EQUIVALENT REGULAR PARTITIONS OF 3-UNIFORM HYPERGRAPHS

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ABSTRACT. The *regularity method* was pioneered by Szemerédi for graphs and is an important tool in extremal combinatorics. Over the last two decades, several extensions to hypergraphs were developed which were based on seemingly different notions of *quasirandom* hypergraphs. We consider the regularity lemmata for 3-uniform hypergraphs of Frankl and Rödl and of Gowers, and present a new proof that the concepts behind these approaches are equivalent.

§1 INTRODUCTION

Szemerédi [12] introduced the regularity method for graphs, which became an important tool in extremal graph theory. The regularity lemma asserts that every large graph $G = (V, E)$ can be approximated by a bounded number of *quasirandom* bipartite subgraphs that are induced by a partition of V. This approximation allows the use of results on quasirandom graphs for the analysis of G, which is a key feature in the success of the regularity method.

Szemerédi's regularity lemma was extended from graphs to k-uniform hypergraphs by Rödl et al. [2,7,11] and Gowers [3,4]. For a fixed k-uniform hypergraph $H = (V, E)$, these regularity lemmata provide well-structured partitions $\mathcal{P}$ of $V^{(k-1)} = \{X \subseteq V : |X| = k-1\}$ where for most edges $e \in E$, the hypergraph H is *quasirandom* on the unique family of classes from $\mathcal{P}$, each containing a $(k-1)$-element subset from e. By quasirandom, we follow either *uniform edge distribution* [2,7,11] or *deviation* [3,4].

For graphs it is well known that both concepts are equivalent quasirandom properties and for 3-uniform hypergraphs a similar equivalence was obtained in joint work with Poerschke [5]. The proof from [5] invokes two applications of the hypergraph regularity lemma. Here, we present a conceptually simpler proof using a single application of the regularity lemma.

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1.1. **Quasirandom bipartite graphs.** We begin our discussion with the notion of quasirandomness that is central to Szemerédi's regularity lemma. For $\varepsilon > 0$ and $d \in [0,1]$, we say a bipartite graph $G = (X \cup Y, E)$ is (ε, d)-*regular* if all subsets $X' \subseteq X$ and $Y' \subseteq Y$ satisfy

$$\big|e(X', Y') - d\,|X'||Y'|\big| \leqslant \varepsilon\,|X||Y|\,, \tag{1}$$

where $e(X', Y')$ denotes the number of edges between X' and Y'. Note that (1) ensures a fairly uniform edge density across the large induced bipartite subgraphs of G, which is a property holding almost surely in the binomial random bipartite graph.

The second notion of quasirandomness considers induced subgraphs on only four vertices. For $\delta > 0$ and $d \in [0,1]$, we say $G = (X \cup Y, E)$ is (δ, d)-*conformant** if

$$\sum_{x_0, x_1 \in X} \sum_{y_0, y_1 \in Y} \prod_{\lambda, \mu \in \{0,1\}} f_{G,d}(x_\lambda, y_\mu) \leqslant \delta\,|X|^2|Y|^2\,,$$

where $f_{G,d}\colon X \times Y \longrightarrow [-1,1]$ is the *d-shifted indicator* of E given by

$$f_{G,d}(x,y) = \mathbb{1}_E(x,y) - d\,.$$

Note that when $d = d(X,Y)$ is the density of G above, $f_{G,d}$ sums to 0 over $X \times Y$.

The aforementioned equivalence is made precise by the following two statements:

(*i*) For all $d \in [0,1]$ and $\varepsilon > 0$, there exists $\delta > 0$ such that every (δ, d)-conformant bipartite graph is (ε, d)-regular.

(*ii*) For all $d \in [0,1]$ and $\delta > 0$, there exists $\varepsilon > 0$ such that every sufficiently large (ε, d)-regular bipartite graph is (δ, d)-conformant.

We briefly sketch the well known proofs of (*i*) and (*ii*).

The proof of the implication in (*i*) starts with the identity

$$e(X', Y') - d\,|X'||Y'| = \sum_{x \in X'} \sum_{y \in Y'} f_{G,d}(x,y) = \sum_{x \in X} \sum_{y \in Y} \mathbb{1}_{X'}(x)\mathbb{1}_{Y'}(y) f_{G,d}(x,y)\,.$$

With two applications of the Cauchy–Schwarz inequality, to separate the indicator functions involving vertices from X and from Y, one can show that

$$\big|e(X', Y') - d\,|X'||Y'|\big|^4 \leqslant |X'|^2 \cdot |Y'|^2 \cdot \sum_{x_0, x_1 \in X} \sum_{y_0, y_1 \in Y} \prod_{\lambda, \mu \in \{0,1\}} f_{G,d}(x_\lambda, y_\mu)$$

and the (ε, d)-regularity follows from the assumed (δ, d)-conformity when $\delta \leqslant \varepsilon^4$.

The proof of implication (*ii*) makes use of the following bounds (see (2) below) on the number of induced copies of subgraphs of the 4-cycle $C_4 = K_{2,2}$. Let F be a spanning subgraph

*We remark that this concept is often called *deviation*. However, referring to such well-behaved graphs as (δ, d)-deviant seemed to be a mismatch and that is why we chose a different name here.

of $K_{2,2}$ with vertex partition $\{x_0, x_1\} \cup \{y_0, y_1\}$. We say a function $\varphi\colon V(F) \longrightarrow V(G)$ is an *induced homomorphism* when $xy \in E(F)$ if, and only if, $\varphi(x)\varphi(y) \in E(G)$. If in addition, φ satisfies $\varphi(x_0)$, $\varphi(x_1) \in X$ and $\varphi(y_0)$, $\varphi(y_1) \in Y$, then φ is a *partite* induced homomorphism of F into G, and we denote the number of such homomorphisms by $\mathrm{ihom}(F, G)$.

If $G = (X \cup Y, E_G)$ is an (ε, d)-regular bipartite graph, then the counting lemma for graphs implies

$$\left|\mathrm{ihom}(F, G) - d^{|E(F)|}(1-d)^{4-|E(F)|}|X|^2|Y|^2\right| \leqslant 4\varepsilon\, |X|^2|Y|^2 \,. \tag{2}$$

The proof of *(ii)* then follows immediately for $\varepsilon \leqslant \delta/64$ from the identity

$$\sum_{x_0,x_1\in X}\ \sum_{y_0,y_1\in Y}\ \prod_{\lambda,\mu\in\{0,1\}} f_{G,d}(x_\lambda, y_\mu) = \sum_{F\subseteq C_4} (1-d)^{|E(F)|}(-d)^{4-|E(F)|}\,\mathrm{ihom}(F, G)\,.$$

by 16 applications of (2), one for every labeled spanning subgraph $F \subseteq K_{2,2}$, and by appealing to the binomial theorem.

1.2. **Quasirandom tripartite hypergraphs.** We continue the discussion above for 3-uniform hypergraphs. In the context of the 3-uniform hypergraph regularity lemma, we consider 3-partite 3-uniform hypergraphs $H = (V, E_H)$, where E_H is a subset of the triangles of an underlying graph $G = (V, E_G)$ on the same vertex set V. To make this precise, we denote by $\mathcal{K}_3(G)$ the set of triples of vertices, which span a graph triangle K_3 in G. We say that $G = (V, E_G)$ *underlies* $H = (V, E_H)$ when $E_H \subseteq \mathcal{K}_3(G)$. Also, for a subgraph $J \subseteq G$, we write $E_H(J) = E_H \cap \mathcal{K}_3(J)$ for the set of hyperedges matching triangles of J, and we set

$$e_H(J) = \left|E_H \cap \mathcal{K}_3(J)\right|.$$

The following notion of a *complex* plays a similar rôle in the hypergraph regularity lemma as a bipartite graph does in the graph regularity lemma.

