Hypergraph regularity and quasi-randomness

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Abstract

Thomason and Chung, Graham, and Wilson were the first to systematically study *quasi-random* graphs and hypergraphs, and proved that several properties of random graphs imply each other in a deterministic sense. Their concepts of quasi-randomness match the notion of ε-regularity from the earlier Szemerédi regularity lemma. In contrast, there exists no "natural" hypergraph regularity lemma matching the notions of quasi-random hypergraphs considered by those authors.

We study several notions of quasi-randomness for 3-uniform hypergraphs which correspond to the regularity lemmas of Frankl and Rödl, Gowers and Haxell, Nagle and Rödl. We establish an equivalence among the three notions of regularity of these lemmas. Since the regularity lemma of Haxell et al. is algorithmic, we obtain algorithmic versions of the lemmas of Frankl–Rödl (a special case thereof) and Gowers as corollaries. As a further corollary, we obtain that the special case of the Frankl–Rödl lemma (which we can make algorithmic) admits a corresponding *counting lemma*. (This corollary follows by the equivalences and that the regularity lemma of Gowers or that of Haxell et al. admits a counting lemma.)

1 Introduction

Thomason [18, 19] and Chung, Graham, and Wilson [5] were the first to systematically study *quasi-random* graphs and hypergraphs, and proved that several properties of random graphs imply each other in a deterministic sense. Recently, and in connection with hypergraph regularity lemmas, related concepts of quasi-randomness for hypergraphs were introduced. We focus to the 3-uniform hypergraph regularity lemmas of Frankl and Rödl [8], Gowers [9] and Haxell, Nagle and Rödl [11, 12]. In this paper, we discuss the relation of these hypergraph concepts to those suggested earlier, and we establish an equivalence among these properties (see Corollary 2.1). As a consequence, we infer algorithmic versions of the regularity lemmas for 3-uniform hypergraphs of Frankl and Rödl and of Gowers (see Corollary 2.2) (using that the lemma of Haxell et al. is algorithmic). Perhaps the most important feature of these three regularity lemmas is that they all admit a corresponding *counting lemma* (which estimates the number of any fixed subhypergraph in an appropriately quasi-random environment). Strictly speaking, our algorithm (and equivalence) for Frankl and Rödl's lemma can only consider a special case (of their lemma) for which no corresponding counting lemma had been obtained before. A further corollary of our work shows that, nonetheless, this special case (which we can make algorithmic) does admit a counting lemma (see Corollary 2.3).

1.1 Quasi-random graphs. We begin our discussion with some results on quasi-random graphs from the papers of Thomason [18, 19] and Chung, Graham and Wilson in their influential paper [5]. We consider the graph properties of *uniform edge distribution* (**disc**), *deviation* (**dev**), and C_4-*minimality* (**cycle**). We say a sequence of graphs $(G_n = (V_n, E_n))_{n\in\mathbb{N}}$ with $|V_n| = n$ and density $e(G_n)/\binom{n}{2} = d$ satisfies property

disc: if $|e(U) - d\binom{|U|}{2}| = o(n^2)$ for every $U \subseteq V_n$,

dev: if

$$\sum_{u,v\in V_n}\Big|\sum_{i,j\in\{0,1\}} d^{2-i-j}(d-1)^{i+j}|N^i(u)\cap N^j(v)|\Big| = o(n^3),$$

cycle: if the number of *ordered* cycles of length four in G_n is at most $d^4n^4 + o(n^4)$,

where we denote by $N^1(u)$ the neighbourhood $N(u)$ of u and by N^0 the set $V_n \setminus N(u)$ of non-adjacent vertices of u, and where an ordered cycle of length 4 is a sequence of distinct vertices (v_1, v_2, v_3, v_4) of V_n where $\{v_i, v_j\} \in E_n$ whenever $|i - j| = 1, 3$. The three properties above are all equivalent [5]. Note that when $d = 1/2$, it follows from the definition that **dev** holds if, and only if, G_n contains (approximately) as many subgraphs of C_4 (the 4-cycle) having oddly many

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edges as it does subgraphs of C_4 having evenly many edges. For densities $d \neq 1/2$, one scales the weights of these subgraphs appropriately. More precisely, for a graph $G_n = (V_n, E_n)$ of density d, we note that **dev** is equivalent to

$$\sum_{u_0,u_1 \in V_n} \sum_{v_0,v_1 \in V_n} \prod_{i \in \{0,1\}} \prod_{j \in \{0,1\}} g(u_i, v_j) = o(n^4), \tag{1.1}$$

where $g(u,v) = 1 - d$ if $\{u,v\} \in E_n$ and $g(u,v) = -d$ if $\{u,v\} \notin E_n$.

The quasi-random concepts above are closely related to the earlier notion of ε-regularity, central to Szemerédi's regularity lemma [17] (see Theorem 3.2). Roughly speaking, the regularity lemma asserts that the vertex set of any graph can be partitioned into a bounded number of classes in such a way that most of its resulting induced bipartite subgraphs satisfy a bipartite version of **disc** (see $\mathbf{disc}_2$ in Definition 1.1) (and so, by the aforementioned equivalence, they also satisfiy bipartite versions of **dev** and **cycle**). The equivalence above was used in [1, 2] to derive the algorithmic version of Szemerédi's regularity lemma. Indeed, naively checking **disc** requires exponential time, while **cycle** (or **dev**) can be verified in polynomial time (and checking **disc** was the central difficulty in making Szemerédi's original proof constructive).

We now consider four approaches to possible generalizations of **disc**, **dev**, and **cycle** to (3-uniform) hypergraphs. The first three approaches will lack important properties which held in the case of graphs. In Section 1.5 we will finally state the appropriate generalization and then in Secion 2 we state our main results.

1.2 Straightforward generalization. The concepts **disc**, **dev**, and **cycle** have natural counterparts for 3-uniform hypergraphs (as well as for k-uniform hypergraphs). It turned out that finding the appropriate generalization is not straightforward. For example, let's say that a 3-uniform, n-vertex, hypergraph H_n with $d\binom{n}{3}$ hyperedges satisfies **weak-disc**, if $|e(U) - d\binom{|U|}{3}| = o(n^3)$ for all subsets $U \subset V(H_n)$, and let's say that H_n satisfies **oct** if its number of ordered *octrahedra* is asymptotically minimal $d^8 n^6 + o(n^6)$. (Here, the *octahedron* is the complete 3-partite 3-uniform hypergraph $K^{(3)}_{2,2,2}$ having two vertices per class, and an ordering of $K^{(3)}_{2,2,2}$ corresponds to a labeling of its vertices.) Then **weak-disc** and **oct** are not equivalent. Indeed, let $H_n = K_3(G(n, 1/2))$ be the 3-uniform hypergraph whose triples correspond to triangles of the random graph $G(n, 1/2)$ on n vertices, where the edges of $G(n, 1/2)$ appear independently with probability 1/2. Then, w.h.p., H_n satisfies **weak-disc** with density $d = 1/8 + o(1)$ and contains $(1/8)^4 n^6 + o(n^6)$ ordered copies of $K^{(3)}_{2,2,2}$. However, all n-vertex 3-uniform hypergraphs of density $d = 1/8$ contain at least $(1/8)^8 n^6 + o(n^6)$ ordered copies of $K^{(3)}_{2,2,2}$, and this lower bound is realized by the random 3-uniform hypergraph on n-vertices whose edges are independently included with probability 1/8. Similar counterexamples exist for the deviation property, which for a 3-uniform hypergraph $H_n = (V_n, E_n)$ of density d is defined as

$$\mathbf{dev}: \sum_{u_0,u_1} \sum_{v_0,v_1} \sum_{w_0,w_1} \prod_{i,j,k \in \{0,1\}} h(u_i, v_j, w_k) = o(n^6), \tag{1.2}$$

where $h(u,v,w) = 1 - d$ if $\{u,v,w\} \in E_n$ and $h(u,v,w) = -d$ if $\{u,v,w\} \notin E_n$.

