ON THE STRUCTURE OF GRAPHS WITH GIVEN ODD GIRTH AND LARGE MINIMUM DEGREE

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Abstract. We study the structure of graphs with high minimum degree conditions and given odd girth. For example, the classical work of Andrásfai, Erdős, and Sós implies that every n-vertex graph with odd girth $2k+1$ and minimum degree bigger than $\frac{2n}{2k+1}$ must be bipartite. We consider graphs with a weaker condition on the minimum degree. Generalizing results of Häggkvist and of Häggkvist and Jin for the cases $k=2$ and 3, we show that every n-vertex graph with odd girth $2k+1$ and minimum degree bigger than $\frac{3n}{4k}$ is homomorphic to the cycle of length $2k+1$. This is best possible in the sense that there are graphs with minimum degree $\frac{3n}{4k}$ and odd girth $2k+1$ which are not homomorphic to the cycle of length $2k+1$. Similar results were obtained by Brandt and Ribe-Baumann.

1. Introduction

We consider finite and simple graphs without loops and for any notation not defined here we refer to the textbooks [3, 4, 9]. In particular, we denote by K_r the complete graph on r vertices, by C_r a cycle of length r, where the length of a cycle or of a path denotes its number of edges. A *homomorphism* from a graph G into a graph H is a mapping $\varphi\colon V(G) \to V(H)$ with the property that $\{\varphi(u), \varphi(w)\} \in E(H)$ whenever $\{u, w\} \in E(G)$. We say that G is *homomorphic* to H if there exists a homomorphism from G into H. Furthermore, a graph G is a *blow-up* of a graph H, if there exists a surjective homomorphism φ from G into H, but for any supergraph of G on the same vertex set the mapping φ is not a homomorphism into H anymore. In particular, a graph G is homomorphic to H if and only if it is a subgraph of a suitable blow-up of H. Moreover, we say a blow-up G of H is *balanced* if the homomorphism φ signifying that G is a blow-up has the additional property that $|\varphi^{-1}(u)| = |\varphi^{-1}(u')|$ for all vertices u and u' of H.

Homomorphisms can be used to capture structural properties of graphs. For example, a graph is k-colourable if and only if it is homomorphic to K_k. Many results in extremal graph theory establish relationships between the minimum degree of a graph and the existence of a given subgraph. The following theorem of Andrásfai, Erdős and Sós [2] is a classical result of that type.

Theorem 1 (Andrásfai, Erdős & Sós). *For every integer $r \geq 3$ and for every n-vertex graph G the following holds. If G has minimum degree $\delta(G) > \frac{3r-7}{3r-4}n$ and G contains no copy of K_r, then G is $(r-1)$-colourable.* □

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In the special case $r = 3$, Theorem 1 states that every triangle-free n-vertex graph with minimum degree greater than $2n/5$ is bipartite, i.e., it is homomorphic to K_2. Several extensions of this result and related questions were studied. For example, motivated by a question of Erdős and Simonovits [10] the chromatic number of triangle-free graphs $G = (V, E)$ with minimum degree $\delta(G) > |V|/3$ was thoroughly investigated in [5, 8, 13, 15, 17] and it was recently shown by Brandt and Thomassé [7] that it is at most four.

Another related line of research (see, e.g., [8, 13, 15, 16]) concerned the question for which minimum degree condition a triangle-free graph G is homomorphic to a graph H of bounded size, which is triangle-free itself. In particular, Häggkvist [13] showed that triangle-free graphs $G = (V, E)$ with $\delta(G) > 3|V|/8$ are homomorphic to C_5. In other words, such a graph G is a subgraph of suitable blow-up of C_5. This can be viewed as an extension of Theorem 1 for $r = 3$, since balanced blow-ups of C_5 show that the degree condition $\delta(G) > 2|V|/5$ is sharp there. Strengthening the assumption of triangle-freeness to graphs of higher odd girth, allows us to consider graphs with a more relaxed minimum degree condition. In this direction Häggkvist and Jin [14] showed that graphs $G = (V, E)$ which contain no odd cycle of length three and five and with minimum degree $\delta(G) > |V|/4$ are homomorphic to C_7.

We generalize those results to arbitrary odd girth. We say a graph G has *odd girth* at least g, if the shortest cycle with odd length has length at least g.

Theorem 2. *For every integer $k \geq 2$ and for every n-vertex graph G the following holds. If G has minimum degree $\delta(G) > \frac{3n}{4k}$ and G has odd girth at least $2k+1$, then G is homomorphic to C_{2k+1}.*

Note that the degree condition given in Theorem 2 is best possible as the following example shows. For an even integer $r \geq 6$ we denote by M_r the so-called *Möbius ladder* (see, e.g., [12]), i.e., the graph obtained by adding all diagonals to a cycle of length r, where a diagonal connects vertices of distance $r/2$ in the cycle. One may check that M_{4k} has odd girth $2k+1$, but it is not homomorphic to C_{2k+1}. Moreover, M_{4k} is 3-regular and, consequently, balanced blow-ups of M_{4k} show that the degree condition in Theorem 2 is best possible when n is divisible by $4k$.