Definition 1.1 (complex)**.** For all reals ε_2, $d_2 > 0$ and every $n \in \mathbb{N}$, an (ε_2, d_2, n)-*complex* $\mathcal{H} = ((X, Y, Z), G, H)$ is a triple satisfying the following properties:

- (*a*) X, Y, and Z are pairwise disjoint vertex sets with $|X| = |Y| = |Z| = n$;
- (*b*) $G = (X \cup Y \cup Z, E_G)$ is a 3-partite graph where each of the induced bipartite subgraphs $G[X, Y]$, $G[X, Z]$, and $G[Y, Z]$ is (ε_2, d_2)-regular;
- (*c*) $H = (X \cup Y \cup Z, E_H)$ is a 3-uniform hypergraph which G underlies, i.e., $E_H \subseteq \mathcal{K}_3(G)$.

More simply, we refer to G as a *triad* and $\mathcal{H}$ as a *complex*.

We now introduce analogues of *regularity* and *conformity* for complexes.

Definition 1.2 (regular complex)**.** Let $\mathcal{H} = ((X, Y, Z), G, H)$ be an (ε_2, d_2, n)-complex. For $\varepsilon_3 > 0$ and $d_3 \in [0, 1]$, we say $\mathcal{H}$ is (ε_3, d_3)-*regular* if all subgraphs $J \subseteq G$ satisfy

$$\left|e_H(J) - d_3\,|\mathcal{K}_3(J)|\right| \leqslant \varepsilon_3 \cdot d_2^3 n^3 .$$

In Definition 1.2, the quantity $d_2^3 n^3$ approximates the number of triangles of G (cf. (b) of Definition 1.1). Similarly, the quantity $d_2^{12} n^6$ below approximates the number of $K_{2,2,2}$'s.

Definition 1.3 (conformant complex)**.** Let $\mathcal{H} = ((X, Y, Z), G, H)$ be an (ε_2, d_2, n)-complex. For $\delta_3 > 0$ and $d_3 \in [0, 1]$, we say $\mathcal{H}$ is (δ_3, d_3)-*conformant* if

$$\sum_{x_0, x_1 \in X} \sum_{y_0, y_1 \in Y} \sum_{z_0, z_1 \in Z} \prod_{\lambda, \mu, \nu \in \{0,1\}} f_{\mathcal{H}, d_3}(x_\lambda, y_\mu, z_\nu) \leqslant \delta_3 \cdot d_2^{12} n^6 ,$$

where $f_{\mathcal{H}, d_3} \colon X \times Y \times Z \longrightarrow [-1, 1]$ is defined by

$$f_{\mathcal{H}, d_3}(x, y, z) = \mathbb{1}_{\mathcal{K}_3(G)}(x, y, z) \cdot (\mathbb{1}_{E_H}(x, y, z) - d_3) . \tag{3}$$

Note that when d_3 is the *relative density* of H

$$d(H \,|\, G) = \begin{cases} \frac{e_H(G)}{|\mathcal{K}_3(G)|}, & \text{if } \mathcal{K}_3(G) \neq \varnothing, \\ 0, & \text{otherwise,} \end{cases}$$

then $f_{\mathcal{H}, d_3}$ sums to 0 over $X \times Y \times Z$.

It was proven in [5] that Definitions 1.2 and 1.3 are equivalent. We will give an alternative proof of this equivalence. First, we will show that conformity implies regularity.

Proposition 1.4. *For all δ_3, d_3, $d_2 > 0$, there exists $\varepsilon_2 > 0$ so that the following holds for all $n \in \mathbb{N}$. If $\mathcal{H}$ is a (δ_3, d_3)-conformant (ε_2, d_2, n)-complex, then $\mathcal{H}$ is $((2\delta_3)^{1/8}, d_3)$-regular.*

Proposition 1.4 follows from three standard applications of the Cauchy–Schwarz inequality (similar to those in the proof of (i) in Section 1.1).

Second, we will show that regularity implies conformity.

Theorem 1.5. *For all δ_3, $d_3 > 0$, there exists $\varepsilon_3 > 0$ so that for every $d_2 > 0$, there exist $\varepsilon_2 > 0$ and $n_0 \in \mathbb{N}$ so that the following holds. If $\mathcal{H}$ is an (ε_3, d_3)-regular (ε_2, d_2, n)-complex with $n \geqslant n_0$, then $\mathcal{H}$ is (δ_3, d_3)-conformant.*

The new proof of Theorem 1.5 is the main contribution here. The main challenge is that its quantification allows for

$$\varepsilon_2 \ll d_2 \ll \varepsilon_3 \ll \delta_3, d_3 ,$$

whereby the density d_2 of the sparse underlying graph G is smaller than the parameter ε_3, which governs the regularity of the hypergraph H. However, this quantification matches the environment obtained by the hypergraph regularity lemmas from [2, 3] and cannot be completely avoided. To overcome this challenge, in [5] two applications of the regularity lemma for hypergraphs were used. We present a simpler proof using only one such application. We close with the following remark.

Remark 1.6. The proof of Theorem 1.5 presented here extends verbatim to the environment of the k-uniform hypergraph regularity lemma from [10, Theorem 2.3]. As a direct consequence for every k, the corresponding counting lemma [9, Theorem 1.3] remains valid for $r = 1$ already, by virtue of the counting lemma from Gowers [4, Corollary 5.3].

Organisation. In Section 2, we prove Proposition 1.4. The proof of Theorem 1.5 is based on estimates similar to (2), where in a regular 3-uniform complex we estimate the number of induced copies of all 3-partite subhypergraphs on vertex classes of size two. Theorem 3.1 of Section 3 provides these bounds, and we deduce Theorem 1.5 in that section. In Section 5, we prove Theorem 3.1. This proof is based on the regularity method for 3-uniform hypergraphs, which we review in Section 4.

§2 Proof of Proposition 1.4: Conformity implies Regularity

The proof of Proposition 1.4 is based on three applications of the Cauchy–Schwarz inequality, and follows lines similar to the proof of (i) in Section 1.1. Let $\mathcal{H} = ((X, Y, Z), G, H)$ be a (δ_3, d_3)-conformant (ε_2, d_2, n)-complex, where $\varepsilon_2 = \varepsilon_2(d_2) > 0$ satisfies

$$(d_2 + \varepsilon_2)^4 \cdot (d_2^2 + 2\varepsilon_2)^2 \cdot (d_2^4 + 4\varepsilon_2) \leqslant 2d_2^{12} \,. \tag{4}$$

Fix a subgraph $J \subseteq G$. Since $f_{\mathcal{H},d_3}(x, y, z)$ in (3) is $\mathbb{1}_{E(H)}(x, y, z) - d_3$ on $xyz \in \mathcal{K}_3(J)$[†], the quantity $e_H(J) - d_3|\mathcal{K}_3(J)|$ equals

$$\sum_{xyz\in\mathcal{K}_3(J)} f_{\mathcal{H},d_3}(x, y, z) = \sum_{x\in X}\sum_{y\in Y} \mathbb{1}_{E_J}(x, y) \sum_{z\in Z} \mathbb{1}_{E_J}(x, z)\mathbb{1}_{E_J}(y, z) f_{\mathcal{H},d_3}(x, y, z) \,.$$

A first application of the Cauchy–Schwarz inequality yields

$$\begin{aligned}\big|e_H(J) - d_3\,|\mathcal{K}_3(J)|\big|^2 &\leqslant \sum_{x\in X}\sum_{y\in Y} \mathbb{1}^2_{E_J}(x, y) \cdot \sum_{x\in X}\sum_{y\in Y}\Big(\sum_{z\in Z} \mathbb{1}_{E_J}(x, z)\mathbb{1}_{E_J}(y, z) f_{\mathcal{H},d_3}(x, y, z)\Big)^2 \\ &= e_J(X, Y) \sum_{z_0,z_1\in Z}\sum_{x\in X}\prod_{\nu\in\{0,1\}} \mathbb{1}_{E_J}(x, z_\nu) \sum_{y\in Y}\prod_{\nu\in\{0,1\}} \mathbb{1}_{E_J}(y, z_\nu) f_{\mathcal{H},d_3}(x, y, z_\nu) \,.\end{aligned}$$

[†]For simplicity, if there is no danger of confusion we sometimes omit parentheses, braces, and commas for 2-element and 3-element sets. In particular, we denote edges $\{u, v\}$, hyperedges $\{u, v, w\}$, or the vertex set of a graph triangle $\{x, y, z\}$ by uv, uvw, and xyz, respectively.