We mention that one can prove a hypergraph regularity lemma whose regularity concept corresponds to **weak-disc**. An unsatisfying feature of such a lemma is that it can't, in principle, admit a corresponding counting lemma. There are no known hypergraph regularity lemmas corresponding to **oct** or **dev**, as we've defined them above.

1.3 A refined approach to disc. Frankl and Rödl suggested the following concept of uniform edge distribution (see also [3, 4]). Say that an n-vertex 3-uniform hypergraph $H_n = (V_n, E_n)$ of density d satisfies **disc** if $||E_n \cap K_3(G)| - d|K_3(G)|| = o(n^3)$ holds for all graphs G with vertex set V_n, where $K_3(G)$ denotes the collection of triples of vertices of V_n which span a triangle K_3 in G. For $d = 1/2$, it was shown in [4] that **disc** (just defined) and **dev** and **oct** (defined above) are all equivalent (see also [13] for $d \neq 1/2$).

In the definition above, we may view the hypergraph $H_n = (V_n, E_n)$ as a subset of the triangles of the complete graph K_n. Similarly to how Szemerédi's regularity lemma partitions the vertex set of a graph, the recent regularity lemmas for 3-uniform hypergraphs also partition the set of pairs of vertices. As a consequence, it is necessary to consider notions of quasi-randomness which involve not only the hypergraph $H_n = (V_n, E_n)$, but also an underlying graph G for which $E_n \subseteq K_3(G)$.

1.4 Absolute quasi-random properties. The discussion above leads to the following concepts, which were partly studied in [13]. To begin our presentation, we state the bipartite versions of **disc**, **dev**, and **cycle** for graphs.

Definition 1.1. *Let $\varepsilon > 0$ and let $G = (U \dot\cup V, E)$ be a bipartite graph with $|U| = |V| = n$ and density $e(G)/n^2 = d_2 \pm \varepsilon$. We say G has the property*

$\mathbf{disc}_2(\varepsilon)$: *if $|e_G(U', V') - d_2|U'||V'|| \leq \varepsilon n^2$ for all $U' \subseteq U$ and $V' \subseteq V$;*

dev$_2(\varepsilon)$: *if*

$$\sum_{u_0,u_1\in U}\sum_{v_0,v_1\in V}\prod_{i\in\{0,1\}}\prod_{j\in\{0,1\}} g(u_i,v_j)\le \varepsilon n^4,$$

where $g(u,v)=1-d_2$ *if* $\{u,v\}\in E$ *and* $g(u,v)=-d_2$ *if* $\{u,v\}\notin E$;

cycle$_2(\varepsilon)$: *if* G *contains at most* $d_2^4\binom{n}{2}^2+\varepsilon n^4$ *4-cycles.*

We now define corresponding notions for 3-uniform hypergraphs H with underlying 3-partite graphs G.

Definition 1.2. *Let* $\varepsilon>0$ *and let* $G=G^{12}\dot\cup G^{13}\dot\cup G^{23}$ *be a 3-partite graph with 3-partition* $V(G)=U\dot\cup V\dot\cup W$, $|U|=|V|=|W|=n$, *and let* H *be a 3-uniform hypergraph where* $E(H)\subseteq K_3(G)$. *Let* G^{ij} *be of density* $d_2\pm\varepsilon$ *for* $1\le i<j\le 3$ *and let* $e(H)=d_3|K_3(G)|$, *i.e.,* H *has relative density* d_3 *w.r.t.* G. *We say* (H,G) *has the property*

disc$_3(\varepsilon)$: *if* G^{ij} *has* ***disc***$_2(\varepsilon)$ *for* $1\le i<j\le 3$ *and* $||E(H)\cap K_3(G')|-d_3|K_3(G')||\le\varepsilon n^3$ *for all subgraphs* G' *of* G;

dev$_3(\varepsilon)$: *if* G^{ij} *has* ***dev***$_2(\varepsilon)$ *for* $1\le i<j\le 3$ *and*

$$\sum_{u_0,u_1\in U}\sum_{v_0,v_1\in V}\sum_{w_0,w_1\in W}\prod_{i,j,k\in\{0,1\}} h_{H,G}(u_i,v_j,w_k)\le\varepsilon n^6,$$

where

$$h_{H,G}(u,v,w)=\begin{cases}1-d_3, & \text{if } \{u,v,w\}\in E(H)\\ -d_3, & \text{if } \{u,v,w\}\in K_3(G)\setminus E(H)\\ 0, & \text{otherwise};\end{cases}$$

oct$_3(\varepsilon)$: *if* G^{ij} *has* ***cycle***$_2(\varepsilon)$ *for* $1\le i<j\le 3$ *and* H *contains at most* $d_3^8d_2^{12}\binom{n}{2}^3+\varepsilon n^6$ *copies of* $K^{(3)}_{2,2,2}$.

We refer to pairs (H,G) satisfying the properties in Definition 1.2 with $\varepsilon\ll d_2,d_3$ as *absolute* quasi-random, since the measure of quasi-randomness ε of the hypergraph H is smaller than the (absolute) density of H, which is essentially $d_3d_2^3$. It was shown in [13] (see also [15, Theorem 2.2]) that for every d_3, d_2, and $\varepsilon>0$ there exists $\delta>0$ such that if a pair (H,G) satisfies $\mathbf{disc}_3(\delta)$, then it also satisfies $\mathbf{oct}_3(\varepsilon)$. In other words, $\mathbf{disc}_3$ implies $\mathbf{oct}_3$, and the arguments from [4] and [13] can be extended to show that indeed all three notions $\mathbf{disc}_3$, $\mathbf{dev}_3$, and $\mathbf{oct}_3$ are equivalent in this sense.

Note that the properties in Definition 1.2 become meaningless if $\varepsilon\ge\min\{d_2,d_3\}$, since then the error term is larger than the main term. However, in all known regularity lemmas, the condition that $\varepsilon<\min\{d_2,d_3\}$ (in fact $\varepsilon\ll\min\{d_2,d_3\}$) cannot be guaranteed. More precisely, the measure of quasi-randomness ε of the 3-uniform hypergraph will typically be larger than the density d_2 of the auxillary underlying graphs in the regular partition of those lemmas. We therefore need a refinement of the properties from Definition 1.2, which leads to the following relative concepts of quasi-randomness. (For a regular partition whose typical "blocks" display $\varepsilon\ll\min\{d_2,d_3\}$, one must perturb the edge set of the input hypergraph, which will be discussed in Theorem 3.3 below (cf. [7, 16]).)

1.5 Relative quasi-random hypergraphs. The recent regularity lemmas for 3-uniform hypergraphs of Frankl–Rödl [8], Gowers [9], and Haxell et al. [11, 12] are based on the following notions of quasi-randomness, in which the quasi-randomness of H and G are measured by ε_3 and ε_2, resp., and where it will typically be the case that $d_3\gg\varepsilon_3\gg d_2\gg\varepsilon_2$.

Definition 1.3. *Let* $\varepsilon_3,\varepsilon_2>0$ *and* $G=G^{12}\dot\cup G^{13}\dot\cup G^{23}$ *be a 3-partite graph with 3-partition* $V(G)=U\dot\cup V\dot\cup W$, $|U|=|V|=|W|=n$, *and let* H *be a 3-uniform hypergraph with* $E(H)\subseteq K_3(G)$. *Let* G^{ij} *be of density* $d_2\pm\varepsilon_2$ *for* $1\le i<j\le 3$ *and let* $e(H)=d_3|K_3(G)|$. *We say* (H,G) *has the property*

disc$_3(\varepsilon_3,\varepsilon_2)$: *if* G^{ij} *has* ***disc***$_2(\varepsilon_2)$ *for* $1\le i<j\le 3$ *and* $||E(H)\cap K_3(G')|-d_3|K_3(G')||\le\varepsilon_3d_2^3n^3$ *for all* $G'\subseteq G$;

dev$_3(\varepsilon_3,\varepsilon_2)$: G^{ij} *has* ***dev***$_2(\varepsilon_2)$ *for* $1\le i<j\le 3$ *and for the function* $h_{H,G}(u,v,w)$, *defined as in Definition 1.2, we have*

$$\sum_{u_0,u_1\in U}\sum_{v_0,v_1\in V}\sum_{w_0,w_1\in W}\prod_{i,j,k\in\{0,1\}} h_{H,G}(u_i,v_j,w_k)\le\varepsilon_3d_2^{12}n^6;$$

oct$_3(\varepsilon_3,\varepsilon_2)$: *if* G^{ij} *has* ***cycle***$_2(\varepsilon_2)$ *for* $1\le i<j\le 3$ *and* H *contains at most* $d_3^8d_2^{12}\binom{n}{2}^3+\varepsilon_3d_2^{12}n^6$ *copies of* $K^{(3)}_{2,2,2}$.

