Monochromatic trees in random graphs

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Abstract

Bal and DeBiasio [*Partitioning random graphs into monochromatic components,* Electron. J. Combin. 24 (2017), no. 1, Paper 1.18] put forward a conjecture concerning the threshold for the following Ramsey-type property for graphs G: every r-colouring of the edge set of G yields r pairwise vertex disjoint monochromatic trees that partition the whole vertex set of G. We determine the threshold for this property for two colours.

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1 Introduction

For a graph $G = (V, E)$ we write $G \longrightarrow \Pi_2$ if for every 2-colouring of E, say with colours red and blue, there exist two monochromatic trees T_1, $T_2 \subseteq G$ such that $V(T_1)\dot{\cup}V(T_2) = V$, i.e., V can be split into two sets each inducing a spanning monochromatic component. Here we allow one of the trees to be empty and we also allow both trees to be monochromatic of the same colour. In [1, Conjecture 8.1] Bal and DeBiasio conjectured that if

$$p = p(n) > (1 + \varepsilon)\left(\frac{2\ln n}{n}\right)^{1/2}$$

for some $\varepsilon > 0$, then *asymptotically almost surely* (*a.a.s.*) the binomial random graph $G(n, p)$ satisfies $G(n, p) \longrightarrow \Pi_2$, i.e.,

$$\lim_{n\to\infty} \mathbb{P}\Big(G(n, p) \longrightarrow \Pi_2\Big) = 1\,.$$

One can observe that, for 2 colours, the conjectured condition on p would be best possible. In fact, if $p < (1-\varepsilon)\left(\frac{2\ln n}{n}\right)^{1/2}$ for some $\varepsilon > 0$, then a.a.s. $G(n, p)$ has diameter at least three (see, e.g., [2, Chapter 10]) and, hence, it contains two non-adjacent vertices u and v with disjoint neighbourhoods. Colouring all edges incident to u or v red and all other edges blue produces a colouring that requires at least three monochromatic trees in any decomposition of $V(G(n, p))$, since u and v cannot be in the same red tree.

Bal and DeBiasio showed that a.a.s. $G(n, p) \longrightarrow \Pi_2$ provided that we have $p > C\left(\frac{\ln n}{n}\right)^{1/3}$ for some suitable constant $C > 1$. We improve on that result by showing that $\left(\frac{\ln n}{n}\right)^{1/2}$ is the threshold for that property.

Theorem 1.1 *If $p = p(n) \gg \left(\frac{\ln n}{n}\right)^{1/2}$, then a.a.s. $G(n, p) \longrightarrow \Pi_2$.*

Combined with the discussion above, Theorem 1.1 implies that $\left(\frac{\ln n}{n}\right)^{1/2}$ is the threshold for the property $G \longrightarrow \Pi_2$. We remark that our proof also yields a semi-sharp threshold, since with not much additional effort we could replace the assumption $p \gg \left(\frac{\ln n}{n}\right)^{1/2}$ by $p > C\left(\frac{\ln n}{n}\right)^{1/2}$ for some suitable constant $C > 1$. In fact, since Theorem 1.1 implies that the threshold function for the monotone graph property $G \longrightarrow \Pi_2$ is not of the form $n^{-\alpha}$ for some rational $\alpha \in \mathbb{Q}_{>0}$ it follows from Friedgut's criterion [3, Theorem 1.4] that $G \longrightarrow \Pi_2$ has indeed a sharp threshold, i.e., there exist constants $c_1 > c_0 > 0$ and a function $c\colon \mathbb{N} \to \mathbb{R}$ with $c_0 < c(n) < c_1$ for every $n \in \mathbb{N}$ such that for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\Big(G(n,p) \longrightarrow \Pi_2\Big) = \begin{cases} 0, & \text{if } p < (1-\varepsilon)c(n)\Big(\frac{\ln n}{n}\Big)^{1/2} \\ 1, & \text{if } p > (1+\varepsilon)c(n)\Big(\frac{\ln n}{n}\Big)^{1/2}. \end{cases}$$

In view of the question of Bal and DeBiasio [1] it remains to show that $c(n)$ is a constant independent of n and that we have $c(n) \equiv \sqrt{2}$. Finally, we remark that Bal and DeBiasio also considered multicoloured extensions of this problem. In fact they conjectured that for every integer $r \geq 1$ the following holds for $G \in G(n,p)$: if $p = p(n) > (1+\varepsilon)\left(\frac{r \ln n}{n}\right)^{1/r}$ for some $\varepsilon > 0$ then, for every r-colouring of $E(G)$, a.a.s. $V(G)$ can be split into at most r sets each inducing a spanning monochromatic component. Unfortunately, for $r \geq 3$, if $p \ll \left(\frac{\ln n}{n}\right)^{1/(r+1)}$ then there is a colouring showing that a.a.s. this property does not hold. For several other interesting questions we refer to [1].