A second application of the Cauchy–Schwarz inequality bounds $|e_H(J) - d_3\,|\mathcal{K}_3(J)||^4$ by

$$\begin{aligned} e_J(X,Y)^2 \cdot \sum_{z_0,z_1\in Z}\sum_{x\in X}\Big(\prod_{\nu\in\{0,1\}}\mathbb{1}_{E_J}(x,z_\nu)\Big)^2 \cdot \sum_{z_0,z_1\in Z}\sum_{x\in X}\Big(\sum_{y\in Y}\prod_{\nu\in\{0,1\}}\mathbb{1}_{E_J}(y,z_\nu)f_{\mathcal{H},d_3}(x,y,z_\nu)\Big)^2 \\ \leqslant e_J(X,Y)^2\cdot \hom\big(K_{1,2},J[X,Z]\big) \\ \cdot \sum_{y_0,y_1\in Y}\sum_{z_0,z_1\in Z}\prod_{\mu,\nu\in\{0,1\}}\mathbb{1}_{E_J}(y_\mu,z_\nu)\sum_{x\in X}\prod_{\mu,\nu\in\{0,1\}}f_{\mathcal{H},d_3}(x,y_\mu,z_\nu)\,, \end{aligned}$$

where $\hom(K_{1,2},J[X,Z])$ denotes the number of (partite) graph homomorphisms of $K_{1,2}$ into $J[X,Z]$. A third application of the Cauchy–Schwarz inequality yields

$$\begin{aligned} \Big|e_H(J) - d_3\,|\mathcal{K}_3(J)|\Big|^8 \leqslant e_J(X,Y)^4\cdot \hom\big(K_{1,2},J[X,Z]\big)^2\cdot\hom\big(K_{2,2},J[Y,Z]\big) \\ \cdot\sum_{x_0,x_1\in X}\sum_{y_0,y_1\in Y}\sum_{z_0,z_1\in Z}\prod_{\lambda,\mu,\nu\in\{0,1\}}f_{\mathcal{H}}(x_\lambda,y_\mu,z_\nu)\,, \end{aligned} \tag{5}$$

where $\hom(K_{2,2},J[Y,Z])$ is defined analogously to $\hom(K_{1,2},J[X,Z])$. Now, the (ε,d_2)-regularity of $G[X,Y]$, $G[X,Z]$, and $G[Y,Z]$ guarantees (see, e.g., (2))

$$\begin{aligned} e_J(X,Y) \leqslant e_G(X,Y) \leqslant (d_2+\varepsilon_2)n^2\,, \\ \hom\big(K_{1,2},J[X,Z]\big)\leqslant \hom\big(K_{1,2},G[X,Z]\big)\leqslant (d_2^2+2\varepsilon_2)n^3\,, \\ \text{and}\quad \hom\big(K_{2,2},J[Y,Z]\big)\leqslant \hom\big(K_{2,2},G[Y,Z]\big)\leqslant (d_2^4+4\varepsilon_2)n^4\,. \end{aligned}$$

Applying these bounds and the (δ_3,d_3)-conformity of $\mathcal{H}$ to (5) implies

$$\Big|e_H(J) - d_3\,|\mathcal{K}_3(J)|\Big|^8 \leqslant (d_2+\varepsilon_2)^4n^8\cdot(d_2^2+2\varepsilon_2)^2n^6\cdot(d_2^4+4\varepsilon_2)n^4\cdot\delta_3d_2^{12}n^6 \overset{(4)}{\leqslant} 2\delta_3d_2^{24}n^{24}\,,$$

which concludes the proof of Proposition 1.4. □

§3 Proof of Theorem 1.5: Regularity implies Conformity

Theorem 1.5 is a consequence of Theorem 3.1 below, which extends (2) to regular complexes $\mathcal{H} = ((X,Y,Z),G,H)$. In particular, Theorem 3.1 asserts that $\mathcal{H}$ admits around the expected number of labeled induced copies of any spanning subhypergraph F of the octahedron $K^{(3)}_{2,2,2}$:

$$V(K^{(3)}_{2,2,2}) = \{x_0,x_1\}\cup\{y_0,y_1\}\cup\{z_0,z_1\}\quad\text{and}\quad E(K^{(3)}_{2,2,2}) = \big\{x_\lambda y_\mu z_\nu\colon \lambda,\mu,\nu\in\{0,1\}\big\}\,.$$

For this, a map $\varphi : V(F)\longrightarrow V(G)$ is a *partite homomorphism* of F into $\mathcal{H}$ when

(1) $\varphi(x_0)$, $\varphi(x_1)\in X$, $\varphi(y_0)$, $\varphi(y_1)\in Y$, and $\varphi(z_0)$, $\varphi(z_1)\in Z$;
(2) $\varphi(x_\lambda)\varphi(y_\mu)\varphi(z_\nu)\in\mathcal{K}_3(G)$ for all $\lambda,\mu,\nu\in\{0,1\}$;
(3) $\varphi(x_\lambda)\varphi(y_\mu)\varphi(z_\nu)\in E(H)$ whenever $x_\lambda y_\mu z_\nu\in E(F)$.

When, additionally, φ satisfies

(3′) $\varphi(x_\lambda)\varphi(y_\mu)\varphi(z_\nu) \in E(H)$ if, and only if, $x_\lambda y_\mu z_\nu \in E(F)$,

we say that φ is a *partite induced homomophism* of F into $\mathcal{H}$. In these contexts, we denote by $\hom(F, \mathcal{H})$ ($\mathrm{ihom}(F, \mathcal{H})$) the number of partite (induced) homomorphisms of F into $\mathcal{H}$.

Theorem 3.1. *For all $\eta > 0$ and $d_3 \in [0, 1]$, there exists $\varepsilon_3 > 0$ so that for every $d_2 > 0$, there exist $\varepsilon_2 > 0$ and $n_0 \geqslant 1$ so that the following holds for every spanning subhypergraph F of $K^{(3)}_{2,2,2}$. If $\mathcal{H}$ is an (ε_3, d_3)-regular (ε_2, d_2, n)-complex with $n \geqslant n_0$, then*

$$\left| \mathrm{ihom}(F, \mathcal{H}) - d_3^{|E(F)|}(1 - d_3)^{8-|E(F)|} d_2^{12} n^6 \right| \leqslant \eta \cdot d_2^{12} n^6 \,. \tag{6}$$

We defer the proof of Theorem 3.1 to Section 5.

Proof of Theorem 1.5. For any given δ_3, $d_3 > 0$ we set $\eta = \delta_3/256$ and for $d_2 > 0$, let $\mathcal{H} = ((X, Y, Z), G, H)$ be an (ε_3, d_3)-regular (ε_2, d_2, n)-complex with $n \geqslant n_0$, where $\varepsilon_3 = \varepsilon_3(\delta_3, d_3, \eta) > 0$, $\varepsilon_2 = \varepsilon_2(\delta_3, d_3, \eta, d_2) > 0$, and $n_0 = n_0(\delta_3, d_3, \eta, d_2, \varepsilon_2) \in \mathbb{N}$ are those parameters guaranteed by Theorem 3.1. It follows from (3) that

$$\sum_{x_0,x_1 \in X} \sum_{y_0,y_1 \in Y} \sum_{z_0,z_1 \in Z} \prod_{\lambda,\mu,\nu \in \{0,1\}} f_{\mathcal{H},d_3}(x_\lambda, y_\mu, z_\nu) = \sum_F (1 - d_3)^{|E(F)|}(-d_3)^{8-|E(F)|} \cdot \mathrm{ihom}(F, \mathcal{H}) \,,$$

where the sum on the right-hand side runs over all 256 labeled spanning subhypergraphs of $K^{(3)}_{2,2,2}$. Applying Theorem 3.1 to all such F bounds the left-hand side from above by

$$256\eta d_2^{12} n^6 + \sum_F (-1)^{8-|E(F)|} \cdot d_3^8 (1 - d_3)^8 d_2^{12} n^6 = 256\eta d_2^{12} n^6,$$

and the (δ_3, d_3)-conformity of $\mathcal{H}$ follows from the choice of η. □

§4 Regularity Method for 3-uniform Hypergraphs

In this section, we state a regularity lemma from [2] (Theorem 4.2 below) and a counting lemma from [6] (Theorem 4.4 below) which we need for proving Theorem 3.1. These require the following notion of a regular complex, which is somewhat stronger than Definition 1.2.