We refer to pairs (H,G) satisfying the properties in Definition 1.3 with $\varepsilon_2\ll d_2\ll\varepsilon_3\ll d_3$ as *relative* quasi-random since here the measure of quasi-randomness ε_3 of the hypergraph H is only smaller than the relative density d_3 of H w.r.t. G.

1.6 Hypergraph regularity lemmas. We state the regularity lemma for 3-uniform hypergraphs of Gowers [9]. The central concept of quasi-randomness in this lemma is $\mathbf{dev}_3$.

Theorem 1.1. *For every* $\varepsilon_3>0$, *every function* $\varepsilon_2\colon\mathbb{N}\to(0,1]$, *and every* $t_0\in\mathbb{N}$, *there exist positive integers* T_0 *and* n_0 *so that for every 3-uniform hypergraph*

$H = (V, E)$ *on* $n \geq n_0$ *vertices, there exist a vertex partition* $V = V_1 \dot\cup \ldots \dot\cup V_t$*, where* $|V_1| \leq \cdots \leq |V_t| \leq |V_1|+1$ *and* $t_0 \leq t \leq T_0$*, and a partition of pairs of the complete bipartite graphs* $K[V_i, V_j]$*,* $1 \leq i < j \leq t$*, given by* $K[V_i, V_j] = G_1^{ij} \dot\cup \ldots \dot\cup G_\ell^{ij}$*, where* $\ell \leq T_0$*, so that the following holds.*

All but $\varepsilon_3 n^3$ *triples* $\{x, y, z\} \in \binom{V}{3}$ *satisfy that whenever* $\{x, y, z\} \in K_3(G_a^{ij} \dot\cup G_b^{jk} \dot\cup G_c^{ik}) = K_3(G_{abc}^{ijk})$*, for some* $1 \leq i < j < k \leq t$ *and* $(a, b, c) \in [\ell]^3$*, then* $(H_{abc}^{ijk}, G_{abc}^{ijk})$ *satisfies* $\boldsymbol{dev}_3(\varepsilon_3, \varepsilon_2(\ell))$ *with relative density* $|H_{abc}^{ijk}|/|K_3(G_{abc}^{ijk})|$ *of* H_{abc}^{ijk} *with respect to* G_{abc}^{ijk} *and the densities of* G_a^{ij}*,* G_b^{jk}*, and* G_c^{ik} *being* $1/\ell$*, where* H_{abc}^{ijk} *has edge set* $E(H) \cap K_3(G_{abc}^{ijk})$*.* □

If we replace $\mathbf{dev}_3$ in Theorem 1.1 by $\mathbf{disc}_3$ or $\mathbf{oct}_3$, then we (resp.) obtain the hypergraph regularity lemmas of Frankl and Rödl [8] and of Haxell et al. [11, 12].

Remark 1.1. *Theorem 1.1 differs slightly from the version proved by Gowers [9] in that the original does not require "most" bipartite graphs* G_a^{ij} *to have density close to* $1/\ell$*. The additional assertion we have stated can be obtained along similar lines to [8].*

We point out that the regularity lemma of Frankl and Rödl is stronger than we have quoted above. It asserts the existence of a partition such that most $(H_{abc}^{ijk}, G_{abc}^{ijk})$ satisfy the following stronger variant $\mathbf{disc}_{3,r}$ of $\mathbf{disc}_3$ (where r can depend on ℓ and t). For H and G as in Definition 1.3 and an integer $r \geq 1$, we say (H, G) satisfies $\mathbf{disc}_{3,r}(\varepsilon_3, \varepsilon_2)$ if

- (i) G^{ij} has $\mathbf{disc}_2(\varepsilon_2)$ for $1 \leq i < j \leq 3$ and
- (ii) $||E(H) \cap \bigcup_{i\in[r]} K_3(G_i)| - d_3|\bigcup_{i\in[r]} K_3(G_i)|| \leq \varepsilon_3 d_2^3 n^3$ for all families of subgraphs $G_1, \ldots, G_r$ of G.

Clearly, $\mathbf{disc}_{3,1} = \mathbf{disc}_3$, but otherwise $\mathbf{disc}_{3,r}$ is stronger than $\mathbf{disc}_3$. Dementieva, Haxell, Nagle and Rödl [6, Theorem 3.5] proved that $\mathbf{oct}_3 \not\Rightarrow \mathbf{disc}_{3,r}$ when r is large.

2 New results

The main new result is the equivalence of the notions of quasi-random hypergraphs from Definition 1.3.

Theorem 2.1. *For all* $d_3, \varepsilon_3 > 0$*, there exists* $\delta_3 > 0$ *such that for all* $d_2, \varepsilon_2 > 0$*, there exist* $\delta_2 > 0$ *and* n_0 *such that the following holds.*

Let $G = G^{12} \dot\cup G^{13} \dot\cup G^{23}$ *be a 3-partite graph with 3-partition* $V(G) = U \dot\cup V \dot\cup W$*,* $|U| = |V| = |W| = n \geq n_0$*, and let* H *be a 3-uniform hypergraph where* $E(H) \subseteq K_3(G)$*. Let* G^{ij} *be of density* $d_2 \pm \delta_2$*,* $1 \leq i < j \leq 3$*, and let* $e(H) = d_3|K_3(G)|$*.*

- (i) *If* (H, G) *satisfies* $\boldsymbol{disc}_3(\delta_3, \delta_2)$*, then it also satisfies* $\boldsymbol{oct}_3(\varepsilon_3, \varepsilon_2)$*, i.e.,* $\boldsymbol{disc}_3 \Rightarrow \boldsymbol{oct}_3$*.*
- (ii) *If* (H, G) *satisfies* $\boldsymbol{oct}_3(\delta_3, \delta_2)$*, then it also satisfies* $\boldsymbol{dev}_3(\varepsilon_3, \varepsilon_2)$*, i.e.,* $\boldsymbol{oct}_3 \Rightarrow \boldsymbol{dev}_3$*.*

We prove the assertions (i) and (ii) of Theorem 2.1 in Sections 3 and 4, resp.

We continue with a few immediate corollaries of our main result. First, the assertion of (i) above directly confirms Conjecture 3.8 of Dementieva et al. [6]. They proved [6, Theorem 3.6] $\mathbf{oct}_3 \Rightarrow \mathbf{disc}_3$, in which case the assertion of (i) above gives $\mathbf{oct}_3 \Leftrightarrow \mathbf{disc}_3$. However, a direct consequence of the counting lemma of Gowers [9, Theorem 6.8] (more precisely, [10, Corollary 5.3]) gives $\mathbf{dev}_3 \Rightarrow \mathbf{oct}_3$. As such, we have the following corollary.

Corollary 2.1. *The properties* $\boldsymbol{disc}_3$*,* $\boldsymbol{dev}_3$*, and* $\boldsymbol{oct}_3$ *are equivalent.* □

Recalling from Dementieva et al. [6] that $\mathbf{oct}_3 \not\Rightarrow \mathbf{disc}_{3,r}$ (when r is large), Corollary 2.1 allows us to extend their work to say that $\mathbf{dev}_3 \not\Rightarrow \mathbf{disc}_{3,r}$.

From the algorithmic regularity lemma of Haxell et al. [11, 12] (based on $\mathbf{oct}_3$), the equivalence above implies algorithmic versions of the 3-uniform hypergraph regularity lemmas of Gowers [9] and Frankl–Rödl [8] (when $r = 1$).

