ON QUANTITATIVE ASPECTS OF A CANONISATION THEOREM FOR EDGE-ORDERINGS

CHRISTIAN REIHER, VOJTĚCH RÖDL, MARCELO SALES, KEVIN SAMES, AND MATHIAS SCHACHT

Dedicated to the memory of Ronald Graham

Abstract. For integers $k \geqslant 2$ and $N \geqslant 2k+1$ there are $k!2^k$ *canonical orderings* of the edges of the complete k-uniform hypergraph with vertex set $[N] = \{1, 2, \dots, N\}$. These are exactly the orderings with the property that any two subsets $A, B \subseteq [N]$ of the same size induce isomorphic suborderings. We study the associated *canonisation* problem to estimate, given k and n, the least integer N such that no matter how the k-subsets of $[N]$ are ordered there always exists an n-element set $X \subseteq [N]$ whose k-subsets are ordered canonically. For fixed k we prove lower and upper bounds on these numbers that are k times iterated exponential in a polynomial of n.

§1. Introduction

We consider numerical aspects of an unpublished result of Klaus Leeb from the early seventies (see [16, 17]). For a natural number N we set $[N] = \{1, 2, \dots, N\}$. Given a set X and a nonnegative integer k we write $X^{(k)} = \{e \subseteq X : |e| = k\}$ for the set of all k-element subsets of X.

1.1. **Ramsey theory.** Recall that for any integers $k, n, r \geqslant 1$, Ramsey's theorem [18] informs us that every sufficiently large integer N satisfies the partition relation

$$N \longrightarrow (n)^k_r \,, \tag{1.1}$$

meaning that for every colouring of $[N]^{(k)}$ with r colours there exists a set $X \subseteq [N]$ of size n such that $X^{(k)}$ is monochromatic. The least number N validating (1.1) is denoted by $\mathrm{R}^{(k)}(n, r)$. The *negative* partition relation $N \longrightarrow\!\!\!\!\!\!/\;\; (n)^k_r$ expresses the fact that (1.1) fails.

For colourings $f : [N]^{(k)} \longrightarrow \mathbb{N}$ with infinitely many colours, however, one can, in general, not even hope to obtain a monochromatic set of size $k+1$. For instance, it may happen that every k-element subset of $[N]$ has its own private colour. Nevertheless, Erdős and Rado [8, 9] established a meaningful generalisation of Ramsey's theorem to colourings

The second author is supported by NSF grant DMS 1764385.

The third author is partially supported by NSF grant DMS 1764385.

The fifth author is supported by the ERC (PEPCo 724903).

with infinitely many colours, even if the ground sets stay finite. In this context it is convenient to regard such colourings as equivalence relations, two k-sets being in the same equivalence class if and only if they are of the same colour. Now Erdős and Rado [8, 9] call an equivalence relation on $[N]^{(k)}$ *canonical* if for any two subsets $A, B \subseteq [N]$ of the same size the equivalence relations induced on $A^{(k)}$ and $B^{(k)}$ correspond to each other via the order preserving map between A and B. So, roughly speaking, an equivalence relation is canonical if you cannot change its essential properties by restricting your attention to a subset. It turns out that for $N \geqslant 2k+1$ there are always exactly 2^k canonical equivalence relations on $[N]^{(k)}$ that can be parametrised by subsets $I \subseteq [k]$ in the following natural way. Let $x, y \in [N]^{(k)}$ be two k-sets and let $x = \{x_1, \dots, x_k\}$ as well as $y = \{y_1, \dots, y_k\}$ enumerate their elements in increasing order. We write $x \equiv_I y$ if and only if $x_i = y_i$ holds for all $i \in I$. Now $\{\equiv_I : I \subseteq [k]\}$ is the collection of all canonical equivalence relations on $[N]^{(k)}$. The *canonical Ramsey theorem* of Erdős and Rado asserts that given two integers k and n there exists an integer $\mathrm{ER}^{(k)}(n)$ such that for every $N \geqslant \mathrm{ER}^{(k)}(n)$ and every equivalence relation $\equiv$ on $[N]^{(k)}$ there exists a set $X \subseteq [N]$ of size n with the property that $\equiv$ is canonical on $X^{(k)}$.

This result sparked the development of *canonical Ramsey theory* in the seventies. Let us commemorate some contributions to the area due to Ronald Graham, who was among the main protagonists of Ramsey theory in the 20^{th} century: Together with Erdős [5], he established a canonical version of van der Waerden's theorem, stating that if for $N \geqslant \mathrm{EG}(k)$ we colour $[N]$ with arbitrarily many colours, then there exists an arithmetic progression of length k which is either monochromatic or receives k distinct colours. With Deuber, Prömel, and Voigt [3] he later obtained a canonical version of the Gallai-Witt theorem, which is more difficult to state — in d dimensions the canonical colourings for a given configuration $F \subseteq \mathbb{Z}^d$ are now parametrised by vector subspaces $U \subseteq \mathbb{Q}^d$ having a basis of vectors of the form $x - y$ with $x, y \in F$. Interestingly, Ronald Graham was also among the small number of co-authors of Klaus Leeb [12, 13] and in joint work with Rothschild they settled a famous conjecture of Rota on Ramsey properties of finite vector spaces.

The canonisation discussed here concerns linear orderings. We say that a linear order $([N]^{(k)}, \prec)$ is *canonical* if for any two sets $A, B \subseteq [N]$ of the same size the order preserving map from A to B induces an order preserving map from $(A^{(k)}, \prec)$ to $(B^{(k)}, \prec)$. It turns out that for $N \geqslant 2k+1$ there are exactly $k!2^k$ canonical orderings of $[N]^{(k)}$ that can uniformly be parametrised by pairs (ε, σ) consisting of a *sign vector* $\varepsilon \in \{-1, +1\}^k$ and a permutation $\sigma \in \mathfrak{S}_k$, i.e., a bijective map $\sigma : [k] \longrightarrow [k]$.

Definition 1.1. *Let $N, k \geqslant 1$ be integers, and let $(\varepsilon, \sigma) \in \{+1, -1\}^k \times \mathfrak{S}_k$ be a pair consisting of a sign vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ and a permutation σ of $[k]$. The ordering $\prec$*

on $[N]^{(k)}$ *associated with* (ε, σ) *is the unique ordering with the property that if*

$$1 \leqslant a_1 < \cdots < a_k \leqslant N \quad \text{and} \quad 1 \leqslant b_1 < \cdots < b_k \leqslant N\,,$$

then

$$\{a_1, \ldots, a_k\} \prec \{b_1, \ldots, b_k\} \iff (\varepsilon_{\sigma(1)} a_{\sigma(1)}, \ldots, \varepsilon_{\sigma(k)} a_{\sigma(k)}) <_{\mathrm{lex}} (\varepsilon_{\sigma(1)} b_{\sigma(1)}, \ldots, \varepsilon_{\sigma(k)} b_{\sigma(k)})\,,$$

where $<_{\mathrm{lex}}$ *denotes the lexicographic order on* $\mathbb{Z}^k$*.*

It is readily seen that such an ordering $\prec$ associated with (ε, σ) does indeed exist, that it is uniquely determined, and, moreover, canonical. Conversely, for $N \geqslant 2k+1$ every canonical ordering is of this form (see Lemma 3.8 below). We proceed by stating the aforementioned unpublished theorem due to Klaus Leeb.