Definition 4.1 (r-regular complex)**.** Let $\mathcal{H} = ((X, Y, Z), G, H)$ be an (ε_2, d_2, n)-complex. For $\varepsilon_3 > 0$, $d_3 \in [0, 1]$, and an integer $r \geqslant 1$, we say $\mathcal{H}$ is (ε_3, d_3, r)-*regular* if all sequences $\boldsymbol{J} = (J_1, \dots, J_r)$ of subgraphs of G satisfy

$$\left| e_H(\boldsymbol{J}) - d_3 \, |\mathcal{K}_3(\boldsymbol{J})| \right| \leqslant \varepsilon_3 \, |\mathcal{K}_3(G)| \,,$$

where $e_H(\boldsymbol{J}) = \left| \bigcup_{i=1}^r E_H(J_i) \right|$ and $\mathcal{K}_3(\boldsymbol{J}) = \bigcup_{i=1}^r \mathcal{K}_3(J_i)$. Moreover, we say $\mathcal{H}$ is (ε_3, r)-*regular* when it is (ε_3, d_3, r)-regular for $d_3 = d(H \,|\, G)$.

We remark that for $r = 1$, Definition 4.1 reduces to Definition 1.2. Otherwise, Definition 4.1 is stronger than Definition 1.2, and for large r it is stronger than Definition 1.3 (see [1]).

The following regularity lemma for complexes is adapted from [2] (see, e.g., [10, Lemma 4.1]).

Theorem 4.2 (Regularity Lemma)**.** *For all constants d_2, $\xi_3 > 0$, integers ℓ_0, $t_0 \geqslant 1$, and functions $\xi_2\colon (0,1] \longrightarrow (0,1]$ and $r\colon (0,1] \times \mathbb{N} \longrightarrow \mathbb{N}$, there exist a constant $\varepsilon_2 > 0$ and integers L_0, T_0, and N_0 so that the following hold.*

Let $\mathcal{H} = ((X, Y, Z), G, H)$ be an (ε_2, d_2, n)-complex with $n \geqslant N_0$, where $T_0!$ divides n. There exist integers ℓ and t with $\ell_0 \leqslant \ell \leqslant L_0$, $t_0 \leqslant t \leqslant T_0$, vertex partitions $\dot\bigcup_{i\in[t]} X_i = X$, $\dot\bigcup_{j\in[t]} Y_j = Y$, and $\dot\bigcup_{k\in[t]} Z_k = Z$, edge-partitions

$$\dot{\bigcup_{i,j\in[t]}}\ \dot{\bigcup_{\alpha\in[\ell]}} P^{ij}_\alpha = E(G[X,Y])\,,\quad \dot{\bigcup_{i,k\in[t]}}\ \dot{\bigcup_{\beta\in[\ell]}} Q^{ik}_\beta = E(G[X,Z])\,,\ \text{and}\ \dot{\bigcup_{j,k\in[t]}}\ \dot{\bigcup_{\gamma\in[\ell]}} R^{jk}_\gamma = E(G[Y,Z])\,,$$

and complexes $\mathcal{H}^{ijk}_{\alpha\beta\gamma} = ((X_i, Y_j, Z_k), G^{ijk}_{\alpha\beta\gamma}, H^{ijk}_{\alpha\beta\gamma})$ for every $(i, j, k, \alpha, \beta, \gamma) \in [t]^3 \times [\ell]^3$, where

$$G^{ijk}_{\alpha\beta\gamma} = \big(X_i \cup Y_j \cup Z_k, P^{ij}_\alpha \cup Q^{ik}_\beta \cup R^{jk}_\gamma\big) \quad \text{and} \quad H^{ijk}_{\alpha\beta\gamma} = \big(X_i \cup Y_j \cup Z_k, E(H) \cap \mathcal{K}_3\big(G^{ijk}_{\alpha\beta\gamma}\big)\big),$$

satisfying the following properties:

- (a) *all $\mathcal{H}^{ijk}_{\alpha\beta\gamma}$ above are $(\xi_2(d_2/\ell), d_2/\ell, n/t)$-complexes;*
- (b) *all but $\xi_3 t^3\ell^3$ many $\mathcal{H}^{ijk}_{\alpha\beta\gamma}$ above are $(\xi_3, r(d_2/\ell, t))$-regular.*

We call the graphs $G^{ijk}_{\alpha\beta\gamma}$ of Theorem 4.2 the *triads* of the regular partition.

In Theorem 4.4 below, we consider a special case of the counting lemma from [6], tailored for counting subhypergraphs of the octahedron $K^{(3)}_{2,2,2}$ within the following *octahedral complexes*.

Definition 4.3 (octahedral complex)**.** For ξ_2, $d > 0$ and $m \in \mathbb{N}$, an *octahedral* (ξ_2, d, m)-*complex* $\mathcal{O} = ((X_0, X_1, Y_0, Y_1, Z_0, Z_1), G, H)$ is a triple satisfying the following properties:

- (i) X_0, X_1, Y_0, Y_1, Z_0, and Z_1 are pairwise disjoint vertex sets of common size m;
- (ii) G is a 3-partite graph with vertex classes $X_0 \cup X_1$, $Y_0 \cup Y_1$, and $Z_0 \cup Z_1$, and H is a 3-partite 3-uniform hypergraph which G underlies;
- (iii) for each $\lambda, \mu, \nu \in \{0, 1\}$, the complex

$$\mathcal{O}^{\lambda\mu\nu} = \big((X_\lambda, Y_\mu, Z_\nu), G^{\lambda\mu\nu} = G[X_\lambda, Y_\mu, Z_\nu], H^{\lambda\mu\nu}\big),$$

where $E(H^{\lambda\mu\nu}) = E(H) \cap \mathcal{K}_3(G^{\lambda\mu\nu})$, is a (ξ_2, d, m)-complex.

Moreover, for $\xi_3 > 0$ and an integer $r \geqslant 1$, we say $\mathcal{O}$ is (ξ_3, r)-*regular* when all $\lambda, \mu, \nu \in \{0, 1\}$ satisfy that $\mathcal{O}^{\lambda\mu\nu}$ is (ξ_3, r)-regular.

Fix a spanning subhypergraph F of the octahedron $K^{(3)}_{2,2,2}$ on the fixed vertex partition

$$V(F) = \{x_0, x_1\} \mathbin{\dot\cup} \{y_0, y_1\} \mathbin{\dot\cup} \{z_0, z_1\},$$

and fix an octahedral complex $\mathcal{O} = ((X_0, X_1, Y_0, Y_1, Z_0, Z_1), G, H)$. A map $\varphi : V(F) \longrightarrow V(G)$ is a *partite homomorphism* of F into $\mathcal{O}$ when all $\lambda, \mu, \nu \in \{0, 1\}$ satisfy the following properties:

(1) $\varphi(x_\lambda) \in X_\lambda$, $\varphi(y_\mu) \in Y_\mu$, and $\varphi(z_\nu) \in Z_\nu$;
(2) $\varphi(x_\lambda)\varphi(y_\mu)\varphi(z_\nu) \in \mathcal{K}_3(G)$;
(3) $\varphi(x_\lambda)\varphi(y_\mu)\varphi(z_\nu) \in E(H)$ whenever $x_\lambda y_\mu z_\nu \in E(F)$.

We denote by $\hom(F, \mathcal{O})$ the number of partite homomorphisms of F into $\mathcal{O}$.