Corollary 2.2. *There exists an algorithm with running time* $O(n^6)$*, which constructs the partitions of vertices and pairs from Theorem 1.1.* □

Strictly speaking, an algorithmic version for $r = 1$ of the Frankl–Rödl regularity lemma was already stated by Dementieva et al. in [6, Theorem 3.10]. However, at the time of that announcement, no corresponding counting lemma was known. By appealing to the counting lemma of Gowers [9] or Haxell et al. [11, 12], the equivalence above implies a counting lemma applicable to the special case $r = 1$.

Corollary 2.3. *For every* $p \in \mathbb{N}$ *and* ξ*,* $d_3 > 0$ *there exists* $\delta_3 > 0$ *such that for every* $d_2 > 0$ *there exist* $\delta_2 > 0$ *and* n_0 *such that the following holds.*

Let $G = \dot\bigcup_{1\leq i<j\leq p} G^{ij}$ *be a* p*-partite graph with vertex partition* $V_1 \dot\cup \ldots \dot\cup V_p$ *where* $|V_1| = \cdots = |V_p| = n \geq n_0$ *and let* H *be a 3-uniform hypergraph with* $E(H) \subseteq K_3(G)$*. Let* G^{ij} *be of density* $d_2 \pm \delta_2$*,* $1 \leq i < j \leq p$ *and let* $e(H^{ijk}) = d_3|K_3(G^{ijk})|$ *for all* $1 \leq i < j < k \leq p$*, where* $G^{ijk} = G[V_i, V_j, V_k]$ *and* $H^{ijk} = H \cap K_3(G^{ijk})$*. Suppose, moreover, that each* H^{ijk} *satisfies* $\boldsymbol{disc}_3(\delta_3, \delta_2)$*,* $1 \leq i < j < k \leq p$*. Then the number* $|K_p(H)|$ *of complete, 3-uniform hypergraphs on* p *vertices in* H *satisfies*

$$|K_p(H)| = (1 \pm \xi) d_3^{\binom{p}{3}} d_2^{\binom{p}{2}} n^p \,. \qquad \square$$

3 Uniform edge distribution implies minimality

In this section, we prove part (i) of Theorem 2.1. The proof is based on the same implication in the "absolute" setting, where roughly speaking we will transfer the known implication $\mathbf{disc}_3 \Rightarrow \mathbf{oct}_3$ from the absolute setting to the relative setting. (Similar ideas were used in [14].) For that we will use Szemerédi's regularity lemma for graphs (see Theorem 3.2) and the *regular approximation lemma* for 3-uniform hypergraphs (see Theorem 3.3). We state these auxilary results in the next section and prove part (i) of Theorem 2.1 in Section 3.2.

3.1 Auxiliary results.

We will use the following proposition, which follows from [13, Theorem 6.5] (see also [15, Theorem 2.2]).

THEOREM 3.1. *For all d_3, $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that the following holds. Let D be a 3-partite, 3-uniform, hypergraph on the vertex partition $U\dot\cup V\dot\cup W$, $|U| = |V| = |W| = n \geq n_0$, and let $e(D) = (d_3 \pm \delta)n^3$. If $(D, K[U, V, W])$ satisfies $\boldsymbol{disc}_3(\delta)$, then $(D, K[U, V, W])$ has $\boldsymbol{oct}_3(\varepsilon)$, where $K[U, V, W]$ denotes the complete tripartite graph on $U\dot\cup V\dot\cup W$.* □

Note that Theorem 3.1 draws the same conclusion as (i) of Theorem 2.1, but in the "absolute" setting. For the transfer of this result to the "relative" setting, we will employ the *regular approximation lemma* for 3-uniform hypergraphs from [16], Theorem 3.3, and Szemerédi's regularity lemma for graphs [17], Theorem 3.2, which we state below (but in opposite order).

THEOREM 3.2. *For all $\mu > 0$ and integers t and M, there exist S_0 and n_0 such that for every family of graphs $F_1, \ldots, F_M$ on the same vertex set V (with $|V| = n \geq n_0$ and n being a multiple of $S_0!$) and for any given partition $V = V_1\dot\cup \ldots \dot\cup V_t$, $|V_i| = n/t$ for $i \in [t]$, there exists a refinement $V = \dot\bigcup_{i\in[t],j\in[s]} V_{i,j}$, with $|V_{i,j}| = n/(ts)$ and $s \leq S_0$, such that for all but $\mu t^2 s^2$ pairs $\{\{i, j\}, \{k, \ell\}\}$, $1 \leq i < j \leq t$, $1 \leq k, \ell \leq s$, the induced bipartite graphs $F_m[V_{i,j}, V_{k,\ell}]$ satisfy $\boldsymbol{disc}_2(\mu)$ for all $m = 1, \ldots, M$.* □

Next we state the regular approximation lemma for 3-uniform hypergraphs (see [16, Lemma 4.2] or [14, Theorem 54]). Roughly speaking, it asserts that for every 3-uniform hypergraph H, there exists a hypergraph $\widetilde{H}$ obtained from H by adding or deleting a few hyperedges from H, so that $\widetilde{H}$ admits a vertex partition and a partition of pairs, as in Theorem 1.1, with the stronger property that for *all* blocks of the partition, the hypergraph $\widetilde{H}$ satisfies the "absolute" $\mathbf{disc}_3$ property from Definition 1.2.

THEOREM 3.3. *For all d_2, $\nu > 0$ and every function $\varrho\colon \mathbb{N}^2 \to (0, 1]$, there exist $\varepsilon_0 > 0$ and T_0 so that the following holds.*

Let $G = G^{12}\dot\cup G^{13}\dot\cup G^{23}$ be a 3-partite graph with 3-partition $V(G) = U \cup V \cup W$, $|U| = |V| = |W| = n \geq n_0$ (where n is a multiple of $T_0!$) and let H be a 3-uniform hypergraph with $E(H) \subseteq K_3(G)$. Let G^{ij} satisfy $\boldsymbol{disc}_2(\varepsilon_0)$ with density d_2 for $1 \leq i < j \leq 3$. Then there exist integers t and $\ell \leq T_0$ and

- *(a) a vertex partition $U = \dot\bigcup_{i\in[t]} U_i$, $V = \dot\bigcup_{j\in[t]} V_j$, and $W = \dot\bigcup_{k\in[t]} U_k$, with $|U_i| = |V_j| = |W_k| = n/t$ for $i, j, k \in [t]$,*
- *(b) a partition of pairs of the induced bipartite graphs $G^{12}[U_i, V_j]$, $G^{13}[U_i, W_k]$, and $G^{23}[V_j, W_k]$, $i, j, k \in [t]$, given by $G^{12}[U_i, V_j] = P_1^{U_i,V_j}\dot\cup \ldots \dot\cup P_\ell^{U_i,V_j}$, $G^{13}[U_i, W_k] = P_1^{U_i,W_k}\dot\cup \ldots \dot\cup P_\ell^{U_i,W_k}$, and $G^{23}[V_j, W_k] = P_1^{V_j,W_k}\dot\cup \ldots \dot\cup P_\ell^{V_j,W_k}$, and*
- *(c) a 3-partite, 3-uniform hypergraph $\widetilde{H}$ on the same vertex set $U\dot\cup V\dot\cup W$*

such that the following holds:

- *(I) $|E(H)\triangle E(\widetilde{H})| \leq \nu n^3$ and*
- *(II) for all $1 \leq i < j < k \leq t$ and $(a, b, c) \in [\ell]^3$ the pair $(\widetilde{H}^{ijk}_{abc}, P^{ijk}_{abc})$ has $\boldsymbol{disc}_3(\varrho(t, \ell))$ with relative density $|E(\widetilde{H}^{ijk}_{abc})|/|K_3(P^{ijk}_{abc})|$ and the densities of the involved bipartite graphs being d_2/ℓ, where $P^{ijk}_{abc} = P_a^{U_i,V_j}\dot\cup P_b^{U_i,W_k}\dot\cup P_c^{V_j,W_k}$ and $\widetilde{H}^{ijk}_{abc} = \widetilde{H} \cap K_3(P^{ijk}_{abc})$.* □

The main difference between Theorems 1.1 and 3.3 concerns the degree of quasi-randomness of $(\widetilde{H}^{ijk}_{abc}, P^{ijk}_{abc})$ (in Theorem 3.3) and $(H^{ijk}_{abc}, G^{ijk}_{abc})$ (in Theorem 1.1). Theorem 3.3 guarantees that, at the cost of altering only a few triples (globally), the measure $\varrho(t, \ell)$ of quasi-randomness can be much smaller than $1/(t\ell)$, while Theorem 1.1 can only guarantee the measure ε_3 of quasi-randomness as a fixed constant (where t and ℓ depend of ε_3). On the other hand, in Theorem 1.1, the quasi-random property holds directly for H, while in Theorem 3.3, it only applies to the changed hypergraph $\widetilde{H}$.