Theorem 1.2 (Leeb)**.** *For all positive integers* k *and* n *there exists an integer* $\mathrm{L}^{(k)}(n)$ *such that for every* $N \geqslant \mathrm{L}^{(k)}(n)$ *and every ordering* $([N]^{(k)}, \prec)$ *there is a set* $X \subseteq [N]$ *of size* n *such that* $(X^{(k)}, \prec)$ *is canonical.*

In the sequel, $\mathrm{L}^{(k)}(n)$ will always refer to the least number with this property. As we shall prove, for fixed $k \geqslant 2$ the dependence of $\mathrm{L}^{(k)}(n)$ on n is k times exponential in a polynomial of n.

1.2. **Bounds.** The study of the Ramsey numbers $\mathrm{R}^{(k)}(n, r)$ has received extensive interest (see e.g. [2, 6, 7, 14]). To state some of the currently known bounds, one recursively defines *tower functions* $t_k \colon \mathbb{N} \longrightarrow \mathbb{N}$ by

$$t_0(x) = x \quad \text{and} \quad t_{i+1}(x) = 2^{t_i(x)}\,.$$

Arguing probabilistically Erdős [4] showed that $\mathrm{R}^{(2)}(n, 2) \geqslant 2^{(n+1)/2}$ and Erdős, Hajnal, and Rado [7] proved later that $\mathrm{R}^{(k)}(n, 2) \geqslant t_{k-2}(cn^2)$ for $k > 2$, where c is a positive constant. They also proved $\mathrm{R}^{(k)}(n, 4) \geqslant t_{k-1}(c_k n)$ for all $k \geqslant 2$, where $c_k > 0$ is an absolute constant. In the other direction, it is known that $\mathrm{R}^{(k)}(n, r) \leqslant t_{k-1}(C_{k,r} n)$.

Estimates on the canonical Ramsey numbers $\mathrm{ER}^{(k)}(n)$ were studied in [15, 19]. Notably, Lefmann and Rödl proved the lower bound $\mathrm{ER}^{(k)}(n) \geqslant t_{k-1}(c_k n^2)$, while Shelah obtained the complementary upper bound $\mathrm{ER}^{(k)}(n) \leqslant t_{k-1}(C_k n^{8(2k-1)})$.

Let us now turn our attention to the Leeb numbers $\mathrm{L}^{(k)}(n)$. For $k = 1$, there exist only two canonical orderings of $[N]^{(1)}$, namely the “increasing” and the “decreasing” one corresponding to the sign vectors $\varepsilon = (+1)$ and $\varepsilon = (-1)$, respectively. Thus the well-known Erdős-Szekeres theorem [10] yields the exact value $\mathrm{L}^{(1)}(n) = (n-1)^2 + 1$. Accordingly, Leeb's theorem can be viewed as a multidimensional version of the Erdős-Szekeres theorem and we refer to [1] for further variations on this theme. Our own results can be summarised as follows.

Theorem 1.3. *If $n \geqslant 4$ and $R \longrightarrow\!\!\!\!\!\!/\;\; (n-1)^2_2$, then $\mathrm{L}^{(2)}(n) > 2^R$. Moreover, if $n \geqslant k \geqslant 3$, then $\mathrm{L}^{(k)}(4n+k) > 2^{\mathrm{L}^{(k-1)}(n)-1}$.*

Due to the known lower bounds on diagonal Ramsey numbers this implies

$$\mathrm{L}^{(2)}(n) \geqslant t_2(n/2) \quad \text{as well as} \quad \mathrm{L}^{(k)}(n) \geqslant t_k(c_k n) \text{ for } k \geqslant 3\,.$$

We offer an upper bound with the same number of exponentiations.

Theorem 1.4. *For every $k \geqslant 2$ there exists a constant C_k such that $\mathrm{L}^{(k)}(n) \leqslant t_k(n^{C_k})$ holds for every $n \geqslant 2k$.*

The case $k = 2$ of Theorems 1.3 and 1.4 was obtained independently by Conlon, Fox, and Sudakov in unpublished work [11].

Organisation. We prove Theorem 1.3 in Section 2. The upper bound, Theorem 1.4, is established in Section 3. Lemma 3.3 from this proof turns out to be applicable to Erdős-Rado numbers as well and Section 4 illustrates this by showing a variant of Shelah's result, namely $\mathrm{ER}^{(k)}(n) \leqslant t_{k-1}(C_k n^{6k})$.

§2. Lower bounds: Constructions

2.1. **Trees and combs.** Our lower bound constructions take inspiration from the negative stepping-up lemma due to Erdős, Hajnal, and Rado [7]. Notice that in order to prove an inequality of the form $\mathrm{L}^{(k)}(n) > N$ one needs to exhibit a special ordering $([N]^{(k)}, \prec)$ for which there is no set $Z \subseteq [N]$ of size n such that $\prec$ orders $Z^{(k)}$ canonically. For us, $N = 2^m$ will always be a power of two and for reasons of visualisability we identify the numbers in $[N]$ with the leaves of a binary tree of height m. We label the levels of our tree from the bottom to the top with the numbers in $[m]$. So the root is at level 1 and the leaves are immediately above level m (see Figure 2.1).

Alternatively, we can identify $[N]$ with the set $\{+1, -1\}^m$ of all ± 1-vectors of length m such that the standard ordering on $[N]$ corresponds to the lexicographic one on $\{+1, -1\}^m$. The literature often works with the number 0 instead of -1 here, but for the application we have in mind our choice turns out to be advantageous.

Now every set $X \subseteq [N]$ generates a tree T_X, which is the subtree of our binary tree given by the union of the paths from the leaves in X to the root (see Figure 2.2).

An essential observation on which both the double-exponential lower bound construction for the four-colour Ramsey number of triples and our lower bound on $\mathrm{L}^{(2)}(n)$ rely is that

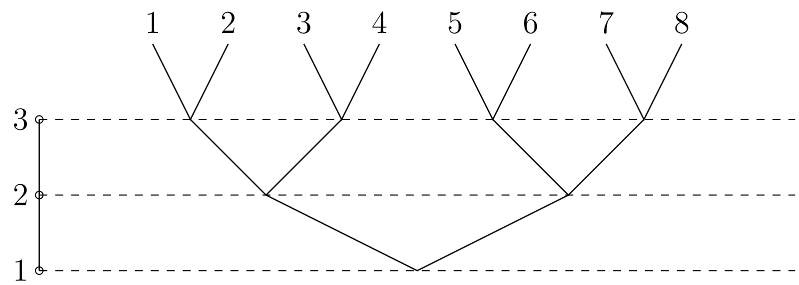

FIGURE 2.1. The binary tree and its levels for $m = 3$.