Theorem 4.4 (Octahedral Counting Lemma)**.** *For every $\vartheta > 0$, there exist $\xi_3 > 0$ and functions $\xi_2 \colon (0, 1] \longrightarrow (0, 1]$, $r \colon (0, 1] \longrightarrow \mathbb{N}$, and $m_0 \colon (0, 1] \longrightarrow \mathbb{N}$ such that for all $d \in (0, 1]$, the following holds.*

For every real constant $d \in (0, 1]$, for every $(\xi_3, r(d))$-regular octahedral $(\xi_2(d), d, m)$-complex $\mathcal{O} = ((X_0, X_1, Y_0, Y_1, Z_0, Z_1), G, H)$ with $m \geqslant m_0(d)$, and for every spanning subhypergraph F of $K^{(3)}_{2,2,2}$, we have

$$\Big| \hom(F, \mathcal{O}) - d^{12} m^6 \prod_{x_\lambda y_\mu z_\nu \in E(F)} d(H \mid G^{\lambda\mu\nu}) \Big| \leqslant \vartheta d^{12} m^6 \,.$$

The essential difference between Theorems 3.1 and 4.4 (aside from counting induced versus non-induced homomorphisms) is the assumed regularity of the given complex. In Theorem 3.1, the given complex $\mathcal{H}$ is (ε_3, d_3)-regular, while in Theorem 4.4, the given octahedral complex $\mathcal{O}$ satisfies the stronger property of being (ξ_3, r)-regular for some large integer r depending on the density of the underlying graph G.

§5 Proof of Theorem 3.1

In this section, we prove Theorem 3.1 in a non-induced but equivalent form.

Theorem 5.1. *For all $\eta > 0$ and $d_3 \in [0, 1]$, there exists $\varepsilon_3 > 0$ so that for every $d_2 > 0$, there exist $\varepsilon_2 > 0$ and $n_0 \geqslant 1$ so that the following holds for every spanning subhypergraph F of $K^{(3)}_{2,2,2}$. If $\mathcal{H}$ is an (ε_3, d_3)-regular (ε_2, d_2, n)-complex with $n \geqslant n_0$, then*

$$\Big| \hom(F, \mathcal{H}) - d_3^{|E(F)|} d_2^{12} n^6 \Big| \leqslant \eta \cdot d_2^{12} n^6 \,.$$

Up to the error η, Theorems 3.1 and Theorems 5.1 are equivalent. Indeed, fixing F above,

$$\hom(F, \mathcal{H}) = \sum_{F'} \mathrm{ihom}(F', \mathcal{H}) \qquad \text{and} \qquad \mathrm{ihom}(F, \mathcal{H}) = \sum_{F'} (-1)^{|E(F')| - |E(F)|} \hom(F', \mathcal{H}) \,,$$

where we sum over all spanning subhypergraphs F' satisfying $F \subseteq F' \subseteq K^{(3)}_{2,2,2}$, and where we use the elementary identities

$$d_3^{|E(F)|} = \sum_{F'} d_3^{|E(F')|}(1-d_3)^{8-|E(F')|} \quad \text{and} \quad d_3^{|E(F)|}(1-d_3)^{8-|E(F)|} = \sum_{F'}(-1)^{|E(F')|-|E(F)|}d_3^{|E(F')|}\,.$$

In the proof of Theorem 5.1, we invoke the regularity method from Section 4. We also use the following standard consequence of the counting lemma for graphs.

Fact 5.2 (counting/extension lemma for graphs)**.** *For all tripartite graphs $G = (X \mathbin{\dot\cup} Y \mathbin{\dot\cup} Z, E_G)$ with $G[X,Y]$, $G[X,Z]$, and $G[Y,Z]$ being (ε,d)-regular we have*

- *(a)* $\big||\mathcal{K}_3(G)| - d^3|X||Y||Z|\big| \leqslant 3\varepsilon|X||Y||Z|$
- *(b) and all but $4\varepsilon^{1/4}|X||Y||Z|$ many triangles of G extend to at most $\big(d^9 + 4\varepsilon^{1/4}\big)|X||Y||Z|$ partite homomorphisms of $K_{2,2,2}$ into G.*

Similarly, given a tripartite graph $G = (X \mathbin{\dot\cup} Y \mathbin{\dot\cup} Z, E_G)$ with vertex classes $X = X_0 \cup X_1$, $Y = Y_0 \cup Y_1$, and $Z = Z_0 \cup Z_1$ with $G[X_\lambda, Y_\mu]$, $G[X_\lambda, Z_\nu]$, and $G[Y_\mu, Z_\nu]$ being (ε,d)-regular for all λ, μ, $\nu \in \{0,1\}$ we have

- *(c)* $\big|\hom(K_{2,2,2},G) - d^{12}|X_0||X_1||Y_0||Y_1||Z_0||Z_1|\big| \leqslant 12\varepsilon|X_0||X_1||Y_0||Y_1||Z_0||Z_1|$*, where in* $\hom(K_{2,2,2},G)$ *we only consider those homomorphisms φ from $V(K_{2,2,2}) = \{x_0,x_1\} \mathbin{\dot\cup} \{y_0,y_1\} \mathbin{\dot\cup} \{z_0,z_1\}$ such that $\varphi(x_0) \in X_0, \dots, \varphi(z_1) \in Z_1$.* □

Note that Fact 5.2 *(c)* also applies in a situation when for example X_0 and X_1 are not disjoint.

Proof of Theorem 5.1. We start by defining all involved constants. Following the quantification of the theorem for given $\eta > 0$ and $d_3 \in [0,1]$ we define

$$\varepsilon_3 = \frac{\eta}{2^{13}}\,. \tag{7}$$

Let $d_2 > 0$ be given. To define the corresponding constant $\varepsilon_2 > 0$, we assemble constants and functions suitable for applications of Theorems 4.2 and 4.4. To that end, set

$$\vartheta = \frac{\eta}{2^{12}}\,. \tag{8}$$

Let $\xi_3 > 0$ and functions $\xi_2\colon (0,1] \longrightarrow (0,1]$, $r\colon (0,1] \longrightarrow \mathbb{N}$ and $m_0\colon (0,1] \longrightarrow \mathbb{N}$ be those parameters guaranteed by Theorem 4.4. W.l.o.g., we may assume that

$$\xi_3 \leqslant \frac{\eta}{2^{15}} \overset{(7)}{=} \frac{\varepsilon_3}{4}\,, \qquad \text{and} \qquad \xi_2(\zeta) \leqslant \frac{\eta}{3\cdot 2^{12}}\zeta^{12} \ \text{ for all } \zeta \in (0,1]\,, \tag{9}$$

and that $m_0(x)$ decreases in x. With constants d_2, $\xi_3 > 0$ fixed above, with fixed integers

$$t_0 = \left\lceil \frac{2^{12}}{\eta} \right\rceil \tag{10}$$

and $\ell_0 = 1$, and with functions $\xi_2(\cdot)$ and $r(\cdot)$ fixed above, Theorem 4.2 guarantees a constant $\varepsilon_2' > 0$ and positive integers L_0, T_0, and N_0. We define the promised constant

$$\varepsilon_2 = \min\left\{\varepsilon_2', \left(\frac{\eta d_2^{12}}{12\cdot 2^{10}}\right)^4, \frac{d_2^{12}}{12T_0}\right\}, \tag{11}$$

and we take the integer n_0 to be sufficiently large whenever needed.

Let $\mathcal{H} = ((X,Y,Z),G,H)$ be an (ε_3, d_3)-regular (ε_2, d_2, n)-complex with $n \geqslant n_0$, where ε_3, ε_2, and n_0 are defined above. It suffices to assume that $T_0!$ divides n since removing up to $T_0!$ vertices from each of X, Y, and Z decreases $\hom(F,\mathcal{H})$ by only $6T_0!n^5 = O(n^5)$ while still resulting in a $(2\varepsilon_3, d_3)$-regular $(2\varepsilon_2, d_2, n)$-complex.

