho_{ij}=\frac{|\mathcal{R}^{ij}|}{|\mathcal{P}^{ij}|}$$

and, analogously, we define $\mathcal{B}^{ij}=\varphi^{-1}(\mathit{blue})\cap\mathcal{P}^{ij}$ and $\beta_{ij}=|\mathcal{B}^{ij}|/|\mathcal{P}^{ij}|$. In view of (3), the assumption on $\delta_2(\mathcal{A},\varphi)$ implies that all ϱ_{ij}, $\beta_{ij}\in[1/3+\varepsilon,2/3-\varepsilon]$. Splitting the interval $[1/3+\varepsilon,2/3-\varepsilon]$ into s intervals of length 2ξ, the size of I yields a subset $J\subseteq I$ of size 5 such that all β_{ij} with $ij\in J^{(2)}$ are in the same interval. Let β be the centre of this interval and set $\varrho = 1 - \beta$. We thus arrive at

$$\beta_{ij} = \beta \pm \xi \qquad \text{and} \qquad \varrho_{ij} = \varrho \pm \xi$$

for all $ij \in J^{(2)}$. Without loss of generality we may assume $\beta \leqslant \varrho$, which implies

$$\frac{1}{3} + \varepsilon \leqslant \beta - \xi < \beta \leqslant \frac{1}{2} \leqslant \varrho < \varrho + \xi \leqslant \frac{2}{3} - \varepsilon\,. \tag{4}$$

For $ijk \in J^{(3)}$ the codegree condition translates for *red* vertices $R^{ij} \in \mathcal{R}^{ij}$ and $R^{ik} \in \mathcal{R}^{ik}$ to

$$\begin{aligned} |N_{\mathcal{B}^{jk}}(R^{ij}, R^{ik})| = d(R^{ij}, R^{jk}) &\geqslant \left(\frac{1}{3} + \varepsilon\right) |\mathcal{P}^{jk}| \\ &\geqslant \left(\frac{1}{3} + \varepsilon\right) \left(\frac{1}{\beta + \xi}\right) |\mathcal{B}^{jk}| \geqslant \left(\frac{1}{3\beta} + \frac{\varepsilon}{2}\right) |\mathcal{B}^{jk}|\,, \end{aligned} \tag{5}$$

where we used $\xi \ll \varepsilon, \beta$ for the last inequality. Similarly, for *blue* vertices we have

$$|N_{\mathcal{R}^{jk}}(B^{ij}, B^{ik})| \geqslant \left(\frac{1}{3\varrho} + \frac{\varepsilon}{2}\right) |\mathcal{R}^{jk}|\,. \tag{6}$$

We may rename the indices in J and assume that $J = \mathbb{Z}/5\mathbb{Z}$. We shall show that $\mathcal{A}$ restricted to J supports a $K_5^{(3)}$. For that we have to find ten vertices $p^{ij} \in \mathcal{P}^{ij}$ one for every $ij \in J^{(2)}$ such that for all of the ten triples $ijk \in J^{(3)}$ the vertices p^{ij}, p^{ik}, and p^{jk} span a hyperedge in the constituent $\mathcal{A}^{ijk}$. For every $i \in J = \mathbb{Z}/5\mathbb{Z}$ we will select $p^{i,i+1}$ from $\mathcal{B}^{i,i+1}$ and $p^{i,i+2}$ from $\mathcal{R}^{i,i+2}$. (In fact, it is easy to see that up to symmetry this choice for the colour classes is unavoidable, as it corresponds to the unique 2-colouring of $E(K_5)$ with no monochromatic triangle.) We stress the colour of our choices by writing $b^{i,i+1}$ and $r^{i,i+2}$ for the chosen p^{ij} and we begin with the selection of r^{14}.