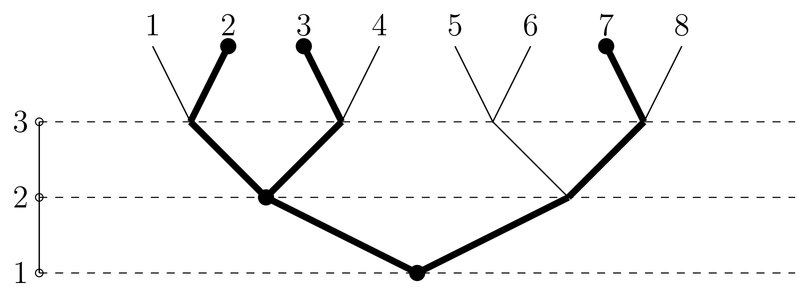

FIGURE 2.2. The auxiliary tree T_X for $X = \{2, 3, 7\}$.

there are two essentially different kinds of triples: those engendering a "left tree" and those engendering a "right tree". Roughly speaking the difference is that when descending from the leaves to the root, the two elements concurring first are the two smaller ones for left trees and the two larger ones for right trees. For instance, Figure 2.2 displays a left tree.

To make these notions more precise we introduce the following notation. Given at least two distinct vectors $x_1, \dots, x_t \in \{-1, +1\}^m$ with coordinates $x_i = (x_{i1}, \dots, x_{im})$ we write

$$\delta(x_1, \dots, x_t) = \min\{\mu \in [m] \colon x_{1\mu} = \dots = x_{t\mu} \text{ is not the case}\} .$$

for the first level where two of them differ.

Now let $xyz \in [N]^{(3)}$ be an arbitrary triple with $x < y < z$ and set $\delta = \delta(x, y, z)$. This means that x, y, and z agree in the first $\delta - 1$ coordinates, while their entries in the δ^{th} coordinate, which we denote by x_δ, y_δ, z_δ, fail to coincide. As a consequence of $x < y < z$ we know $x_\delta \leqslant y_\delta \leqslant z_\delta$. Thus $x_\delta = -1$, $z_\delta = +1$, and there remain two possibilities: Depending on whether $y_\delta = -1$ or $y_\delta = +1$ we say that xyz forms a *left tree* or a *right tree*, respectively. Equivalently, xyz yields a left tree whenever $\delta(x, y) > \delta(y, z)$ and a right tree whenever $\delta(x, y) < \delta(y, z)$.

When the triples are coloured with $\{\text{left}, \text{right}\}$ depending on whether they form left trees or right trees, the monochromatic sets are called *combs*. More explicitly, for

$$1 \leqslant x_1 < \dots < x_t \leqslant N$$

the set $X = \{x_1, \dots, x_t\}$ is a *left comb* if $\delta(x_1, x_2) > \delta(x_2, x_3) > \dots > \delta(x_{t-1}, x_t)$ and a *right comb* if $\delta(x_1, x_2) < \delta(x_2, x_3) < \dots < \delta(x_{t-1}, x_t)$ (see Figure 2.3). For instance, the empty set, every singleton, and every pair are both a left comb and a right comb. Since every triple is either a left tree or a right tree, triples are combs in exactly one of the two directions. Some quadruples fail to be combs.

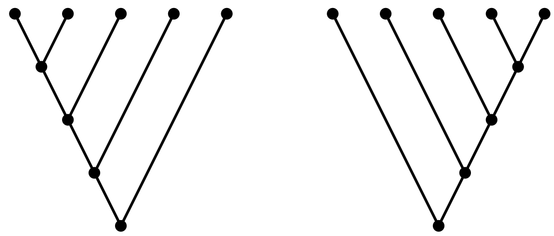

FIGURE 2.3. A left comb and a right comb.

2.2. **Ordering pairs.** Let us recall that for Ramsey numbers the negative stepping-up lemma shows, e.g., that if $m \longarrownot\longrightarrow (n-1)^2_2$, then $2^m \longarrownot\longrightarrow (n)^3_4$. As we shall reuse the argument, we provide a brief sketch. Let $f\colon [2^m]^{(3)} \longrightarrow \{\text{left}, \text{right}\}$ be the colouring telling us whether a triple generates a left tree or a right tree and let $g\colon [m]^{(2)} \longrightarrow \{+1, -1\}$ be a colouring exemplifying the negative partition relation $m \longarrownot\longrightarrow (n-1)^2_2$. Define a colouring

$$h\colon [2^m]^{(3)} \longrightarrow \{\text{left}, \text{right}\} \times \{+1, -1\}$$

of the triples with four colours by

$$h(xyz) = \big(f(xyz), g\big(\delta(xy), \delta(yz)\big)\big)$$

whenever $1 \leqslant x < y < z \leqslant 2^m$. If $Z \subseteq [2^m]$ is monochromatic with respect to h, then, in particular, it needs to be monochromatic for f, i.e., Z is a comb. Thus, if $Z = \{z_1, \dots, z_{|Z|}\}$ with $z_1 < z_2 < \dots < z_{|Z|}$, then $\{\delta(z_i, z_{i+1})\colon i < |Z|\}$ is a set of size $|Z| - 1$ whose pairs are monochromatic with respect to g, for which reason $|Z| - 1 < n - 1$. In other words, h establishes the desired negative partition relation $2^m \longarrownot\longrightarrow (n)^3_4$.

In the light of this reasoning it suffices to construct, in the same setup, an ordering $([2^m]^{(2)}, \dashv)$ such that for every triple $e \in [2^m]^{(3)}$ the colour $h(e)$ from the foregoing paragraph determines the "isomorphism type" of $(e^{(2)}, \dashv)$ and vice versa. Let xyz with $1 \leqslant x < y < z \leqslant 2^m$ be an arbitrary triple and set $\xi = \delta(xy)$ as well as $\eta = \delta(yz)$. Our goal when constructing $\dashv$ is that

if xyz is a	and	then
left tree	$g(\eta, \xi) = +1$	$xy \prec xz \prec yz$,
left tree	$g(\eta, \xi) = -1$	$xy \prec yz \prec xz$,
right tree	$g(\xi, \eta) = +1$	$yz \prec xy \prec xz$,
right tree	$g(\xi, \eta) = -1$	$yz \prec xz \prec xy$.

If this can be accomplished, then every set $Z \subseteq [2^m]$, for which $(Z^{(2)}, \prec)$ is ordered canonically, is monochromatic with respect to the colouring h and, therefore, the proof of $\mathrm{L}^{(2)}(n) > 2^m$ will be complete.

Let us introduce one final piece of notation. Given a vector $x = (x_1, \dots, x_m) \in \{-1, +1\}^m$ and $\delta \in [m]$ we define the vector $x \ltimes \delta \in \{-1, +1\}^{m-\delta}$ by

$$x \ltimes \delta = \big(x_{\delta+1} g(\delta, \delta+1), \dots, x_m g(\delta, m)\big)\,.$$

Now let $\prec$ be an arbitrary ordering with the property that for any two distinct pairs xy and $x'y'$ with $x < y$ and $x' < y'$ the following statements hold.

(*1*) If $\delta(xy) \neq \delta(x'y')$, then

$$xy \prec x'y' \iff \delta(x'y') < \delta(xy)\,.$$

(*2*) If $\delta = \delta(xy) = \delta(x'y')$ and at least one of $x \ltimes \delta \neq x' \ltimes \delta$ or $y \ltimes \delta \neq y' \ltimes \delta$ holds, then

$$xy \prec x'y' \iff (x \ltimes \delta) \circ (y \ltimes \delta) <_{\mathrm{lex}} (x' \ltimes \delta) \circ (y' \ltimes \delta)\,,$$

where $\circ$ indicates concatenation, i.e., that we are comparing $(2m - 2\delta)$-tuples lexicographically.