For every fixed spanning subhypergraph $F \subseteq K^{(3)}_{2,2,2}$ we shall establish

$$\left|\hom(F,\mathcal{H}) - d_3^{|E(F)|} d_2^{12} n^6\right| \leqslant \frac{2^{|E(F)|}}{2^8}\eta \cdot d_2^{12} n^6 . \tag{12}$$

Note that (12) holds when F is the empty (spanning) subhypergraph of $K^{(3)}_{2,2,2}$, since then $\hom(F,\mathcal{H}) = \hom(K_{2,2,2}, G)$, for which *(c)* of Fact 5.2 yields

$$\left|\hom(F,\mathcal{H}) - d_3^0 \cdot d_2^{12} n^6\right| \leqslant 12\varepsilon_2 n^6 \overset{(11)}{\leqslant} \frac{2^0}{2^8}\eta \cdot d_2^{12} n^6 .$$

We assume, for a contradiction, that there exists an edge-minimal non-empty spanning subhypergraph F of $K^{(3)}_{2,2,2}$ for which (12) fails. W.l.o.g., we assume that $x_0y_0z_0 \in E(F)$ and we set $F^- = F - x_0y_0z_0$ to be the subhypergraph of F obtained by removing the hyperedge $x_0y_0z_0$. Since (12) fails for F but holds for F^-, we deduce that

$$\left|\hom(F,\mathcal{H}) - d_3 \cdot \hom(F^-,\mathcal{H})\right| > \frac{2^{|E(F)|} - d_3 \cdot 2^{|E(F^-)|}}{2^8}\eta \cdot d_2^{12} n^6 \geqslant \frac{\eta}{2^8} \cdot d_2^{12} n^6 . \tag{13}$$

We shall use the discrepancy in (13) to establish the existence of a subgraph $J_0 \subseteq G$ violating the regularity of $\mathcal{H}$:

$$\left|e_H(J_0) - d_3\,|\mathcal{K}_3(J_0)|\right| > \varepsilon_3 \cdot d_2^3 n^3 . \tag{14}$$

The proof of the existence of J_0 consist of four steps. First, we apply Theorem 4.2 and locate a triad in the regular partition where (13) carries over (in an appropriately scaled way) to copies of F^- and F that extend hyperedges supported by that triad (see (18) below).

Step 1: Applying the regularity lemma. We apply Theorem 4.2 to $\mathcal{H}$ with the chosen parameters d_2, ξ_3, ℓ_0, t_0, $\xi_2(\cdot)$ and $r(\cdot)$. Theorem 4.2 guarantees integers $\ell_0 \leqslant \ell \leqslant L_0$ and $t_0 \leqslant t \leqslant T_0$, vertex partitions $\dot\bigcup_{i\in[t]} X_i = X$, $\dot\bigcup_{j\in[t]} Y_j = Y$, and $\dot\bigcup_{k\in[t]} Z_k = Z$, edge-partitions

$$\dot\bigcup_{i,j\in[t]} \dot\bigcup_{\alpha\in[\ell]} P^{ij}_\alpha = E(G[X,Y]), \quad \dot\bigcup_{i,k\in[t]} \dot\bigcup_{\beta\in[\ell]} Q^{ik}_\beta = E(G[X,Z]), \quad\text{and}\quad \dot\bigcup_{j,k\in[t]} \dot\bigcup_{\gamma\in[\ell]} R^{jk}_\gamma = E(G[Y,Z]),$$

and complexes $\mathcal{H}^{ijk}_{\alpha\beta\gamma}$, $(i,j,k,\alpha,\beta,\gamma) \in [t]^3 \times [\ell]^3$, where

$$G^{ijk}_{\alpha\beta\gamma} = \left(X_i \cup Y_j \cup Z_k, P^{ij}_{\alpha} \cup Q^{ik}_{\beta} \cup R^{jk}_{\gamma}\right) \quad \text{and} \quad H^{ijk}_{\alpha\beta\gamma} = \left(X_i \cup Y_j \cup Z_k, E(H) \cap \mathcal{K}_3(G^{ijk}_{\alpha\beta\gamma})\right),$$

which satisfy properties (*a*) and (*b*) of its conclusion. We set

$$m = \frac{n}{t}, \qquad \xi_2 = \xi_2\Big(\frac{d_2}{\ell}\Big), \qquad \text{and} \qquad r = r\Big(\frac{d_2}{\ell}\Big)$$

and note that $m \geqslant n/T_0 \geqslant m_0(d_2/\ell)$ and by (9) we have

$$\xi_2 \leqslant \frac{\eta}{3 \cdot 2^{12}} \cdot \Big(\frac{d_2}{\ell}\Big)^{12}. \tag{15}$$

We remove hyperedges xyz from H when they belong to some $H^{ijk}_{\alpha\beta\gamma}$ for which $\mathcal{H}^{ijk}_{\alpha\beta\gamma}$ is not (ξ_3, r)-regular, and we let H' and $\mathcal{H}'$ denote the resulting hypergraph and complex. By (*b*) of Theorem 4.2 and (*a*) of Fact 5.2,

$$|E(H) \smallsetminus E(H')| \leqslant \xi_3 t^3 \ell^3 \cdot ((d_2/\ell)^3 m^3 + 3\xi_2 m^3) \overset{(15)}{\leqslant} 2\xi_3 d_2^3 n^3\,. \tag{16}$$

Consequently, (*b*) of Fact 5.2 applied to G and (16) implies

$$\begin{aligned}&\big|\hom(F,\mathcal{H}') - d_3 \cdot \hom(F^-,\mathcal{H}')\big| \\ &\qquad\geqslant \big|\hom(F,\mathcal{H}) - d_3 \cdot \hom(F^-,\mathcal{H})\big| - \left(2\xi_3 d_2^3 n^3 \cdot (d_2^9 n^3 + 4\varepsilon_2^{1/4} n^3) + 4\varepsilon_2^{1/4} n^3 \cdot n^3\right).\end{aligned}$$

Thus, inequality (13) can be transferred from $\mathcal{H}$ to $\mathcal{H}'$ by

$$\big|\hom(F,\mathcal{H}') - d_3 \cdot \hom(F^-,\mathcal{H}')\big| > \frac{\eta}{2^8} d_2^{12} n^6 - 2\xi_3 d_2^{12} n^6 - 12\varepsilon_2^{1/4} n^6 \overset{(9),(11)}{\geqslant} \frac{\eta}{2^9} \cdot d_2^{12} n^6\,. \tag{17}$$

Next we shall find a triad $G^{ijk}_{\alpha\beta\gamma}$ such that a similar (appropriately scaled) inequality like (17) holds for the homomorphisms of F and F^- in H' that map the three vertices x_1, y_1, z_1 to $\mathcal{K}_3(G^{ijk}_{\alpha\beta\gamma})$ and the other three vertices x_0, y_0, z_0 (which span the additional hyperedge in F) outside X_i, Y_j, and Z_k. For that we denote by $\hom(F,\mathcal{H}' \mid G^{ijk}_{\alpha\beta\gamma})$ (respectively by $\hom(F^-,\mathcal{H}' \mid G^{ijk}_{\alpha\beta\gamma})$) the number of those injective partite homomorphisms. It follows from (*c*) of Fact 5.2 that are at most

$$3t^5\ell^{12} \cdot \left((d_2/\ell)^{12} m^6 + 12\xi_2 m^6\right) \overset{(15)}{\leqslant} \frac{4}{t_0} \cdot d_2^{12} n^6 \overset{(10)}{\leqslant} \frac{\eta}{2^{10}} \cdot d_2^{12} n^6$$

homomorphism from $K_{2,2,2}$ in G with two vertices contained in the same vertex class from the vertex partitions $\dot\bigcup_{i\in[t]} X_i$, $\dot\bigcup_{j\in[t]} Y_j$, or $\dot\bigcup_{k\in[t]} Z_k$. Consequently, summing $\hom(F,\mathcal{H}' \mid G^{ijk}_{\alpha\beta\gamma})$ over all $t^3\ell^3$ triads $G^{ijk}_{\alpha\beta\gamma}$ of the regular partition yields

$$\sum_{i,j,k\in[t]} \sum_{\alpha,\beta,\gamma\in[\ell]} \hom(F,\mathcal{H}' \mid G^{ijk}_{\alpha\beta\gamma}) \geqslant \hom(F,\mathcal{H}') - \frac{\eta}{2^{10}} \cdot d_2^{12} n^6$$

and the same inequality holds for F^-. Therefore, (17) implies

$$\Big|\sum_{i,j,k\in[t]}\sum_{\alpha,\beta,\gamma\in[\ell]}\hom(F,\mathcal{H}'\mid G^{ijk}_{\alpha\beta\gamma})-d_3\cdot\sum_{i,j,k\in[t]}\sum_{\alpha,\beta,\gamma\in[\ell]}\hom(F^-,\mathcal{H}'\mid G^{ijk}_{\alpha\beta\gamma})\Big|>\frac{\eta}{2^{10}}\cdot d_2^{12}n^6\,.$$

Since there are $t^3\ell^3$ triads, by the pigeonhole principle there exists a triad $G^{ijk}_{\alpha\beta\gamma}$ such that

$$\big|\hom(F,\mathcal{H}'\mid G^{ijk}_{\alpha\beta\gamma})-d_3\cdot\hom(F^-,\mathcal{H}'\mid G^{ijk}_{\alpha\beta\gamma})\big|>\frac{\eta}{2^{10}}\cdot\frac{d_2^{12}n^6}{\ell^3t^3}\,. \tag{18}$$

We may assume that $i=j=k=t$ and $\alpha=\beta=\gamma=\ell$ and this concludes the first step.