Applying (6) to all pairs of vertices $B^{15} \in \mathcal{B}^{15}$ and $B^{45} \in \mathcal{B}^{45}$ implies that the total number of hyperedges in $\mathcal{A}^{145}$ crossing the sets $\mathcal{R}^{14}$, $\mathcal{B}^{15}$, and $\mathcal{B}^{45}$ is at least

$$|\mathcal{B}^{15}||\mathcal{B}^{45}| \cdot \left(\frac{1}{3\varrho} + \frac{\varepsilon}{2}\right) |\mathcal{R}^{14}|\,.$$

Consequently, we can fix some vertex $r^{14} \in \mathcal{R}^{14}$ such that

$$|N_{\mathcal{B}^{15} \times \mathcal{B}^{45}}(r^{14})| \geqslant \left(\frac{1}{3\varrho} + \frac{\varepsilon}{2}\right) |\mathcal{B}^{15}||\mathcal{B}^{45}|\,. \tag{7}$$

The following claim fixes the four vertices b^{12}, b^{34} and r^{13}, r^{24}.

Claim 1. *There exist blue vertices $b^{12} \in \mathcal{B}^{12}$, $b^{34} \in \mathcal{B}^{34}$ and red vertices $r^{13} \in \mathcal{R}^{13}$, $r^{24} \in \mathcal{R}^{24}$ such that*

(*i*) *$b^{12}r^{14}r^{24}$ and $r^{13}r^{14}b^{34}$ are hyperedges in $\mathcal{A}$ and*

(*ii*) $|N_{\mathcal{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathcal{B}^{23}}(r^{24}, b^{34})| \geqslant \left(1 - \frac{1}{3\beta}\right) |\mathcal{B}^{23}|$.

Proof. Owing to (5) for every $R^{13} \in \mathcal{R}^{13}$ we have $d(R^{13}, r^{14}) \geqslant \left(\frac{1}{3\beta} + \frac{\varepsilon}{2}\right)|\mathcal{B}^{34}|$ and, hence, there is a vertex $b^{34} \in \mathcal{B}^{34}$ such that

$$|N_{\mathcal{R}^{13}}(r^{14}, b^{34})| \geqslant \left(\frac{1}{3\beta} + \frac{\varepsilon}{2}\right) |\mathcal{R}^{13}| \geqslant \frac{\varrho}{3\beta}|\mathcal{P}^{13}|\,. \tag{8}$$

Similarly, we can fix a vertex $r^{24} \in \mathcal{R}^{24}$ such that

$$|N_{\mathcal{B}^{23}}(r^{24}, b^{34})| \geqslant \frac{1}{3\varrho}|\mathcal{B}^{23}|\,. \tag{9}$$

Recalling that $|\mathcal{R}^{13}| \leqslant (\varrho + \xi)|P^{13}|$ for every $B^{12} \in \mathcal{B}^{12}$ and $B^{23} \in \mathcal{B}^{23}$ we have

$$\begin{aligned}\big|N_{\mathcal{R}^{13}}(B^{12}, B^{23}) \cap N_{\mathcal{R}^{13}}(r^{14}, b^{34})\big| &\geqslant \left(\frac{1}{3} + \varepsilon\right)|\mathcal{P}^{13}| + \big|N_{\mathcal{R}^{13}}(r^{14}, b^{34})\big| - |\mathcal{R}^{13}| \\ &\geqslant \big|N_{\mathcal{R}^{13}}(r^{14}, b^{34})\big| - \left(\varrho + \xi - \frac{1}{3} - \varepsilon\right)|\mathcal{P}^{13}| \\ &\overset{(8)}{\geqslant} \left(1 - 3\beta + \frac{\beta}{\varrho}\right)\big|N_{\mathcal{R}^{13}}(r^{14}, b^{34})\big|\,.\end{aligned}$$

Hence, the number of hyperedges crossing $N_{\mathcal{B}^{12}}(r^{14}, r^{24})$, $N_{\mathcal{B}^{23}}(r^{24}, b^{34})$, and $N_{\mathcal{R}^{13}}(r^{14}, b^{34})$ is at least

$$|N_{\mathcal{B}^{12}}(r^{14}, r^{24})||N_{\mathcal{B}^{23}}(r^{24}, b^{34})| \cdot \left(1 - 3\beta + \frac{\beta}{\varrho}\right)|N_{\mathcal{R}^{13}}(r^{14}, b^{34})|\,.$$