It remains to show that these two properties of $\prec$ imply the four statements in the above table. Let us return to this end to our triple xyz with $1 \leqslant x < y < z \leqslant 2^m$ and $\xi = \delta(xy)$, $\eta = \delta(yz)$. If xyz is a left tree, then $\delta(xz) = \delta(yz) < \delta(xy)$, for which reason (*1*) implies $xy \prec xz, yz$ and it remains to compare the latter two pairs. Since $x \ltimes \eta \neq y \ltimes \eta$, the second rule declares that $xz \prec yz$ holds if and only if $x \ltimes \eta <_{\mathrm{lex}} y \ltimes \eta$. Now x and y agree in the first $\xi - 1$ coordinates and continue with $x_\xi = -1$, $y_\xi = +1$. Thus $x \ltimes \eta$ and $y \ltimes \eta$ agree in their first $\xi - \eta - 1$ coordinates and proceed with $x_\xi g(\eta, \xi)$, $y_\xi g(\eta, \xi)$, respectively. So altogether $xz \prec yz$ is equivalent to $g(\eta, \xi) = +1$ and we have confirmed the two upper rows of the table. The case where xyz is a right tree is discussed similarly. Summarising, we have thereby proved the first assertion in Theorem 1.3.

2.3. **Negative stepping-up lemma for Leeb numbers.** Throughout this subsection we fix two integers $k \geqslant 3$ and $n \geqslant k$. We set $m = \mathrm{L}^{(k-1)}(n) - 1$ and study orderings on $[2^m]^{(k)}$, where 2^m is again identified with $\{+1, -1\}^m$. We are to exhibit an ordering $([2^m]^{(k)}, \prec)$ without canonically ordered subsets of size $4n + k$. Preparing ourselves we consider the following concept.

Definition 2.1. *A set $Z \subseteq [2^m]$ is said to be impeded (see Figure 2.4) if there exist two elements $x < y$ in Z with the following properties.*

- *(a) At least k members of Z lie between x and y.*
- *(b) Setting $\delta = \delta(x, y)$ there exist $x', x'', y', y'' \in Z$ such that $x < x' < x'' < y'' < y' < y$ and $x''_\delta = -1$ while $y''_\delta = +1$.*

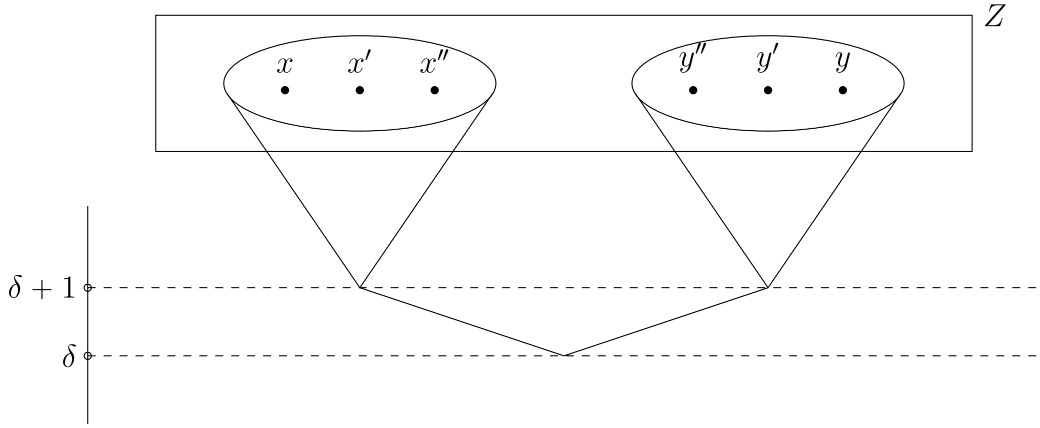

FIGURE 2.4. An impeded set Z.

Roughly speaking, every unimpeded set contains a large comb.

Lemma 2.2. *Every set $Z \subseteq [2^m]$ is either impeded or possesses two subsets $L, R \subseteq Z$ with $|L| + |R| \geqslant \frac{1}{2}(|Z| - k + 1)$ forming a left comb and a right comb, respectively.*

Proof. Otherwise let Z be a minimal counterexample. Since every set with at most two elements is a two-sided comb, we have $|Z| \geqslant k + 2$. Now the vectors $x = \min(Z)$ and $y = \max(Z)$ satisfy part (a) of Definition 2.1. Let x', x'' be the second and third-smallest vector in Z and, similarly, let y' and y'' be the second and third largest one. If both $x''_\delta = -1$ and $y''_\delta = +1$ would hold, where $\delta = \delta(x, y)$, then the set Z were impeded, contrary to its choice.

Thus at least one of the cases $x''_\delta = +1$ and $y''_\delta = -1$ occurs and for reasons of symmetry we may suppose that $x''_\delta = +1$. The set $Z_\star = Z \setminus \{x, x'\}$ cannot be impeded for then Z was impeded as well. So by the minimality of Z there exist sets $L, R_\star \subseteq Z_\star$ forming a left comb and a right comb, respectively, and satisfying $|L| + |R_\star| \geqslant \frac{1}{2}(|Z_\star| - k + 1)$.

For any two elements $r < r'$ from $R_\star$ we have $\delta(x, r) = \delta < \delta(r, r')$ and, consequently, $R = \{x\} \cup R_\star$ is a right comb. But now $|L| + |R| = |L| + |R_\star| + 1 \geqslant \frac{1}{2}(|Z| - k + 1)$ yields a final contradiction. □

We proceed with some further notation. Let $<_{\text{lex}}$ denote the lexicographic ordering on $[2^m]^{(k)}$. So explicitly for $X, Y \in [2^m]^{(k)}$ we have $X <_{\text{lex}} Y$ if and only if $\sum_{x \in X} 2^{-x}$ is larger than $\sum_{y \in Y} 2^{-y}$.

For every set $X \in [2^m]^{(k)}$ we determine a natural number $\varphi(X)$ according to the following rule: Consider $x^- = \min(X)$, $x^+ = \max(X)$ and compute $\delta = \delta(x^-, x^+)$. Notice that all elements of X agree in their first $\delta - 1$ coordinates, while in the δ^{th} coordinate $x^-_\delta = -1$ and $x^+_\delta = +1$. Let $\varphi(X)$ count the number of members of X having a -1 as their δ^{th} coordinate. Notice that if X is a left comb, then $\varphi(X) = |X| - 1$, while right combs satisfy $\varphi(X) = 1$.

Next, if B is either a left or a right comb, we know that $\pi(B) = \{\delta(xy) \colon xy \in B^{(2)}\} \subseteq [m]$ is a set of size $|B| - 1$ whose elements can be found by applying δ to the $|B| - 1$ pairs of consecutive members of B. We call $\pi(B)$ the *projection* of B.