Step 2: Further restricting the considered copies of F^- and F. In the second step, we further restrict the set of copies of F and F^- that we consider in (18). For that, fix $1\leqslant i\leqslant t-1$. We wish to select a fixed bipartite graph $P^{it}_{\alpha_i}$ among the ℓ such with vertex bipartition $X_i\cup Y_t$. More generally, for all $i,\,j,\,k\in[t-1]$ we wish to respectively select

$$P^{it}_{\alpha_i}\,,\ P^{tj}_{\alpha'_j}\,,\quad Q^{it}_{\beta_i}\,,\ Q^{tk}_{\beta'_k}\,,\quad\text{and}\quad R^{jt}_{\gamma_j}\,,\ R^{tk}_{\gamma'_h}$$

from the partition of pairs. To make these selections, for $\vec{a}=(\vec{\alpha},\vec{\alpha}',\vec{\beta},\vec{\beta}',\vec{\gamma},\vec{\gamma}')\in[\ell]^{6(t-1)}$ where

$$\begin{aligned}\vec{\alpha}&=(\alpha_1,\dots,\alpha_{t-1})\,, & \vec{\beta}&=(\beta_1,\dots,\beta_{t-1})\,, & \vec{\gamma}&=(\gamma_1,\dots,\gamma_{t-1})\,,\\ \vec{\alpha}'&=(\alpha'_1,\dots,\alpha'_{t-1})\,, & \vec{\beta}'&=(\beta'_1,\dots,\beta'_{t-1})\,, & \vec{\gamma}'&=(\gamma'_1,\dots,\gamma'_{t-1})\,,\end{aligned}$$

we denote by $\hom(F,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell},\vec{a})$ (respectively $\hom(F^-,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell},\vec{a})$) the number of partite homomorphisms φ from F (resp. F^-) to $\mathcal{H}'$ that satisfy

$$\varphi(x_1)\in X_t\,,\quad\varphi(y_1)\in Y_t\,,\quad\varphi(z_1)\in Z_t\,,\quad\varphi(x_1)\varphi(y_1)\varphi(z_1)\in\mathcal{K}_3(G^{ttt}_{\ell\ell\ell}) \tag{19}$$

(as before), and also that for some fixed indices $i,\,j,\,k\in[t-1]$,

$$\varphi(x_0)\in X_i\,,\quad\varphi(y_0)\in Y_j\,,\quad\varphi(z_0)\in Z_k\,, \tag{20}$$

and

$$\begin{aligned}\varphi(x_0)\varphi(y_1)&\in P^{it}_{\alpha_i}\,, & \varphi(x_0)\varphi(z_1)&\in Q^{it}_{\beta_i}\,, & \varphi(y_0)\varphi(z_1)&\in R^{jt}_{\gamma_j}\,,\\ \varphi(x_1)\varphi(y_0)&\in P^{tj}_{\alpha'_j}\,, & \varphi(x_1)\varphi(z_0)&\in Q^{tk}_{\beta'_k}\,, & \varphi(y_1)\varphi(z_0)&\in R^{tk}_{\gamma'_k}\,.\end{aligned}$$

Homomorphisms φ satisfying (19) alone are counted by $\hom(F,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell})$, and for each such there are precisely $\ell^{6(t-2)}$ vectors $\vec{a}\in[\ell]^{6(t-1)}$ so that φ also satisfies (20). Consequently,

$$\hom(F,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell})\cdot\ell^{6(t-2)}=\sum_{\vec{a}\in[\ell]^{6(t-1)}}\hom(F,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell},\vec{a})$$

and the same identity holds for F^-. Applying these identities to (18) yields

$$\frac{1}{\ell^{6(t-2)}}\left|\sum_{\vec{a}\in[\ell]^{6(t-1)}}\Big(\hom(F,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell},\vec{a})-d_3\cdot\hom(F^-,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell},\vec{a})\Big)\right|>\frac{\eta}{2^{10}}\cdot\frac{d_2^{12}n^6}{\ell^3t^3}\,,$$

and applying the triangle inequality further yields

$$\sum_{\vec{a}\in[\ell]^{6(t-1)}}\big|\hom(F,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell},\vec{a})-d_3\cdot\hom(F^-,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell},\vec{a})\big|>\frac{\eta}{2^{10}}\cdot\frac{d_2^{12}n^6}{\ell^3t^3}\cdot\ell^{6(t-2)}.$$

Averaging over the $\ell^{6(t-1)}$ terms above yields some vector $\vec{a}\in[\ell]^{6(t-1)}$ that satisfies

$$\Big|\hom(F,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell},\vec{a})-d_3\cdot\hom(F^-,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell},\vec{a})\Big|>\frac{\eta}{2^{10}}\cdot\frac{d_2^{12}n^6}{\ell^3t^3}\cdot\frac{\ell^{6(t-2)}}{\ell^{6(t-1)}}=\frac{\eta}{2^{10}}\cdot\frac{d_2^{12}n^6}{\ell^9t^3}\,. \tag{21}$$

In Step 1, we fixed the triad $G^{ttt}_{\ell\ell\ell}$ to satisfy (18), and in Step 2, we fixed the vector $\vec{a}$ to satisfy (21). We now observe that every triad $G^{ijk}_{\alpha\beta\gamma}$ with $i,j,k\in[t-1]$ determines a unique octahedral complex (see Definition 4.3)

$$\mathcal{O}^{ijk}_{\alpha\beta\gamma}=\big((X_i,X_t,Y_j,Y_t,Z_k,Z_t),\hat{G}^{ijk}_{\alpha\beta\gamma},\hat{H}^{ijk}_{\alpha\beta\gamma}\big)\,, \tag{22}$$

where the edge-set of $\hat{G}^{ijk}_{\alpha\beta\gamma}$ is given by the edges of the graph

$$P^{ij}_{\alpha}\cup P^{it}_{1}\cup P^{tj}_{1}\cup P^{tt}_{\ell}\cup Q^{ik}_{\beta}\cup Q^{it}_{1}\cup Q^{tk}_{1}\cup Q^{tt}_{\ell}\cup R^{jk}_{\gamma}\cup R^{jt}_{1}\cup R^{tk}_{1}\cup R^{tt}_{\ell}\,,$$

and where $E(\hat{H}^{ijk}_{\alpha\beta\gamma})=E(H')\cap\mathcal{K}_3(\hat{G}^{ijk}_{\alpha\beta\gamma})$. It follows by these constructions that

$$\hom(F,\mathcal{H}'\mid G^{ttt}_{\ell\ell\ell},\vec{a})=\sum_{i,j,k\in[t-1]}\sum_{\alpha,\beta,\gamma\in[\ell]}\hom(F,\mathcal{O}^{ijk}_{\alpha\beta\gamma})\,,$$

and the same identity holds for F^-. We may therefore rewrite (21) to say

$$\left|\sum_{i,j,k\in[t-1]}\sum_{\alpha,\beta,\gamma\in[\ell]}\Big(\hom(F,\mathcal{O}^{ijk}_{\alpha\beta\gamma})-d_3\cdot\hom(F^-,\mathcal{O}^{ijk}_{\alpha\beta\gamma})\Big)\right|>\frac{\eta}{2^{10}}\cdot\frac{d_2^{12}n^6}{\ell^9t^3}\,. \tag{23}$$

In Step 3, we will invoke Theorem 4.4 to evaluate the differences above.