3.2 Proof of (i) of Theorem 2.1.

We now prove assertion (i) of Theorem 2.1.

Proof. ($\mathbf{disc}_3 \Rightarrow \mathbf{oct}_3$) Let $d_3, \varepsilon_3 > 0$ be given and let δ' be the constant ensured by Theorem 3.1 for d_3 and $\varepsilon' = \varepsilon_3/4$. Without loss of generality, we may assume that $\delta' \leq \varepsilon' d_3^8/8$. For Theorem 2.1, we set $\delta_3 = \delta'/4$ and let $\delta_0 \ll \delta_3$. Then, for given d_2 and $\varepsilon_2 > 0$, we set

$$0 < \nu \ll \min\{\delta_3 d_3 d_2^3, \varepsilon_3 d_3^8 d_2^{12}/4, \delta' d_2^3/2\}$$

and

$$0 < \varrho(t,\ell) \ll \left(\frac{\delta_0}{S_0(\mu \ll \delta_0/\ell,\, M = 3t^2\ell,\, t)}\right)^3,$$

i.e., $\varrho(t,\ell)$ tends faster to 0 (when t and ℓ tend to infinity) than $(\delta_0/S_0)^3$, where $S_0(t,\ell)$ is given by Szemerédi's regularity lemma, Theorem 3.2, applied with $0 < \mu \ll \delta_0/\ell$, $M = 3t^2\ell$, and t. Finally, let

$$0 < \delta_2 \ll \varepsilon_0 \times \min_{t\in[T_0],\ell\in[T_0]} \varrho(t,\ell),$$

where ε_0 and T_0 are given by the regular approximation lemma, Theorem 3.3, applied with ν and $\varrho(\cdot,\cdot)$. Moreover, we choose δ_2 small enough so that $\mathbf{disc}_2(\delta_2) \Rightarrow \mathbf{cycle}_2(\varepsilon_2)$ for bipartite graphs of density d_2. For these constants and sufficiently large n let (H,G) be a pair satisfying $\mathbf{disc}_3(\delta_3,\delta_2)$ as given in Theorem 2.1. We have to show that (H,G) satisfies $\mathbf{oct}_3(\varepsilon_3,\varepsilon_2)$.

We first apply Theorem 3.3, with ν and $\varrho(t,\ell)$ above, to H and G and obtain integers t and $\ell \le T_0$, a vertex partition, a partition of pairs, and a hypergraph $\widetilde{H}$ as stated in (a)–(c) in Theorem 3.3 with properties (I) and (II).

We want to apply Theorem 3.1. For this we construct a "dense" 3-partite, 3-uniform, hypergraph D on the same vertex set $U\dot\cup V\dot\cup W$, which we view as a subhypergraph of $K_3(K[U,V,W])$ the triangles of $K[U,V,W]$. Roughly speaking, we will construct D by "mimicking" the partition of vertices and pairs of $\widetilde{H}$, which we obtained from Theorem 3.3. For that we will consider the same vertex partition, but replace every graph P^{U_i,V_j} of density d_2/ℓ (similarly, P^{U_i,W_k} and P^{V_j,W_k}) by a random graph B^{U_i,V_j} of density $1/\ell$ and for every B^{ijk}_{abc} we let the edges of D be a random subset of $K_3(B^{ijk}_{abc})$ with a relative density matching the one of $\widetilde{H}^{ijk}_{abc}$ w.r.t. P^{ijk}_{abc}.

As a consequence of this construction the hypergraph D will have absolute density $d_3 \pm \nu$ (note H only has relative density d_3 w.r.t. G) and we will show that $(D,K[U,V,W])$ satisfies $\mathbf{disc}_3(\delta')$ (see Claim 1). Hence, Theorem 3.1 implies that $(D,K[U,V,W])$ will also satisfy $\mathbf{oct}_3(\varepsilon_3/4)$, which estimates the number of octahedra in D. On the other hand, we will show that the construction of D yields $\#\{K^{(3)}_{2,2,2} \subseteq D\} \times d_2^{12} \approx \#\{K^{(3)}_{2,2,2} \subseteq \widetilde{H}\}$ (see Claim 2). From that we will infer that (H,G) satisfies $\mathbf{oct}_3(\varepsilon_3,\varepsilon_2)$, since $|E(H)\triangle E(\widetilde{H})| \le \nu n^3 \le \varepsilon_3 d_3^8 d_2^{12} n^3/4$. We now give the details of this plan.

For the construction of D, we will "mimic" the partition of vertices and pairs which we obtained for H after we applied Theorem 3.3. Recall we take the vertex set of D the same as of H, i.e., $U\dot\cup V\dot\cup W$, where there exists a partition of $U = U_1\dot\cup\ldots\dot\cup U_t$, $V = V_1\dot\cup\ldots\dot\cup V_t$, and $W = W_1\dot\cup\ldots\dot\cup W_t$. Now for all $i,\,j \in [t]$, consider a random partition of the edge set of $K[U_i,V_j]$ into ℓ parts $K[U_i,V_j] = B_1^{U_i,V_j}\dot\cup\ldots\dot\cup B_\ell^{U_i,V_j}$. Define the graphs $B_b^{U_i,W_k}$ and $B_c^{V_j,W_k}$ for $i,j,k \in [t]$ and $b,c \in [\ell]$ analogously. We may think of the graph $B_a^{U_i,V_j}$ as playing a similar role for D as $P_a^{U_i,V_j}$ does for $\widetilde{H}$. Note, however, that the density of $B_a^{U_i,V_j}$ is $\sim 1/\ell$, while the density of $P_a^{U_i,V_j}$ is $\sim d_2/\ell$.

To define the edges of D, fix $i,j,k \in [t]$ and $a,b,c \in [\ell]$ and set $B^{ijk}_{abc} = B_a^{U_i,V_j}\dot\cup B_b^{U_i,W_k}\dot\cup B_c^{V_j,W_k}$. Let D^{ijk}_{abc}, the subhypergraph of D induced on $K_3(B^{ijk}_{abc})$, be a random subset of $K_3(B^{ijk}_{abc})$, where each triple $\{u,v,w\} \in K_3(B^{ijk}_{abc})$ is chosen to be an edge in D^{ijk}_{abc} independently with probability $d(\widetilde{H}|P^{ijk}_{abc}) = |E(\widetilde{H}^{ijk}_{abc})|/|K_3(P^{ijk}_{abc})|$. In other words, we construct D in such a way that the relative density of D w.r.t. B^{ijk}_{abc}, i.e., $d(D|B^{ijk}_{abc})$, is very close to $d(\widetilde{H}|P^{ijk}_{abc})$, i.e., the relative density of $\widetilde{H}$ w.r.t. P^{ijk}_{abc}. We will verify two claims, Claim 1 and 2, for D.