Finally, instead of a colouring we shall apply an appropriate ordering $([m]^{(k-1)}, \prec)$ senkrecht. Because of $m < \mathrm{L}^{(k-1)}(n)$ this ordering can be chosen without canonical subsets of size n.

We now state four desirable properties we would like an ordering $([2^m]^{(k)}, \dashv)$ to satisfy for any two distinct sets $X, Y \in [2^m]^{(k)}$.

(*1*) If $\varphi(X)$ is odd and $\varphi(Y)$ is even, then $X \dashv Y$.
(*2*) If X and Y are two combs in the same direction having different projections, then $X \dashv Y \iff \pi(X) \prec \pi(Y)$.
(*3*) If X and Y are left combs with $\pi(X) = \pi(Y)$, then $X \dashv Y \iff Y <_{\text{lex}} X$.
(*4*) If X and Y are right combs with $\pi(X) = \pi(Y)$, then $X \dashv Y \iff X <_{\text{lex}} Y$.

It is not hard to see that these properties do not contradict each other. Indeed, in order to satisfy (*1*) one partitions $[2^m]^{(k)}$ into two sets depending on the parity of φ and puts the odd class before the even class. The remaining rules (*2*)–(*4*) only concern the internal orderings within these two classes. Similarly, (*2*) tells us to sub-partition the combs within both classes according to their projections and (*3*), (*4*) only give further instructions how to compare combs within the same sub-class.

Now it suffices to show for no ordering $([2^m]^{(k)}, \dashv)$ satisfying (*1*)–(*4*) there is a set $Z \subseteq [2^m]$ of size $4n + k$ such that $(Z^{(k)}, \dashv)$ is ordered canonically. Assume contrariwise that $\dashv$ and Z have these properties. If $L \subseteq Z$ is a left comb, then so is every $X \in L^{(k)}$ and by (*2*) it follows that $\pi(L)^{(k-1)}$ is ordered canonically by $\prec$. So our choice of $\prec$

yields $|\pi(L)| \leqslant n-1$ and, consequently, $|L| \leqslant n$. For the same reason, every right comb $R \subseteq Z$ has at most the size n. But now Lemma 2.2 tells us that Z is impeded. Pick $x, x', x'', y'', y', y \in Z$ as in Definition 2.1.

Special case: $k = 3$

Recall that every triple is either a left tree or a right tree. Owing to *(1)* the right trees precede the left trees with respect to $\dashv_3$.

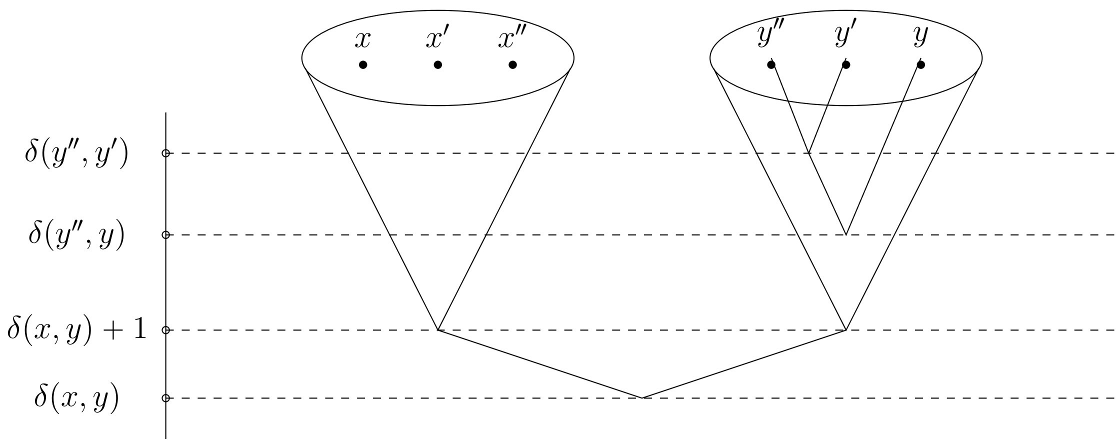

FIGURE 2.5. The case $k = 3$.

Now consider the quintuples $A = \{x, x', x'', y', y\}$ and $B = \{x, x', y'', y', y\}$, whose elements have just been listed in increasing order. By canonicity we know that $(A^{(3)}, \dashv_3)$ and $(B^{(3)}, \dashv_3)$ are isomorphic via the map that sends x'' to y'' and keeps the four other elements fixed. Since $xx''y'$ and $xx''y$ are left trees with the same projection, *(3)* yields $xx''y \dashv_3 xx''y'$. This statement in $(A^{(3)}, \dashv_3)$ translates to $xy''y \dashv_3 xy''y'$. As we are now comparing right trees, *(4)* discloses that $xy''y$ and $xy''y'$ cannot have the same projection, for which reason $\delta(y''y) \neq \delta(y''y')$. In other words, $y''y'y$ is a left tree and $\delta(y''y) = \delta(y'y)$. It follows that the right trees $xy''y$ and $xy'y$ have the same projection, so another application of *(4)* leads to $xy''y \dashv_3 xy'y$. When transferred back to A this statement becomes $xx''y \dashv_3 xy'y$, so we have found a left tree coming before a right tree, contrary to *(1)*. This concludes the discussion of the case $k = 3$.

General case: $k \geqslant 4$

We may assume that x', x'' are the elements of Z immediately succeeding x and, similarly, that y'', y' are immediately preceeding y. According to part *(a)* of Definition 2.1 there is a set $W \subseteq Z$ with $|W| = k-2$, $\min(W) = x''$, and $\max(W) = y''$. Consider the sets $A = W \cup \{x, x', y\}$ and $B = W \cup \{x, y', y\}$ of size $k+1$. By canonicity we know that the orderings $(A^{(k)}, \dashv_3)$ and $(B^{(k)}, \dashv_3)$ are in the following sense essentially the same.

For $i \in [k+1]$ let A_i and B_i denote the sets of size k arising from A and B by removing their i^{th} elements, respectively. Now if $A_i \lhd A_j$ holds for some $i, j \in [k+1]$, then $B_i \lhd B_j$ follows. Set $\varphi = \varphi(A)$ and observe that $3 \leqslant \varphi \leqslant k-1$ due to the presence of x, x', x'', y, and y''. Moreover

$$\varphi(A_1) = \dots = \varphi(A_\varphi) = \varphi - 1, \qquad \varphi(A_{\varphi+1}) = \dots = \varphi(A_{k+1}) = \varphi,$$
$$\varphi(B_1) = \dots = \varphi(B_{\varphi-1}) = \varphi - 2, \quad \varphi(B_\varphi) = \dots = \varphi(B_{k+1}) = \varphi - 1\,.$$

In view of *(1)*, this leads to the equivalences

$$A_1 \lhd A_{k+1} \iff \varphi \text{ is even} \iff B_{k+1} \lhd B_1\,,$$

which contradict the canonicity of $\lhd$. Altogether, this completes the proof of Theorem 1.3.