Step 3: Applying the octahedral counting lemma. Theorem 4.4 expresses each $\hom(F,\mathcal{O}^{ijk}_{\alpha\beta\gamma})$ and $\hom(F^-,\mathcal{O}^{ijk}_{\alpha\beta\gamma})$ in (23) as products of densities of triads of $\mathcal{O}^{ijk}_{\alpha\beta\gamma}$. We express these same densities in terms of the following piece-wise defined weight function $w\colon V(G)\cup E(G)\longrightarrow[0,1]$. First, we weight all vertices $v\in V(G)$ and edges $e\in E(G)$ incident to $X_t\cup Y_t\cup Z_t$ with $w(v)=w(e)=1$. Then, we weight remaining vertices and edges of G systematically by the following constant functions: for each $(i,j,k,\alpha,\beta,\gamma) \in [t-1]^3 \times [\ell]^3$, set

$$w|_{X_i} \equiv \begin{cases} d(H' \mid G^{itt}_{11\ell}) & \text{if } x_0y_1z_1 \in E(F)\,, \\ 1 & \text{otherwise,} \end{cases}$$

$$w|_{Y_j} \equiv \begin{cases} d(H' \mid G^{tjt}_{1\ell 1}) & \text{if } x_1y_0z_1 \in E(F)\,, \\ 1 & \text{otherwise,} \end{cases}$$

$$w|_{Z_k} \equiv \begin{cases} d(H' \mid G^{ttk}_{\ell 11}) & \text{if } x_1y_1z_0 \in E(F)\,, \\ 1 & \text{otherwise,} \end{cases}$$

$$w|_{P^{ij}_\alpha} \equiv \begin{cases} d(H' \mid G^{ijt}_{\alpha 11}) & \text{if } x_0y_0z_1 \in E(F)\,, \\ 1 & \text{otherwise,} \end{cases}$$

$$w|_{Q^{ik}_\beta} \equiv \begin{cases} d(H' \mid G^{itk}_{1\beta 1}) & \text{if } x_0y_1z_0 \in E(F)\,, \\ 1 & \text{otherwise,} \end{cases}$$

$$\text{and} \quad w|_{R^{jk}_\gamma} \equiv \begin{cases} d(H' \mid G^{tjk}_{11\gamma}) & \text{if } x_1y_0z_0 \in E(F)\,, \\ 1 & \text{otherwise.} \end{cases}$$

Finally, we define the weight of the triad $G^{ijk}_{\alpha\beta\gamma}$ by the product of the six values given to its vertex classes and edge sets, i.e., we set

$$w(G^{ijk}_{\alpha\beta\gamma}) = w(X_i)w(Y_j)w(Z_k)w(P^{ij}_\alpha)w(Q^{ik}_\beta)w(R^{jk}_\gamma)\,. \tag{24}$$

The number of copies of F (resp. F^-) in $\mathcal{O}^{ijk}_{\alpha\beta\gamma}$ also depends on $d(H' \mid G^{ttt}_{\ell\ell\ell})$ if $x_1y_1z_1$ is an edge of F and we set

$$w_{111} = \begin{cases} d(H' \mid G^{ttt}_{\ell\ell\ell}) & \text{if } x_1y_1z_1 \in E(F)\,, \\ 1 & \text{otherwise.} \end{cases}$$

We can now use the weights defined above to rewrite (23). Since H' is (ξ_3, r)-regular w.r.t. every triad of the regular partition we obtain from Theorem 4.4

$$\left| \hom(F, \mathcal{H}' \mid G^{ttt}_{\ell\ell\ell}, \vec{a}) - \sum_{i,j,k\in[t-1]} \sum_{\alpha,\beta,\gamma\in[\ell]} d(H' \mid G^{ijk}_{\alpha\beta\gamma}) w(G^{ijk}_{\alpha\beta\gamma}) w_{111} \cdot \left(\frac{d_2}{\ell}\right)^{12} m^6 \right|$$
$$\leqslant (t-1)^3\ell^3 \cdot \vartheta \left(\frac{d_2}{\ell}\right)^{12} m^6 \leqslant \vartheta \frac{d_2^{12} n^6}{\ell^9 t^3}\,.$$

Similarly, for F^- we arrive at

$$\left|\hom(F^-,\mathcal{H}' \mid G^{ttt}_{\ell\ell\ell},\vec{a}) - \sum_{i,j,k\in[t-1]}\sum_{\alpha,\beta,\gamma\in[\ell]} w(G^{ijk}_{\alpha\beta\gamma})w_{111}\cdot\Big(\frac{d_2}{\ell}\Big)^{12}m^6\right| \leqslant \vartheta\frac{d_2^{12}n^6}{\ell^9t^3}\,.$$

This way we can rewrite (21) and after dividing both sides with $(d_2/\ell)^9m^3$ we obtain

$$\left|\sum_{i,j,k\in[t-1]}\sum_{\alpha,\beta,\gamma\in[\ell]}\big(d(H'\mid G^{ijk}_{\alpha\beta\gamma})-d_3\big)w(G^{ijk}_{\alpha\beta\gamma})w_{111}\cdot\Big(\frac{d_2}{\ell}\Big)^3m^3\right| > \Big(\frac{\eta}{2^{10}}-2\vartheta\Big)\cdot d_2^3n^3 \overset{(8)}{\geqslant} \frac{\eta}{2^{11}}\cdot d_2^3n^3\,.$$

It follows that $w_{111}>0$ and, since by definition $w_{111}\leqslant 1$, we may divide both sides by w_{111} and, owing to another application of *(a)* of Fact 5.2 for every triad $G^{ijk}_{\alpha\beta\gamma}$ considered in the sum, we can replace $(d_2/\ell)^3m^3$ by $|\mathcal{K}_3(G^{ijk}_{\alpha\beta\gamma})|\pm 3\xi_2m^3$. This way we obtain

$$\begin{aligned}\left|\sum_{i,j,k\in[t-1]}\sum_{\alpha,\beta,\gamma\in[\ell]}\big(d(H'\mid G^{ijk}_{\alpha\beta\gamma})-d_3\big)w(G^{ijk}_{\alpha\beta\gamma})\big|\mathcal{K}_3(G^{ijk}_{\alpha\beta\gamma})\big|\right| &> \frac{\eta}{2^{11}}\cdot d_2^3n^3-3(t-1)^3\ell^3\xi_2m^3\\ &\overset{(15)}{\geqslant} \frac{\eta}{2^{12}}\cdot d_2^3n^3 \overset{(7)}{=} 2\varepsilon_3\cdot d_2^3n^3\,.\end{aligned}$$

Rewriting the left-hand side by summing over all triangles of $G'=G[X\smallsetminus X_t, Y\smallsetminus Y_t, Z\smallsetminus Z_t]$ instead over all triads $G^{ijk}_{\alpha\beta\gamma}\subseteq G'$ and expanding $w(G^{ijk}_{\alpha\beta\gamma})$ according to (24) tells us

$$\left|\sum_{xyz\in\mathcal{K}_3(G')}\big(\mathbb{1}_{E(H')}(x,y,z)-d_3\big)w(x)w(y)w(z)w(xy)w(xz)w(yz)\right| > 2\varepsilon_3\cdot d_2^3n^3\,. \tag{25}$$

Step 4: Determining the promised subgraph $J_0\subseteq G$. Inequality (25) shows that there exists a weighted subgraph of G' such that the weighted version of Definition 1.2 fails. Since all weights are in $[0,1]$, we may view them as a probability distribution over all subgraph of $J\subseteq G'$ and the left-hand side of (25) corresponds the expected value of $|e_{H'}(J)-d_3\,|\mathcal{K}_3(J)||$. Consequently, there exists a concrete subgraph $J\subseteq G'\subseteq G$ such that

$$\big|e_{H'}(J)-d_3\,|\mathcal{K}_3(J)|\big| > 2\varepsilon_3\cdot d_2^3n^3\,.$$

Finally, (16) allows us to move back from H' to H and we get the desired inequality

$$\big|e_H(J)-d_3\,|\mathcal{K}_3(J)|\big| \geqslant \big|e_{H'}(J)-d_3\,|\mathcal{K}_3(J)|\big| - 2\xi_3d_2^3n^3 \overset{(9)}{>} \varepsilon_3\cdot d_2^3n^3\,,$$

which yields the desired contradiction to the (ε_3,d_3)-regularity of the (ε_2,d_2,n)-complex $\mathcal{H}$ and concludes the proof of Theorem 3.1. $\square$

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