Claim 1. *$(D,K[U,V,W])$ satisfies $\boldsymbol{disc}_3(\delta')$ and $e(D) = (d_3 \pm \delta'/2)n^3$ with probability $1-o(1)$.*

Proof. Consider an arbitrary subgraph F of $K[U,V,W]$, which we view as the union of $3t^2\ell$ graphs of the form

$$F_a^{U_i,V_j} = F \cap B_a^{U_i,V_j},$$
$$F_b^{U_i,W_k} = F \cap B_b^{U_i,W_k}, \text{ and } F_c^{V_j,W_k} = F \cap B_c^{V_j,W_k}.$$

We apply Szemeredi's regularity lemma, Theorem 3.2, to all such $3t^2\ell$ graphs. This way we obtain a refinement of the vertex partition on $U\dot\cup V\dot\cup W$, and each $F_a^{U_i,V_j}$ is split into s^2 (typically) quasi-random bipartite graphs. For each of these $3t^2\ell s^2$ graphs, say

$$\begin{aligned} F_a^{U_{i,p},V_{j,q}} &= F_a^{U_i,V_j}[U_{i,p},V_{j,q}] \\ &\subseteq B_a^{U_i,V_j}[U_{i,p},V_{j,q}] = B_a^{U_{i,p},V_{j,q}} \end{aligned}$$

with $p,q \in [s]$, we consider a random subgraph $Q_a^{U_{i,p},V_{j,q}} \subseteq P_a^{U_{i,p},V_{j,q}} = P_a^{U_i,V_j}[U_{i,p},V_{j,q}]$, where we include every edge of $P_a^{U_{i,p},V_{j,q}}$ independently with probability $e(F_a^{U_{i,p},V_{j,q}})/e(B_a^{U_{i,p},V_{j,q}})$, i.e., $Q_a^{U_{i,p},V_{j,q}}$ has approximately the same relative density compared to $P_a^{U_{i,p},V_{j,q}}$, as the graph $F_a^{U_{i,p},V_{j,q}}$ has w.r.t. $B_a^{U_{i,p},V_{j,q}}$.

Finally, we consider the union of all such $Q_a^{U_{i,p},V_{j,q}}$. So let

$$\begin{aligned} Q = &\bigcup_{i,j\in[t]}\bigcup_{p,q\in[s]}\bigcup_{a\in[\ell]} Q_a^{U_{i,p},V_{j,q}} \\ &\cup \bigcup_{i,k\in[t]}\bigcup_{p,r\in[s]}\bigcup_{b\in[\ell]} Q_b^{U_{i,p},W_{k,r}} \\ &\cup \bigcup_{j,k\in[t]}\bigcup_{q,r\in[s]}\bigcup_{c\in[\ell]} Q_c^{V_{j,q},W_{k,r}} \end{aligned}$$

be the union of all these random graphs. We will show that w.h.p.

$$\big||K_3(Q)| - d_2^3|K_3(F)|\big| \le \tfrac{\delta'}{8}d_2^3n^3 \tag{3.3}$$

and

$$\Big||E(\widetilde{H}) \cap K_3(Q)| - d_2^3|E(D)\cap K_3(F)|\Big| \le \tfrac{\delta'}{8}d_2^3n^3 \tag{3.4}$$

From (3.3) and (3.4) we infer

$$\begin{aligned}
&||E(D)\cap K_3(F)| - d_3|K_3(F)||\\
&\quad\le \left|\frac{|E(\widetilde{H})\cap K_3(Q)|}{d_2^3} - \frac{d_3|K_3(Q)|}{d_2^3}\right| + \frac{\delta'}{4}n^3\\
&\quad\le \left|\frac{|E(H)\cap K_3(Q)|}{d_2^3} - \frac{d_3|K_3(Q)|}{d_2^3}\right| + \frac{\delta'}{4}n^3 + \frac{\nu}{d_2^3}n^3\\
&\quad\le \frac{\delta'}{4}n^3 + \frac{\delta'}{4}n^3 + \frac{\nu}{d_2^3}n^3 \le \delta' n^3\,,
\end{aligned}$$

since (H,G) satisfies $\mathbf{disc}_3(\delta_3,\delta_2)$ with $\delta_3 \le \delta'/4$ and since $|E(H)\triangle E(\widetilde{H})| \le \nu n^3 \le \delta' d_2^3 n^3/2$. Since F was an arbitrary subgraph of $K[U,V,W]$, this implies that $(D, K[U,V,W])$ satisfies $\mathbf{disc}_3(\delta')$.

For the proof of (3.3) we consider tripartite graphs

$$F_{abc}^{ijk,pqr} = F_a^{U_{i,p},V_{j,q}} \dot\cup F_b^{U_{i,p},W_{k,r}} \dot\cup F_c^{V_{j,q},W_{k,r}}$$

and

$$Q_{abc}^{ijk,pqr} = Q_a^{U_{i,p},V_{j,q}} \dot\cup Q_b^{U_{i,p},W_{k,r}} \dot\cup Q_c^{V_{j,q},W_{k,r}}\,.$$

Suppose the bipartite subgraphs of $F_{abc}^{ijk,pqr}$ satisfy $\mathbf{disc}_2(\mu(\ell))$ (all but $\mu t^2 s^2$ do) and have density δ_0/ℓ. Then we can appeal to the counting lemma for graph triangles and infer that the number of triangles in $F_{abc}^{ijk,pqr}$ satisfies

$$(1\pm\xi_\mu)\frac{e(F_a^{U_{i,p},V_{j,q}})\cdot e(F_b^{U_{i,p},W_{k,r}})\cdot e(F_c^{V_{j,q},W_{k,r}})}{(n/(st))^3}\,,$$

where $\xi_\mu \to 0$ as $\mu \to 0$. On the other hand, since P^{U_i,V_j} satisfies $\mathbf{disc}_2(\varrho(t,\ell))$, we have that $P^{U_{i,p},V_{j,q}}$ satisfies $\mathbf{disc}_2(s\cdot\varrho(t,\ell))$ with density $d_2/\ell \pm (s\cdot\varrho(t,\ell)+\delta_2)$. Consequently, since $Q_a^{U_{i,p},V_{j,q}}$ is a random subgraph it satisfies $\mathbf{disc}_2(s\cdot\varrho(t,\ell)+o(1))$ (as long as the density of $F_a^{U_{i,p},V_{j,q}}$ is $\gg 1/\log n$). Moreover, if the density of $F_a^{U_{i,p},V_{j,q}}$ is at least δ_0/ℓ, we have that

$$e(Q^{U_{i,p},V_{j,q}}) = (d_2 \pm (s\cdot\varrho(t,\ell)+\delta_2+o(1)))e(F^{U_{i,p},V_{j,q}}).$$

Consequently, if the bipartite subgraphs of $F_{abc}^{ijk,pqr}$ have density δ_0/ℓ, then we have, again due to the triangle counting lemma,

$$\begin{aligned}
|K_3(Q_{abc}^{ijk,pqr})| &= (1\pm(\zeta_{s\cdot\varrho}+\delta_2))d_2^3 \times \cdots\\
&\quad\times \frac{e(F_a^{U_{i,p},V_{j,q}})e(F_b^{U_{i,p},W_{k,r}})e(F_c^{V_{j,q},W_{k,r}})}{(n/(st))^3}\,,
\end{aligned}$$

where $\zeta_{s\cdot\varrho} \to 0$ as $s\varrho \to 0$. In other words, we have shown that if the bipartite subgraphs of $F_{abc}^{ijk,pqr}$ satisfy $\mathbf{disc}_2(\mu(\ell))$ and have density δ_0/ℓ, then $|K_3(Q_{abc}^{ijk,pqr})| = (1\pm(\zeta_{s\cdot\varrho}+\xi_\mu+\delta_2+o(1)))d_2^3|K_3(F_{abc}^{ijk,pqr})|$. Finally, the first assertion of (3.3) follows from the choice of δ_0, $\mu(\ell) \ll \delta_0/\ell$, $\varrho(t,\ell) \ll 1/S_0$, and the fact that all but $\mu t^2 s^2$ bipartite graphs $F_a^{U_{i,p},V_{j,q}}$ satisfy $\mathbf{disc}_2(\mu(\ell))$.