We also remark that Theorem 2 implies that every graph with odd girth at least $2k+1$ and minimum degree bigger than $\frac{3n}{4k}$ contains an independent set of size at least $\frac{kn}{2k+1}$. This answers affirmatively a question of Albertson, Chan, and Haas [1].
Similar results were obtained by Brandt and Ribe-Baumann.

2. Sketch of the proof

In the proof of Theorem 2 we consider an edge-maximal graph and show that it is either a bipartite graph or a blow-up of a $(2k+1)$-cycle. We say that a graph G with odd girth at least $2k+1$ is *edge-maximal* if adding any edge to G yields an odd cycle of length at most $2k-1$. We denote by $\mathcal{G}_{n,k}$ all edge-maximal n-vertex graphs satisfying the assumptions of the main theorem, i.e., for integers $k \geq 2$ and n we set

$$\mathcal{G}_{n,k} = \{G = (V, E)\colon\ |V| = n\,,\ \delta(G) > \tfrac{3n}{4k}\,,\\ \text{and } G \text{ is edge-maximal with odd girth } 2k+1\}\,.$$

The proof of the theorem relies on two lemmas, Lemmas 3 and 5 below, which state that certain configurations cannot occur in such edge-maximal graphs.

Lemma 3. *Let Φ denote the graph obtained from C_6 by adding exactly one diagonal. For all integers $k \geq 2$ and n and for every $G \in \mathcal{G}_{n,k}$ we have that G does not contain an induced copy of Φ.*

Proof (sketch). Suppose, contrary to the assertion, that $G = (V, E)$ contains Φ in an induced way. Since G is edge-maximal, the non-existence of a diagonal must be forced by the existence of an even path which, together with the missing diagonal, would yield an odd cycle of length at most $2k - 1$. One can show that such a path must have length exactly $2k - 2$ and that it must be edge-disjoint from Φ. Since there are two missing diagonals and since one can show that the related paths are also disjoint, the resulting configuration Φ' has $4k$ vertices. Finally one shows that no vertex in G can be joined to four vertices of Φ', which leads to a contradiction to the minimum degree condition of G. □

We remark that the above lemma can also be deduced from [14, Lemma 2], where is shown that $G \in \mathcal{G}_{n,k}$ cannot contain a cycle of length $4k$ with two consecutive diagonals. The next lemma states that graphs $G \in \mathcal{G}_{n,k}$ contain no graph from the following family, which can be viewed as tetrahedra with three faces formed by cycles of length $2k + 1$.

Definition 4 ($(2k + 1)$-tetrahedra)**.** *Given $k \geq 2$ we denote by $\mathcal{T}_k$ those subdivisions T of K_4 satisfying*

- (i) *three triangles S_1, S_2, and S_3 of K_4 are subdivided in T into cycles of length $2k + 1$*
- (ii) *two of the three edges contained in two of the triangles S_1, S_2, and S_3 are subdivided in T into paths of length at least two.*

In the context of graphs with given odd girth "odd subdivisions" of K_4 already appeared in [11].

Lemma 5. *For all integers $k \geq 2$ and n and for every $G \in \mathcal{G}_{n,k}$ we have that G does not contain any $T \in \mathcal{T}_k$ as a (not necessarily induced) subgraph.*

Proof (sketch). Similarly to the previous lemma, one can show that if such a $T \in \mathcal{T}_k$ is contained in G, then we get a contradiction to the minimum degree condition. In fact, (i) of the definition of $\mathcal{T}_k$ implies that all four triangles of K_4 must be subdivided into a cycle of odd length in T. Since all these cycles must have length at least $2k + 1$ it follows that T consists of at least $4k$ vertices. Finally, some case analysis shows that any vertex in G can be joined to at most three vertices in T, contradicting the assumption on the minimum degree of G. □

In the proof of the main theorem, we assume that G is not bipartite and show that G is a blow-up of a $(2k+1)$-cycle. In particular, we show that if a vertex of G is not contained in a maximal blow-up, then it gives rise to one of the forbidden configurations of Lemmas 3 and 5.

Proof of Theorem 2 (sketch). Suppose G is not bipartite. The edge-maximality of G implies that it contains a cycle of length C_{2k+1}. Let B be a vertex-maximal blow-up of a $(2k+1)$-cycle contained in G with vertex classes $A_0, \ldots, A_{2k}$. We will show $B = G$. Suppose, for a contradiction, that there exists a vertex $x \in V \setminus V(B)$. Owing to the odd girth assumption on G, the vertex x can have neighbours in at most two of the vertex classes of B and if there are two such classes, then within B each vertex in one class has distance two from the vertices in the other class.