2 Outline of the proof of the main result

We introduce some further notation and classify the two-colourings into two classes (see Definition 2.1 below). For a colouring $\varphi\colon E \to \{\text{red}, \text{blue}\}$ of the edges of a graph $G = (V, E)$ we write $\varphi \longrightarrow \Pi_2$ to indicate that there exist two monochromatic trees T_1, $T_2 \subseteq G$ such that $V(T_1)\dot\cup V(T_2) = V$. In particular, $G \longrightarrow \Pi_2$ if $\varphi \longrightarrow \Pi_2$ holds for all 2-colourings φ of E. We denote the two edge disjoint spanning monochromatic subgraphs induced by φ by G^{φ}_{red} and $G^{\varphi}_{\text{blue}}$, i.e., $G^{\varphi}_{\text{red}} = \big(V, \varphi^{-1}(\text{red})\big)$ and $G^{\varphi}_{\text{blue}} = \big(V, \varphi^{-1}(\text{blue})\big)$. For a vertex $v \in V$ we consider its red- and blue-*neighbourhood*

$$N^{\varphi}_{\text{red}}(v) = \{u \in N(v)\colon \varphi(\{v,u\}) = \text{red}\} \tag{1}$$

and

$$N^{\varphi}_{\text{blue}}(v) = \{u \in N(v)\colon \varphi(\{v,u\}) = \text{blue}\} \tag{2}$$

and the corresponding degrees $d^{\varphi}_{\text{red}}(v) = |N^{\varphi}_{\text{red}}(v)|$ and $d^{\varphi}_{\text{blue}}(v) = |N^{\varphi}_{\text{blue}}(v)|$. We roughly classify the vertices depending on these degrees by defining the following sets

$$R^{\varphi} = \Big\{v \in V\colon d^{\varphi}_{\text{red}}(v) > \tfrac{1}{3}d(v)\Big\} \quad\text{and}\quad B^{\varphi} = \Big\{v \in V\colon d^{\varphi}_{\text{blue}}(v) > \tfrac{1}{3}d(v)\Big\}. \tag{3}$$

These sets might not be disjoint, but every vertex is a member of at least one of them and vertices v in the symmetric difference of these sets have at least $2d(v)/3$ neighbours in one colour. In the proof of Theorem 1.1 we consider two cases depending, whether there is a monochromatic path between some vertex in R^φ and a different vertex in B^φ.

Definition 2.1 Let $G = (V, E)$ be a graph and $\varphi\colon E \to \{\text{red}, \text{blue}\}$. We say φ is *extremal* if there is a pair of distinct vertices $r \in R^\varphi$ and $b \in B^\varphi$ for which no monochromatic r-b-path exists. If no such pair of vertices exists, then we say φ is *non-extremal*.

For the proof of Theorem 1.1 we consider non-extremal and extremal colourings φ separately. Before we proceed, let us remark that the property $G \longrightarrow \Pi_2$ is an increasing property, that is, if G is a spanning subgraph of G' and $G \longrightarrow \Pi_2$ holds, then $G' \longrightarrow \Pi_2$ also holds. This implies that it suffices to prove Theorem 1.1 under the additional hypothesis that $p = o(1)$.

In the remainder of this extended abstract let $G = (V, E) \in G(n, p)$ and let ε be a sufficiently small constant. Owing to $p \gg \left(\frac{\ln n}{n}\right)^{1/2}$ we know that a.a.s. every vertex $v \in V(G)$ has degree $d_G(v) = (1 \pm \varepsilon)pn$ and every pair of distinct vertices u, $w \in V(G)$ has $|N_G(u) \cap N_G(w)| = (1 \pm \varepsilon)p^2 n$ joint neighbours. Moreover, given $\varphi\colon E \to \{\text{red}, \text{blue}\}$, we suppress the superscript φ in terms G^φ_{red}, $N^\varphi_{\text{red}}(v)$, $d^\varphi_{\text{red}}(v)$, R^φ, and their blue counterparts.

2.1 Non-extremal colourings

The following proposition addresses the case when φ is non-extremal.

Proposition 2.2 (Non-extremal case) *If $p = p(n) \gg ((\ln n)/n)^{1/2}$ and $p = o(1)$, then a.a.s. $G \in G(n, p)$ satisfies $\varphi \to \Pi_2$ for every non-extremal colouring $\varphi\colon E(G) \to \{\text{red}, \text{blue}\}$.*

In the proof of Proposition 2.2 we shall make use of the following simple observation, which is closely related to the fact that every 2-colouring of the edges of the complete graph yields a monochromatic spanning tree.