§3. Upper bounds: the canonisation procedure

3.1. **Preparation.** The proof of Theorem 1.4 starts with some synchronisation principles for colourings $f\colon X^{(r)} \longrightarrow \Gamma$, where $X \subseteq \mathbb{N}$, $r \geqslant 2$, and Γ denotes a finite *set of colours*. A special rôle will be played by the set of colours

$$\Phi = \{-1, +1\}\,. \tag{3.1}$$

Given a colouring $f\colon X^{(r)} \longrightarrow \Phi$ we define the *opposite colouring* $(-f)\colon X^{(r)} \longrightarrow \Phi$ by $(-f)(x) = -(f(x))$ for all $x \in X^{(r)}$. For simplicity we call a colouring $f\colon X^{(r)} \longrightarrow \Phi$, where $r \geqslant 1$ and $X \subseteq \mathbb{N}$, *increasing* if $f(x_1, \dots, x_{r-1}, y) \leqslant f(x_1, \dots, x_{r-1}, y')$ holds for all $x_1 < \dots < x_{r-1} < y < y'$ from X and *decreasing* if its opposite colouring is increasing. We refer to f as *monotone* if it is either increasing or decreasing. Two colourings $f\colon X^{(r)} \longrightarrow \Phi$ and $g\colon Y^{(s)} \longrightarrow \Phi$ are said to be *monotone in the same direction* if either both are increasing or both are decreasing. Similarly, f and g are *monotone in opposite directions* if f and $-g$ are monotone in the same direction.

Lemma 3.1. *Suppose that $s, t \geqslant 1$ are positive integers and that $X \subseteq \mathbb{N}$ is a set of size $|X| \geqslant (s-1)(t-1)+1$. If the colouring $f\colon X^{(2)} \longrightarrow \Phi$ is monotone, then either there is a monochromatic set $A \in X^{(s)}$ for the colour -1 or a monochromatic set $B \in X^{(t)}$ for the colour $+1$.*

Proof. Due to the symmetry between f and $-f$ we may suppose that f is increasing. We argue by induction on t. The base case $t = 1$ offers no difficulty, for every set of size one is monochromatic in both colours. Now suppose that our claim is already known to be true for $t-1$ in place of t, where $t \geqslant 2$. Set $z = \max(X)$ and consider the set $N = \{x \in X\colon f(xz) = -1\}$.

First Case: $|N| \geqslant s-1$

Now it suffices to prove that $A = N \cup \{z\}$ is monochromatic in colour -1. So let $x < y$ from A be arbitrary and observe that $x \in N$. If $y = z$, then $f(xy) = -1$ follows from the definition of N. Otherwise we have $y < z$ and the assumption that f is increasing implies $f(xy) \leqslant f(xz) = -1$, whence $f(xy) = -1$.

Second Case: $|N| < s-1$

Now the set $X_\star = X \smallsetminus (N \cup \{z\})$ satisfies $|X_\star| \geqslant |X| - (s-1) \geqslant (s-1)(t-2)+1$. Thus the induction hypothesis applies to $X_\star$ and yields either a monochromatic set A of size s for -1 or a monochromatic set B' of size $t-1$ for $+1$. In the former case we are done immediately and in the latter case the definition of N entails that $B = B' \cup \{z\}$ is monochromatic for the colour $+1$. □

This iterates as follows.

Lemma 3.2. *Given a natural number n and a nonempty finite index set I let $X \subseteq \mathbb{N}$ be a set with $|X| \geqslant |I|!n^{|I|+1}$. If $(f_i)_{i\in I}$ denotes a family of monotone colourings $f_i\colon X^{(2)} \longrightarrow \Phi$, then there is a set $Z \subseteq X$ of size n which is monochromatic with respect to every f_i.*

Proof. Replacing some of the colourings f_i by their opposites, if necessary, we may assume that all of them are increasing. We argue by induction on $|I|$, the base case $|I| = 1$ being dealt with in the foregoing lemma. Now suppose that $|I| \geqslant 2$ and that the lemma has already been established for every set $I' \subsetneq I$ instead of I. Given a set X and a family of increasing colourings $(f_i)_{i\in I}$ of the pairs of elements from X we define an auxiliary colouring $f\colon X^{(2)} \longrightarrow \Phi$ by $f = \min\{f_i\colon i \in I\}$, i.e.,

$$f(xy) = +1 \iff \forall i \in I \;\, f_i(xy) = +1\,.$$

Evidently, f is increasing as well. So by Lemma 3.1 there either exists a set $A \subseteq X$ with $|A| \geqslant |I|!n^{|I|}$ that is monochromatic for f in colour -1 or there is a set $Z \subseteq X$ of size n that is monochromatic for f in colour $+1$. In the latter case Z monochromatically has colour $+1$ with respect to every f_i and thus it possesses the desired property.

So suppose from now on that the former case occurs, i.e., that we have obtained a set A of size $|A| \geqslant |I|!n^{|I|}$ that is monochromatic with respect to f in colour -1. Put $z = \max(A)$ and $X_i = \{a \in A\colon f_i(az) = -1\}$ for every $i \in I$. Since $f(az) = -1$ holds for all $a \in A \smallsetminus \{z\}$, we know that $\bigcup_{i\in I} X_i = A \smallsetminus \{z\}$. Owing to the Schubfachprinzip this implies $|X_{i(\star)}| \geqslant (|I|-1)!n^{|I|}$ for some $i(\star) \in I$. As in the first case of the proof of Lemma 3.1 one can show that the set $X_{i(\star)} \cup \{z\}$ monochromatically receives the colour -1 with respect to $f_{i(\star)}$. It remains to apply the induction hypothesis to this set and to $I' = I \smallsetminus \{i(\star)\}$. □

Let us now return to an arbitrary colouring $f\colon X^{(r)} \longrightarrow \Gamma$, where $X \subseteq \mathbb{N}$, $r \geqslant 2$, and Γ denotes a set of colours. We say that f is *governed* by another colouring $f'\colon X^{(\ell)} \longrightarrow \Gamma$, where $\ell \in [r-1]$, if $f(x_1, \dots, x_r) = f'(x_1, \dots, x_\ell)$ holds for all $x_1 < \dots < x_r$ from X. Standard references call f *end-homogeneous* if it is governed by some $(r-1)$-ary function f' in this sense, but for us it will be useful to have words expressing the relationship between f and f'.

Lemma 3.3. *Given integers $n \geqslant r \geqslant 2$ and $s \geqslant 3$, a set of colours Γ, and a nonempty finite index set I let $X \subseteq \mathbb{N}$ be a set of size $|X| \geqslant (2|\Gamma|)^{n^{r-1}} \cdot (2|I|n^{s-2})^{|I|n^{s-2}}$. If $f\colon X^{(r)} \longrightarrow \Gamma$ is an arbitrary colouring and $(g_i)_{i\in I}$ with $g_i\colon X^{(s)} \longrightarrow \Phi$ is a family of monotone colourings, then there exist a set $Z \subseteq X$ of size n, a colouring $f'\colon Z^{(r-1)} \longrightarrow \Gamma$ and a family $(g'_i)_{i\in I}$ of monotone colourings $g'_i\colon Z^{(s-1)} \longrightarrow \Phi$ such that*

(*a*) *f' governs the restriction of f to $Z^{(r)}$,*
(*b*) *for every $i \in I$ the colouring g'_i governs the restriction of g_i to $Z^{(s)}$,*
(*c*) *and for every $i \in I$ the colourings g_i, g'_i are monotone in opposite directions.*

Proof. As usual, we may suppose that every g_i is increasing. An integer $\nu \in [n]$ is said to be *secure* if there exist sets $Z_\nu, A_\nu \subseteq X$ with $|Z_\nu| = \nu$ and $\max(Z_\nu) < \min(A_\nu)$, a function $f'\colon Z_\nu^{(r-1)} \longrightarrow \Gamma$, and a family $(g'_i)_{i\in I}$ of decreasing colourings $g'_i\colon Z_\nu^{(s-1)} \longrightarrow \Phi$ with the following properties.