Noting that, if the bipartite subgraphs of $F_{abc}^{ijk,pqr}$ satisfy $\mathbf{disc}_2(\mu(\ell))$ and have density δ_0/ℓ, then $(\widetilde{H}_{abc}^{ijk,pqr}, P_{abc}^{ijk,pqr})$ satisfies $\mathbf{disc}_3(s^3\varrho(t,\ell)/\delta_0^3)$ and appealing to the random construction of D, we infer that $d(\widetilde{H}|Q_{abc}^{ijk,pqr}) = d(D|F_{abc}^{ijk,pqr}) \pm s^3\varrho(t,\ell)/\delta_0^3 + o(1)$ and the second assertion of (3.3) follows from the discussion above. □

Claim 2. *With probability* $1-o(1)$ *we have*

$$\#\{K_{2,2,2}^{(3)} \subseteq \widetilde{H}\} \le (1+o(1))d_2^{12} \times \#\{K_{2,2,2}^{(3)} \subseteq D\}.$$

Proof. Apply the counting lemma from [13, Theorem 6.5] to $\widetilde{H}$ to count the number of octahedra. More precisely, apply the dense counting lemma to $\widetilde{H}$ induced on every selection of six vertex classes $U_{i_1}, U_{i_2}, V_{j_1}, V_{j_2}, W_{k_1}, W_{k_2}$ and 12 graphs $P_{a_1}^{U_{i_1},V_{j_1}},\dots,P_{a_4}^{U_{i_2},V_{j_2}},\dots,P_{c_4}^{V_{j_2},W_{k_2}}$. There are $t^6\ell^{12}$ such choices, and for each such choice, we get an estimate on the number of octahedra of $\widetilde{H}$ induced on that choice. Moreover, for each such choice, we will consider the corresponding such selection with the bipartite graphs $P_a^{X,Y}$ replaced by the corresponding graph $B_a^{X,Y}$. For such a selection of "B-graphs", we can estimate the number of octahedra in D induced on those B-graphs (due to the randomness in the construction of D). The number of octahedra in H and D for a corresponding choice of B- and P-graphs will be equal up to a factor of d_2^{12}. Repeating this analysis for all appropriate $t^6\ell^{12}$ choices then yields the claim. □

Finally, we deduce $\mathbf{oct}_3(\varepsilon_3,\varepsilon_2)$ for (H,G) from the claims above. Because of Claim 1 and Theorem 3.1, we have that, w.h.p., (H,G) satisfies $\mathbf{oct}_3(\varepsilon')$, i.e., the number of of octahedra in D is at most

$$(d_3+\delta')^8 1^{12}\tbinom{n}{2}^3 + \varepsilon' n^6 = (d_3+\delta')^8\tbinom{n}{2}^3 + \varepsilon' n^6\,.$$

Hence, we infer from the choice of $\delta' \le \varepsilon' d_3^8/8$ and Claim 2 that $(\widetilde{H},G)$ satisfies $\mathbf{oct}_3(2\varepsilon'+o(1),\varepsilon_2)$, in particular, $\widetilde{H}$ contains at most $d_3^8 d_2^{12}\binom{n}{2}^3 + (2\varepsilon'+o(1))n^6$ octahedra. Note that G^{ij} satisfies $\mathbf{cycle}_2(\varepsilon_2)$ due to the choice of δ_2. Now it follows that (H,G) satisfies $\mathbf{oct}_3(\varepsilon_3,\varepsilon_2)$, since $\varepsilon' \le \varepsilon_3/4$ and since $|E(H)\triangle E(\widetilde{H})| \le \nu n^3 \le \varepsilon_3 d_3^8 d_2^{12} n^3/4$, which yields that H contains at most $\varepsilon_3 d_3^8 d_2^{12} n^3/4 \times n^3$ octahedra more than $\widetilde{H}$. □

4 Minimality implies small deviation

In this section, we prove assertion (ii) of Theorem 2.1. The proof is based on the counting lemma from Haxell et al. [12] and on the equivalence of $\mathbf{disc}_3$ and $\mathbf{oct}_3$ (which was established in Section 3 using the result from Dementieva et al. [6, Theorem 3.6]). More precisely, we first use these tools to derive the following induced counting lemma for subhypergraphs of the octahedron. For a suboctahedron $O \subseteq K^{(3)}_{2,2,2}$ with vertex classes $\{x_0, x_1\}$, $\{y_0, y_1\}$, and $\{z_0, z_1\}$ and a hypergraph H and a graph G with $E(H) \subseteq K_3(G)$ we say a copy of O on vertex pairs $\{u_0, u_1\}$, $\{v_0, v_1\}$, and $\{w_0, w_1\}$ is *induced in H (w.r.t. G)*, if $\{u_i, v_j, w_k\} \in K_3(G)$ for all $i, j, k = 0, 1$ and $\{u_i, v_j, w_k\} \in E(H)$ if and only if $\{x_i, y_j, z_k\} \in E(O)$.

PROPOSITION 4.1. *For all ξ, $d_3 > 0$, there exists $\delta_3 > 0$ such that for all $d_2 > 0$ there exist $\delta_2 > 0$ and n_0 such that the following holds.*

Let $G = G^{12} \dot\cup G^{13} \dot\cup G^{23}$ be a 3-partite graph with 3-partition $V(G) = U \dot\cup V \dot\cup W$, $|U| = |V| = |W| = n \geq n_0$ and let H be a 3-uniform hypergraph with $E(H) \subseteq K_3(G)$. Let G^{ij} be of density $d_2 \pm \delta_2$ for $1 \leq i < j \leq 3$ and let $e(H) = d_3|K_3(G)|$. If (H, G) satisfies $\mathbf{oct}_3(\delta_3, \delta_2)$, then for every suboctahedron $O \subseteq K^{(3)}_{2,2,2}$, the number of (partite) labeled, induced copies of O in H w.r.t. G satisfies

$$\#\{O \subseteq H \text{ induced w.r.t. } G\} = (1 \pm \xi) d_3^{e(O)} (1 - d_3)^{8 - e(O)} d_2^{12} n^6 .$$

Before we prove Proposition 4.1, we derive part (ii) of Theorem 2.1 from it.

Proof. ($\mathbf{oct}_3 \Rightarrow \mathbf{dev}_3$) Let $d_3, \varepsilon_3 > 0$ be given. We choose $\delta_3 > 0$ small enough so that Propositition 4.1 holds for $\xi \leq \varepsilon_3(d_3(1 - d_3)/2)^8/2$. Then for given d_2 and $\varepsilon_2 > 0$, we let $\delta_2 > 0$ be small enough for Propositition 4.1 and so that every bipartite graph of density d_2 with $\mathbf{cycle}_2(\delta_2)$ also satisfies $\mathbf{dev}_2(\varepsilon_2)$. Finally, let n_0 be large enough so that Propositition 4.1 and $\mathbf{cycle}_2(\delta_2) \Rightarrow \mathbf{dev}_2(\varepsilon_2)$ hold.

For a given pair (H, G) satisfying $\mathbf{oct}_3(\delta_3, \delta_2)$, we apply Propositition 4.1 for every (spanning) suboctahedron $O \subseteq K^{(3)}_{2,2,2}$, and since

$$\sum_{u_0, u_1 \in U} \sum_{v_0, v_1 \in V} \sum_{w_0, w_1 \in W} \prod_{i,j,k \in \{0,1\}} h_{H,G}(u_i, v_j, w_k) = O(n^5) + \sum_{O \subseteq K^{(3)}_{2,2,2}} (-d_3)^{8-e(O)} (1 - d_3)^{e(O)} \times \times \#\{O \subseteq H \text{ induced w.r.t. } G\} ,$$

we obtain

$$\sum_{u_0, u_1 \in U} \sum_{v_0, v_1 \in V} \sum_{w_0, w_1 \in W} \prod_{i,j,k \in \{0,1\}} h_{H,G}(u_i, v_j, w_k) = O(n^5) + d_3^8 (1 - d_3)^8 d_2^{12} n^6 \sum_O ((-1)^{8-e(O)} \pm \xi) \leq O(n^5) + \frac{\varepsilon_3}{2} d_2^{12} n^6 ,$$

where we used $\sum_{O \subseteq K^{(3)}_{2,2,2}} (-1)^{8-e(O)} = 0$. Therefore, the pair (H, G) satisfies $\mathbf{dev}_3(\varepsilon_3, \varepsilon_2)$ if n is sufficiently large. □

It is left to prove Proposition 4.1.