Lemma 2.3 *Let $G = (V, E)$ be a graph and $\varphi\colon E \to \{\text{red}, \text{blue}\}$. If for a subset $U \subseteq V$ all pairs of vertices u, $u' \in U$ are connected by a monochromatic path, then there exists a monochromatic tree T with $V(T) \supseteq U$.*

Outline of the proof of Proposition 2.2. Let $\varphi\colon E \to \{\text{red}, \text{blue}\}$ be a non-extremal colouring. If one of the sets R or B, say R, is empty, then it follows from the degree condition of G that every vertex in $V(G)$ satisfies

$d_{\text{blue}}(v) \geq (2/3-\varepsilon)pn$. Hence, it is not hard to show that in this case there is a blue spanning tree of G and $\varphi \longrightarrow \Pi_2$.

Since φ is non-extremal, between every vertex $r \in R$ and every $b \in B$ there exists a monochromatic r-b-path. In particular, vertices contained in the intersection $R \cap B$ are connected to every other vertex by a monochromatic path. By considering a monochromatic component C containing the most number of vertices, one can use Lemma 2.3 and the non-extremality of φ to obtain monochromatic components $C_{\text{red}} \subseteq G_{\text{red}}$ and $C_{\text{blue}} \subseteq G_{\text{blue}}$ covering V, i.e.,

$$V(C_{\text{blue}}) \cup V(C_{\text{red}}) = V\,. \tag{4}$$

It is left to deduce the proposition from (4). For that let $C_{\text{red}} \subseteq G_{\text{red}}$ and $C_{\text{blue}} \subseteq G_{\text{blue}}$ satisfy (4). We may assume that both components are maximal, i.e., every vertex in the complement of C_{red} has only blue neighbours in C_{red} and, analogously, every vertex in the complement of C_{blue} has only red neighbours in C_{blue}. We consider the symmetric difference of C_{red} and C_{blue} and let $O_{\text{red}} = V(C_{\text{red}}) \smallsetminus V(C_{\text{blue}})$ and $O_{\text{blue}} = V(C_{\text{blue}}) \smallsetminus V(C_{\text{red}})$ be the two parts of the symmetric difference. Note that the maximal choice of C_{red} and C_{blue} implies that there is no edge between O_{red} and O_{blue}. In fact, there is not even a monochromatic path between O_{red} and O_{blue}, since every edge leaving O_{red} is blue and every edge entering O_{blue} is red. Owing to the assumption that every vertex in R is connected by a monochromatic path with every vertex in B we arrive at one of the following two cases

(I) $O_{\text{red}} = \varnothing$ or $O_{\text{blue}} = \varnothing$,

(II) $O_{\text{red}} \cup O_{\text{blue}} \subseteq R \smallsetminus B$ or $O_{\text{red}} \cup O_{\text{blue}} \subseteq B \smallsetminus R$.

Note that case (I) asserts that one of the parts of the symmetric difference of C_{red} and C_{blue} is empty, which combined with (4) implies the existence of a monochromatic spanning tree in G. For case (II), assuming without loss of generality that $O_{\text{red}} \cup O_{\text{blue}} \subseteq R \smallsetminus B$, one can show that there is a red spanning tree on O_{blue}, which combined with a red spanning tree on C_{red} concludes the proof of Proposition 2.2. □

2.2 Extremal colourings

In this section we consider extremal colourings φ. Together Propositions 2.2 and 2.4 establish Theorem 1.1.

Proposition 2.4 (Extremal case) *If $p = p(n) \gg ((\ln n)/n)^{1/2}$ and $p = o(1)$, then a.a.s. $G \in G(n,p)$ satisfies $\varphi \longrightarrow \Pi_2$ for every extremal colouring $\varphi\colon E(G) \to \{\text{red}, \text{blue}\}$.*

Outline of the proof of Proposition 2.4. Let $\varphi\colon E \to \{\text{red}, \text{blue}\}$ be a fixed extremal colouring. Let $r \in R$ and $b \in B$ be two distinct vertices for which no monochromatic r-b-path exists. We shall build a red and a blue tree with *roots* r and b. We sometimes refer to r as the *red root* and to b as the *blue root*. The trees will be built in two stages. In the first stage every vertex $v \in V \smallsetminus \{r, b\}$ will be assigned a *preferred colour* $\varrho(v)$, which indicates its "preference". In fact, the preferred colour $\varrho(v)$ will be chosen in such a way that v can be connected in the 'right colour' to r or b in a robust way, that is, there will be 'many' $\varrho(v)$-coloured paths from v to the root of colour $\varrho(v)$. Some vertices, which we call *joker vertices*, can be used to connect other vertices to both roots in a robust way. Because of this property, they play an important rôle in our proof. The preferred colours will be assigned vertex by vertex and earlier choices may influence those chosen later. However, in this process it might turn out that a later vertex v needs to be connected to the blue tree through an earlier vertex u with $\varrho(u) = \text{red}$ (thus u would in principle belong to the red tree that we are building). To resolve such conflicts, we finalise the choices in a second round after every vertex has chosen its preferred colour and, in fact, here some vertices may get connected to the tree opposite to its preferred colour (e.g., because of v above we may decide to override u's preference ($\varrho(u) = \text{red}$) and connect u to the blue tree). The technical details can be found in the full version of this work [4]. □

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