(*i*) If the numbers $z_1 < \dots < z_{r-1}$ are in Z_ν and $z_r > z_{r-1}$ is in $Z_\nu \cup A_\nu$, then $f(z_1, \dots, z_r) = f'(z_1, \dots, z_{r-1})$.
(*ii*) If $i \in I$, the numbers $z_1 < \dots < z_{s-1}$ are in Z_ν, and $z_s > z_{s-1}$ is in $Z_\nu \cup A_\nu$, then $g_i(z_1, \dots, z_s) = g'_i(z_1, \dots, z_{s-1})$.
(*iii*) If $i \in I$, the numbers $z_1 < \dots < z_{s-1}$ are in Z_ν, and $g'_i(z_1, \dots, z_{s-1}) = -1$, then $g_i(z_1, \dots, z_{s-2}, y, y') = -1$ holds for all $y, y' \in A_\nu$.
(*iv*) Finally, $|A_\nu| \geqslant (2|\Gamma|)^{n^{r-2}(n-\nu)} \cdot (2|I|n^{s-2})^{|I|n^{s-2}-\Omega}$, where Ω denotes the number of pairs (i, Z) such that $i \in I$, $\max(Z_\nu) \in Z \in Z_\nu^{(s-1)}$, and $g'_i(Z) = -1$.

Observe that (*i*) and (*ii*) are fairly standard conditions in arguments concerning end-homogeneity. Except for the presence of Ω so is (*iv*). Clause (*iii*) interacts well with the demand that the functions g'_i be decreasing.

We contend that 1 is secure. If $r \geqslant 3$ this can be seen by taking $Z_1 = \{\min(X)\}$ and $A_1 = X \smallsetminus Z_1$. In the special case $r = 2$ we still set $Z_1 = \{\min(X)\}$ but now we need to pay some attention to condition (*i*). The Schubfachprinzip yields a colour $\gamma \in \Gamma$ and a set $A_1 \subseteq X \smallsetminus Z_1$ of size $|A_1| \geqslant (|X| - 1)/r$ such that $f(\min(X), a_1) = \gamma$ holds for all $a_1 \in A_1$. Since A_1 is at least as large as (*iv*) demands, 1 is indeed secure.

Notice that if n turns out to be secure, then any set Z_n exemplifying this will have the desired properties. So it suffices to prove that the largest secure number, denoted by $\nu(\star)$ in the sequel, is equal to n.

Assume for the sake of contradiction that $\nu(\star) < n$ and let the sets $Z_{\nu(\star)}$, $A_{\nu(\star)}$, the function $f'\colon Z_{\nu(\star)}^{(r-1)} \longrightarrow \Gamma$, and the family $(g'_i)_{i\in I}$ of decreasing colourings $g'_i\colon Z_{\nu(\star)}^{(s-1)} \longrightarrow \Phi$ witness the security of $\nu(\star)$. Due to $\Omega \leqslant |I|\binom{\nu(\star)-1}{s-2} < |I|n^{s-2}$ the second factor in *(iv)* is integral and we may assume without loss of generality that

$$|A_{\nu(\star)}| = (2|\Gamma|)^{n^{r-2}(n-\nu(\star))} \cdot (2|I|n^{s-2})^{|I|n^{s-2}-\Omega} .$$

Now the plan is to reach a contradiction by establishing the security of $\nu(\star)+1$. To this end we shall add a new element to $Z_{\nu(\star)}$ and extend the functions f' and g'_i appropriately. The conditions that seems hardest to maintain is *(iii)*. In fact, this task requires some preparations that have to be carried out before one decides which element gets added to $Z_{\nu(\star)}$.

Set $z_{\nu(\star)} = \max(Z_{\nu(\star)})$ and define

$$L = \left\{(i, Z) \in I \times Z_{\nu(\star)}^{(s-2)} \colon \text{ If } z_{\nu(\star)} \notin Z, \text{ then } g'_i(Z \cup \{z_{\nu(\star)}\}) = +1\right\} .$$

A pair of sets (W, D) with $W \subseteq L$ and $D \subseteq A_{\nu(\star)}$ is said to be a *candidate* if

- for all $(i, Z) \in W$ and all $dd' \in D^{(2)}$ we have $g_i(Z \cup \{d, d'\}) = -1$,
- and $|D| \geqslant (2|I|n^{s-2})^{-|W|}|A_{\nu(\star)}|$.

For instance, $(\varnothing, A_{\nu(\star)})$ is a candidate and, consequently, we can pick a candidate $(W_\star, D_\star)$ for which $|W_\star|$ is maximal. Since $\Omega + |W_\star| \leqslant |I|\binom{\nu(\star)}{s-2} \leqslant |I|n^{s-2}$, we may again assume that

$$|D_\star| = (2|I|n^{s-2})^{-|W_\star|}|A_{\nu(\star)}| = (2|\Gamma|)^{n^{r-2}(n-\nu(\star))} \cdot (2|I|n^{s-2})^{|I|n^{s-2}-\Omega-|W_\star|} .$$

Now we partition $D_\star$ into two consecutive halves, i.e., we write $D_\star = D' \,\dot\cup\, D''$ such that $|D'| = |D''| = \frac{1}{2}|D_\star|$ and $D' < D''$. Put $d''_\star = \min(D'')$ and consider for every pair $(i, Z) \in L \setminus W_\star$ the set

$$X(i, Z) = \left\{d' \in D' \colon g_i(Z \cup \{d', d''_\star\}) = -1\right\} .$$

Assume for the sake of contradiction that there exists a pair $(i, Z) \in L \setminus W_\star$ such that $|X(i, Z)| \geqslant (2|I|n^{s-2})^{-1}|D_\star|$. Now for all $y < y'$ from $X(i, Z)$ the assumption that g_i be increasing yields $g_i(Z \cup \{y, y'\}) \leqslant g_i(Z \cup \{y, d''_\star\}) = -1$, whence $g_i(Z \cup \{y, y'\}) = -1$. In other words, $\big(W_\star \cup \{(i, Z)\}, X(i, Z)\big)$ is a candidate contradicting the maximality of $|W_\star|$.

We have thereby proved $|X(i, Z)| < (2|I|n^{s-2})^{-1}|D_\star|$ for all pairs $(i, Z) \in L \setminus W_\star$. Since $|L \setminus W_\star| \leqslant |L| \leqslant |I|n^{s-2}$, this implies

$$\Bigg|\bigcup_{(i,Z)\in L\setminus W_\star} X(i, Z)\Bigg| < \tfrac{1}{2}|D_\star| = |D'| .$$

Therefore, we can pick an element

$$z_{\nu(\star)+1} \in D' \smallsetminus \bigcup_{(i,Z)\in L\smallsetminus W_\star} X(i,Z)$$

and set $Z_{\nu(\star)+1} = Z_{\nu(\star)} \cup \{z_{\nu(\star)+1}\}$.