Consequently, there exist $b^{12} \in N_{\mathcal{B}^{12}}(r^{14}, r^{24})$ and $r^{13} \in N_{\mathcal{R}^{13}}(r^{14}, b^{34})$ such that

$$\begin{aligned}|N_{\mathcal{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathcal{B}^{23}}(r^{24}, b^{34})| &\geqslant \left(1 - 3\beta + \frac{\beta}{\varrho}\right)|N_{\mathcal{B}^{23}}(r^{24}, b^{34})| \\ &\overset{(9)}{\geqslant} \left(\frac{1}{3\varrho} - \frac{\beta}{\varrho} + \frac{\beta}{3\varrho^2}\right)|\mathcal{B}^{23}| \\ &\geqslant \left(1 - \frac{1}{3\beta}\right)|\mathcal{B}^{23}|\,,\end{aligned}$$

where the last inequality follows from the identity $\varrho = 1 - \beta$. □

The next claim fixes the four vertices b^{15}, b^{45} and r^{25}, r^{35}. Together with Claim 1 this fixes all vertices except b^{23} and both claims guarantee those seven hyperedges supporting a $K_5^{(3)}$ that do not involve b^{23}.

Claim 2. *There exist blue vertices $b^{15} \in \mathcal{B}^{15}$, $b^{45} \in \mathcal{B}^{45}$ and red vertices $r^{25} \in \mathcal{R}^{25}$, $r^{35} \in \mathcal{R}^{35}$ such that $b^{12}b^{15}r^{25}$, $r^{13}b^{15}r^{35}$, $r^{14}b^{15}b^{45}$, $r^{24}r^{25}b^{45}$, and $b^{34}r^{35}b^{45}$ are hyperedges in $\mathcal{A}$.*

Proof. Consider the following sets of pairs in $\mathcal{B}^{15} \times \mathcal{B}^{45}$.

$$\begin{aligned} G_1 &= \{(B^{15}, B^{45}) \in \mathcal{B}^{15} \times \mathcal{B}^{45} \colon N_{\mathcal{R}^{25}}(b^{12}, B^{15}) \cap N_{\mathcal{R}^{25}}(r^{24}, B^{45}) \neq \varnothing\} \\ \text{and}\quad G_2 &= \{(B^{15}, B^{45}) \in \mathcal{B}^{15} \times \mathcal{B}^{45} \colon N_{\mathcal{R}^{35}}(b^{13}, B^{15}) \cap N_{\mathcal{R}^{35}}(b^{34}, B^{45}) \neq \varnothing\}\,.\end{aligned}$$

Note that for every $B^{15} \in \mathcal{B}^{15}$ there is some $R^{25} \in N_{\mathcal{R}^{25}}(b^{12}, B^{15})$ and we have

$$|N_{\mathcal{B}^{45}}(r^{24}, R^{25})| \overset{(5)}{\geqslant} \frac{1}{3\beta}|\mathcal{B}^{45}|\,.$$

Clearly, $\{B^{15}\} \times N_{\mathcal{B}^{45}}(r^{24}, R^{25}) \subseteq G_1$ and, hence, we establish

$$|G_1| \geqslant \frac{1}{3\beta}|\mathcal{B}^{15}||\mathcal{B}^{45}|\,. \tag{10}$$

A symmetric argument yields the same bound for G_2. Combining (10) and the same bound for G_2 with (7) leads to

$$|G_1| + |G_2| + |N_{\mathcal{B}^{15}\times\mathcal{B}^{45}}(r^{14})| \geqslant \left(\frac{2}{3\beta} + \frac{1}{3\varrho} + \frac{\varepsilon}{2}\right)|\mathcal{B}^{15}||\mathcal{B}^{45}| \overset{(4)}{>} 2\,|\mathcal{B}^{15}||\mathcal{B}^{45}|\,.$$

Consequently, we can fix a pair $(b^{15}, b^{45}) \in G_1 \cap G_2 \cap N_{\mathcal{B}^{15}\times\mathcal{B}^{45}}(r^{14})$. Moreover, having fixed b^{15} and b^{45} this defines a vertex $r^{25} \in \mathcal{R}^{25}$ from the non-empty intersection considered in the definition of G_1. Similarly, G_2 leads to our choice of $r^{35} \in \mathcal{R}^{35}$.