Proof. We use the equivalence of $\mathbf{disc}_3$ and $\mathbf{oct}_3$ in the following way. Suppose (H, G) satisfies $\mathbf{disc}_3(\varepsilon_3, \varepsilon_2)$ for some densities d_3 and d_2. Then it follows directly from the definition of $\mathbf{disc}_3$ that for the complement of H w.r.t. G, i.e., $\overline{H} = (V(H), K_3(G) \setminus E(H))$, $(\overline{H}, G)$ satisfies $\mathbf{disc}_3(\varepsilon_3, \varepsilon_2)$ for densities $\bar{d}_3 = 1 - d_3$ and d_2. Hence, we infer from the equivalence of $\mathbf{disc}_3$ and $\mathbf{oct}_3$ that if (H, G) satisfies $\mathbf{oct}_3(\delta_3, \delta_2)$, then $(\overline{H}, G)$ satisfies $\mathbf{oct}_3(\delta'_3, \delta_2)$ for some $\delta'_3(\delta_3) \to 0$ as $\delta_3 \to 0$.

For the proof of Proposition 4.1 we may choose the constants so that

$$\min\{\xi, d_3, 1 - d_3\} \gg \xi' \gg \delta'_3 \geq \delta_3 \gg d_2 \gg \delta_2 .$$

By the discussion above, we may assume that for the given pair (H, G) with $\mathbf{oct}_3(\delta_3, \delta_2)$, we have that $(\overline{H}, G)$ satisfies $\mathbf{oct}_3(\delta'_3, \delta_2)$.

For a given suboctahedron $O \subseteq K^{(2)}_{2,2,2}$, we "double" (H, G) according to O. More precisely, let the three vertex classes of O be $\{x_0, x_1\}$, $\{y_0, y_1\}$, and $\{z_0, z_1\}$ and let U, V, W be the vertex classes of H and G. First we construct a new 6-partite graph G' with vertex classes $U_i = U \times \{i\}$, $V_j = V \times \{j\}$, and $W_k = W \times \{k\}$ with $i, j, k = 0, 1$, i.e., we take two copies of every original vertex class. Moreover, let $\{(u, i), (v, j)\}$ be an edge in G' if, and only if, $\{u, v\} \in E(G)$ (similarly for $\{(u, i), (w, k)\}$ and $\{(v, j), (w, k)\}$). In other words, we obtain G' from G by cloning every vertex and replacing every edge by a C_4 on the corresponding cloned vertices. Note that the construction of G' is independent of O. Next we define the edges of H' as follows: for $u \in U$, $v \in V$, $w \in W$, and $i, j, k = 0, 1$, let

$$\{(u, i), (v, j), (w, k)\} \in E(H') \Leftrightarrow \{u, v, w\} \in \begin{cases} E(H), & \{x_i, y_j, z_k\} \in E(O), \\ K_3(G) \setminus E(H), & \{x_i, y_j, z_k\} \notin E(O). \end{cases}$$

In other words, (H', G') was constructed so that $(H'[U_i, V_j, W_k], G'[U_i, V_j, W_k])$ is a copy of (H, G) if $\{x_i, y_j, z_k\} \in E(O)$ and a copy of $(\overline{H}, G)$ otherwise.

In any case, from the discussion above, we know that $(H'[U_i, V_j, W_k], G'[U_i, V_j, W_k])$ satisfies $\mathbf{oct}_3(\delta_3', \delta_2)$. Hence, the counting lemma from [12] implies that the number of crossing copies of $K^{(3)}_{2,2,2}$ in H' satisfies $(1 \pm \xi')d_3^{e(O)}(1-d_3)^{8-e(O)}d_2^{12}n^6$. Noting, that, due to the construction of H', this equals the number of (partite) labeled, induced copies of O in H w.r.t. G minus an error of $O(n^5)$ (for copies in H' which use two copies of the same vertex, e.g., $(u,1)$ and $(u,2)$), we conclude the proposition. □

5 Concluding remarks

The main result asserts that for 3-uniform hypergraphs the properties $\mathbf{disc}_3$, $\mathbf{dev}_3$, and $\mathbf{oct}_3$ are equivalent. We believe the same result holds for k-uniform hypergraphs. Such equivalences would be useful to obtain algorithmic regularity lemmas for k-uniform hypergraphs. We believe those results hold, which is work in progress.

References

[1] N. Alon, R. A. Duke, H. Lefmann, V. Rödl, and R. Yuster, *The algorithmic aspects of the regularity lemma (extended abstract)*, 33rd Annual Symposium on Foundations of Computer Science (Pittsburgh, Pennsylvania), IEEE Comput. Soc. Press, 1992, pp. 473–481. 1.1

[2] ———, *The algorithmic aspects of the regularity lemma*, J. Algorithms **16** (1994), no. 1, 80–109. 1.1

[3] F. R. K. Chung, *Quasi-random classes of hypergraphs*, Random Structures Algorithms **1** (1990), no. 4, 363–382. 1.3

[4] F. R. K. Chung and R. L. Graham, *Quasi-random hypergraphs*, Random Structures Algorithms **1** (1990), no. 1, 105–124. 1.3, 1.4

[5] F. R. K. Chung, R. L. Graham, and R. M. Wilson, *Quasi-random graphs*, Combinatorica **9** (1989), no. 4, 345–362. 1, 1.1

[6] Y. Dementieva, P. E. Haxell, B. Nagle, and V. Rödl, *On characterizing hypergraph regularity*, Random Structures Algorithms **21** (2002), no. 3-4, 293–335, Random structures and algorithms (Poznan, 2001). 1.6, 2, 2, 2, 4

[7] G. Elek and B. Szegedy, *Limits of hypergraphs, removal and regularity lemmas. A non-standard approach*, submitted. 1.4

[8] P. Frankl and V. Rödl, *Extremal problems on set systems*, Random Structures Algorithms **20** (2002), no. 2, 131–164. 1, 1.5, 1.6, 1.1, 2

[9] W. T. Gowers, *Quasirandomness, counting and regularity for 3-uniform hypergraphs*, Combin. Probab. Comput. **15** (2006), no. 1–2, 143–184. 1, 1.5, 1.6, 1.1, 2, 2, 2

[10] ———, *Hypergraph regularity and the multidimensional Szemerédi theorem*, Ann. of Math. (2) **166** (2007), no. 3, 897–946. 2

[11] P. E. Haxell, B. Nagle, and V. Rödl, *An algorithmic version of the hypergraph regularity method*, 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005), 23-25 October 2005, Pittsburgh, PA, USA, Proceedings, IEEE Computer Society, 2005, pp. 439–448. 1, 1.5, 1.6, 2, 2

[12] ———, *An algorithmic version of the hypergraph regularity method*, SIAM J. Comput. **37** (2008), no. 6, 1728–1776. 1, 1.5, 1.6, 2, 2, 4, 4

[13] Y. Kohayakawa, V. Rödl, and J. Skokan, *Hypergraphs, quasi-randomness, and conditions for regularity*, J. Combin. Theory Ser. A **97** (2002), no. 2, 307–352. 1.3, 1.4, 1.4, 3.1, 3.2

[14] B. Nagle, V. Rödl, and M. Schacht, *The counting lemma for regular k-uniform hypergraphs*, Random Structures Algorithms **28** (2006), no. 2, 113–179. 3, 3.1

[15] V. Rödl and M. Schacht, *Regular partitions of hypergraphs: Counting lemmas*, Combin. Probab. Comput. **16** (2007), no. 6, 887–901. 1.4, 3.1

[16] ———, *Regular partitions of hypergraphs: Regularity lemmas*, Combin. Probab. Comput. **16** (2007), no. 6, 833–885. 1.4, 3.1, 3.1

[17] E. Szemerédi, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, vol. 260, CNRS, Paris, 1978, pp. 399–401. 1.1, 3.1

[18] A. Thomason, *Pseudorandom graphs*, Random graphs '85 (Poznań, 1985), North-Holland, Amsterdam, 1987, pp. 307–331. 1, 1.1

[19] ———, *Random graphs, strongly regular graphs and pseudorandom graphs*, Surveys in combinatorics 1987 (New Cross, 1987), London Math. Soc. Lecture Note Ser., vol. 123, Cambridge Univ. Press, Cambridge, 1987, pp. 173–195. 1, 1.1