Suppose first that x has neighbours in two classes A_{i-1} and A_{i+1}. If we are able to prove that x is adjacent to all the vertices in the two classes, then x can be included in A_i, i.e., the class of B which has distance one to both the aforementioned classes. If this is not the case, then by symmetry we may assume that there exists some vertex $b_{i-1} \in A_{i-1}$ which is not a neighbour of x. Fix vertices $a_{i-2} \in A_{i-2}$ and $a_i \in A_i$ arbitrarily and let $a_{i-1} \in A_{i-1}$ and $a_{i+1} \in A_{i+1}$ be neighbours of x. This fixes a cycle of length six in G, namely $xa_{i+1}a_ib_{i-1}a_{i-2}a_{i-1}x$, with one diagonal $\{a_{i-1}, a_i\}$. Owing to Lemma 3, there must be at least one more diagonal and one can easily show that this diagonal must be $\{b_{i-1}, x\}$, since $\{a_{i+1}, a_{i-2}\}$ is a "shortcut" in the blow-up which would create a cycle of length $2k-1$ in G.

Now consider the case that x has neighbours in only one class of B, say A_i. Let $a_i \in A_i$ be a neighbour of x and fix a cycle $a_0a_1 \dots a_{2k}$ in B containing a_i. Due to the edge-maximality, the non-existence of the edges $\{x, a_{i-2}\}$ and $\{x, a_{i+2}\}$ is forced by two paths which, together with the missing edges, would create short odd cycles. One can check that such paths have length exactly $2k-2$ and, together with the fixed cycle, they form a graph $T \in \mathcal{T}_k$, contradicting Lemma 5. Recalling that G is connected due to its edge-maximality, this concludes the proof of Theorem 2. $\square$

3. Conluding remarks

Extremal case in Theorem 2. A more careful analysis yields that the unique n-vertex graph with odd girth at least $2k+1$ and minimum degree exactly $\frac{3n}{4k}$, which is not homomorphic to C_{2k+1}, is the balanced blow-up of the Möbius ladder M_{4k}. In fact, the proofs of Lemmas 3 and 5 can be adjusted in such a way that they either exclude the existence of Φ resp. T in G or they yield a copy of M_{4k} in G. In the former case, one can repeat the proof of Theorem 2 based on those lemmas and obtains that G is homomorphic to C_{2k+1}. In the latter case, one uses the degree assumption to deduce that G is isomorphic to a balanced blow-up of M_{4k}.

Open questions. It would be interesting to study the situation, when we further relax the degree condition in Theorem 2. It seems plausible that if G has odd girth at least $2k+1$ and $\delta(G) \geq (\frac{3}{4k} - \varepsilon)n$ for sufficiently small $\varepsilon > 0$, then the graph G is homomorphic to M_{4k}. In fact, this could be true until $\delta(G) > \frac{4n}{6k-1}$. At this point blow-ups of the $(6k-1)$-cycle with all chords connecting two vertices of distance $2k$ in the cycle added, would show that this is best possible. For $k = 2$ such a result was proved by Chen, Jin, and Koh [8] and for $k = 3$ it was obtained by Brandt and Ribe-Baumann [6].

More generally, for $\ell \geq 2$ and $k \geq 3$ let $F_{\ell,k}$ be the graph obtained from a cycle of length $(2k-1)(\ell-1)+2$ by adding all chords which connect vertices with distance of the form $j(2k-1)+1$ in the cycle for some $j = 1, \dots, \lfloor (\ell-1)/2 \rfloor$. Note that $F_{2,k} = C_{2k+1}$ and $F_{3,k} = M_{4k}$. For every $\ell \geq 2$ the graph $F_{\ell,k}$ is ℓ-regular, has odd girth $2k+1$, and it has chromatic number three. Moreover, $F_{\ell+1,k}$ is not homomorphic to $F_{\ell,k}$, but contains it as a subgraph.

A possible generalization of the known results would be the following: *if an n-vertex graph G has odd girth at least $2k+1$ and minimum degree bigger than $\frac{\ell n}{(2k-1)(\ell-1)+2}$, then it is homomorphic to $F_{\ell-1,k}$.* However, this is known to be false for $k = 2$ and $\ell > 10$, since such a graph G may contain a copy of the Grötzsch graph which (due to having chromatic number four) is not homomorphically embeddable into any $F_{\ell,k}$. However, in some sense this is the only exception for $k = 2$ and

$\ell > 10$. In fact, with the additional condition $\chi(G) \leq 3$ the statement is known to be true for $k = 2$ (see, e.g., [8]). To our knowledge it is not known what happens for $k > 2$ and it would be interesting to study this further.

The discussion above motivates the following question, which asks for an extension of a result of Łuczak for triangle-free graphs from [16]. Note that for fixed k the degree of $F_{\ell,k}$ divided by its number of vertices tends to $\frac{1}{2k-1}$ as $\ell \to \infty$. Is it true that every n-vertex graph with odd girth at least $2k+1$ and minimum degree at least $(\frac{1}{2k-1}+\varepsilon)n$ can be mapped homomorphically into a graph H which also has odd girth at least $2k+1$ and $V(H)$ is bounded by a constant $C = C(\varepsilon)$ independent of n? Łuczak proved this for $k = 2$ and we are not aware of a counterexample for larger k.

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