Since $(b^{15}, b^{45}) \in N_{\mathcal{B}^{15}\times\mathcal{B}^{45}}(r^{14})$, the hyperedge $r^{14}b^{15}b^{45}$ exists in $\mathcal{A}$ and the other four hyperedges are a result of the definition of G_1 and G_2. □

As mentioned above, Claims 1 and 2 fix all vertices except $b^{23} \in \mathcal{B}^{23}$ and all hyperedges not involving b^{23}. For the three remaining hyperedges it suffices to show that

$$N_{\mathcal{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathcal{B}^{23}}(r^{24}, b^{34}) \cap N_{\mathcal{B}^{23}}(r^{25}, r^{35}) \neq \varnothing\,.$$

Claim 1 *(ii)* and (5) imply

$$\begin{aligned}
&\left|N_{\mathcal{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathcal{B}^{23}}(r^{24}, b^{34}) \cap N_{\mathcal{B}^{23}}(r^{25}, r^{35})\right| \\
&\qquad \geqslant \left|N_{\mathcal{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathcal{B}^{23}}(r^{24}, b^{34})\right| + \left|N_{\mathcal{B}^{23}}(r^{25}, r^{35})\right| - \left|\mathcal{B}^{23}\right| \\
&\qquad \overset{(5)}{\geqslant} \left(1 - \frac{1}{3\beta} + \frac{1}{3\beta} + \frac{\varepsilon}{2} - 1\right)|\mathcal{B}^{23}| > 0\,.
\end{aligned}$$

Hence a choice for $b^{23} \in N_{\mathcal{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathcal{B}^{23}}(r^{24}, b^{34}) \cap N_{\mathcal{B}^{23}}(r^{25}, r^{35})$ exists and, therefore, $\mathcal{A}$ restricted to J supports a $K_5^{(3)}$. □

References

[1] P. Erdős and V. T. Sós, *On Ramsey-Turán type theorems for hypergraphs*, Combinatorica **2** (1982), no. 3, 289–295, DOI 10.1007/BF02579235. MR698654 (85d:05185) ↑1.1

[2] Chr. Reiher, *Extremal problems in uniformly dense hypergraphs*, European J. Combin. **88** (2020), 103117, 22, DOI 10.1016/j.ejc.2020.103117. MR4111729 ↑1.1, 2

[3] Chr. Reiher, V. Rödl, and M. Schacht, *Some remarks on $\pi_{\wedge}$*, Connections in discrete mathematics, Cambridge Univ. Press, Cambridge, 2018, pp. 214–239. MR3821841 ↑1.1

[4] ______, *On a Turán problem in weakly quasirandom 3-uniform hypergraphs*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 5, 1139–1159, DOI 10.4171/JEMS/784. MR3790065 ↑1.1

[5] ______, *Embedding tetrahedra into quasirandom hypergraphs*, J. Combin. Theory Ser. B **121** (2016), 229–247, DOI 10.1016/j.jctb.2016.06.008. MR3548293 ↑1.1

[6] H. Towsner, *σ-algebras for quasirandom hypergraphs*, Random Structures Algorithms **50** (2017), no. 1, 114–139, DOI 10.1002/rsa.20641. MR3583029 ↑1.1

Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany

Email address: {soeren.berger,simon.piga,christian.reiher,mathias.schacht}@uni-hamburg.de

Department of Mathematics, Emory University, Atlanta, USA

Email address: vrodl@emory.edu