With the help of this set we intend to convince ourselves that $\nu(\star)+1$ is secure, which will contradict our maximal choice of $\nu(\star)$ and thus conclude the proof of Lemma 3.3. Towards this goal, we need to specify an appropriate set $A_{\nu(\star)+1}$, a function $f''\colon Z^{(r-1)}_{\nu(\star)+1} \longrightarrow \Gamma$, and a family $(g''_i)_{i\in I}$ of decreasing colourings $g''_i\colon Z^{(s-1)}_{\nu(\star)+1} \longrightarrow \Phi$ such that the conditions *(i)*–*(iv)* hold in this setting. These choices will be made in such a way that $A_{\nu(\star)+1} \subseteq D''$, the function f'' extends f', and for every $i \in I$ the colouring g''_i extends g'_i.

Our first step consists in determining $A_{\nu(\star)+1}$ and f''. To this end, we define for every $d'' \in D''$ the function $h_{d''}\colon Z^{(r-2)}_{\nu(\star)} \longrightarrow \Gamma$ by

$$h_{d''}(Z) = f(Z \cup \{z_{\nu(\star)+1}, d''\}) \quad \text{for all } Z \in Z^{(r-2)}_{\nu(\star)}\,.$$

Since $|Z^{(r-2)}_{\nu(\star)}| \leqslant n^{r-2}$, there are at most $|\Gamma|^{n^{r-2}}$ possible functions from $Z^{(r-2)}_{\nu(\star)}$ to Γ and the Schubfachprinzip yields a function $h_\star\colon Z^{(r-2)}_{\nu(\star)} \longrightarrow \Gamma$ together with a set $A_{\nu(\star)+1} \subseteq D''$ satisfying $h_{d''} = h_\star$ for all $d'' \in A_{\nu(\star)+1}$ and $|A_{\nu(\star)+1}| \geqslant |\Gamma|^{-n^{r-2}}|D''|$. For later use we remark that

$$|A_{\nu(\star)+1}| \geqslant \frac{|D''|}{|\Gamma|^{n^{r-2}}} = \frac{|D_\star|}{2|\Gamma|^{n^{r-2}}} \geqslant (2|\Gamma|)^{n^{r-2}(n-\nu(\star)-1)} \cdot (2|I|n^{s-2})^{|I|n^{s-2}-\Omega-|W_\star|}\,.$$

Now we define the extension $f'' \supseteq f'$ by $f''(Z \cup \{z_{\nu(\star)+1}\}) = h_\star(Z)$ for all $Z \in Z^{(r-2)}_{\nu(\star)}$ and observe that *(i)* clearly remains valid.

Proceeding with the extensions $g''_i \supseteq g'_i$ for $i \in I$ we set

$$g''_i(Z \cup \{z_{\nu(\star)+1}\}) = \begin{cases} +1 & \text{if } (i,Z) \in L \smallsetminus W_\star \\ -1 & \text{otherwise} \end{cases}$$

for all $i \in I$ and $Z \in Z^{(s-2)}_{\nu(\star)}$. Let us start the discussion of these colourings by verifying that they are indeed decreasing. In other words, we want to check that if $i \in I$ and $Z \in (Z_{\nu(\star)} \smallsetminus \{z_{\nu(\star)}\})^{(s-2)}$ satisfy $g''(Z \cup \{z_{\nu(\star)}\}) = -1$, then $g''(Z \cup \{z_{\nu(\star)+1}\}) = -1$ follows. To this end we just need to observe that $g''(Z \cup \{z_{\nu(\star)}\}) = -1$ implies $(i,Z) \notin L$.

For the confirmation of *(ii)* it suffices to argue that if $i \in I$, $Z \in Z^{(s-2)}_{\nu(\star)}$, and $d'' \in D''$, then $g_i(Z \cup \{z_{\nu(\star)+1}, d''\}) = g''_i(Z \cup \{z_{\nu(\star)+1}\})$. If $(i,Z) \in W_\star$, then the first bullet in the definition of $(W_\star, D_\star)$ being a candidate yields indeed $g_i(Z \cup \{z_{\nu(\star)+1}, d''\}) = -1$. Next, if $(i,Z) \in L \smallsetminus W_\star$, then $z_{\nu(\star)+1} \notin X(i,Z)$ entails $g_i(Z \cup \{z_{\nu(\star)+1}, d''_\star\}) = +1$ and thus the assumption that g_i be increasing and $d''_\star = \min(D'') \leqslant d''$ reveal $g_i(Z \cup \{z_{\nu(\star)+1}, d''\}) = +1$, as desired. Finally, suppose that $(i,Z) \notin L$, whence $z_{\nu(\star)} \notin Z$ and $g'_i(Z \cup \{z_{\nu(\star)}\}) = -1$.

Since g_i' satisfies *(iii)* and $z_{\nu(\star)+1}, d'' \in A_{\nu(\star)}$, we have indeed $g_i(Z \cup \{z_{\nu(\star)+1}, d''\}) = -1$. Altogether, we have thereby proved that the colourings g_i'' still obey condition *(ii)*.

To handle the new cases of *(iii)* we show that if $Z \in Z_{\nu(\star)}^{(s-2)}$ and either $(i, Z) \in W_\star$ or $(i, Z) \notin L$, then $g_i(Z \cup \{y, y'\}) = -1$ holds for all $yy' \in D_\star^{(2)}$. For $(i, Z) \in W_\star$ this is a direct consequence of $(W_\star, D_\star)$ being a candidate, so we may suppose $(i, Z) \notin L$ from now on, which, let us recall, means $z_{\nu(\star)} \notin Z$ and $g_i'(Z \cup \{z_{\nu(\star)}\}) = -1$. So the desired conclusion is already covered by g_i' satisfying *(iii)*.

Finally, we need to verify the lower bound on $|A_{\nu(\star)+1}|$ promised by *(iv)*. Denoting the number of pairs (i, Z) such that $i \in I$, $z_{\nu(\star)+1} \in Z \in Z_{\nu(\star)+1}^{(s-1)}$, and $g_i''(Z) = -1$ by Ω' it suffices to check that $\Omega' = \Omega + |W_\star|$. Since Ω' counts the number of pairs $(i, Z) \in I \times Z_{\nu(\star)}^{(s-2)}$ satisfying $g_i''(Z \cup \{z_{\nu(\star)+1}\}) = -1$, or in other words $Z \notin L \smallsetminus W_\star$, we have indeed

$$\begin{aligned}
\Omega' &= |I \times Z_{\nu(\star)}^{(s-2)} \smallsetminus (L \smallsetminus W_\star)| \\
&= |I \times Z_{\nu(\star)}^{(s-2)} \smallsetminus L| + |W_\star| \\
&= \left|\left\{(i, Z) \in I \times Z_{\nu(\star)}^{(s-2)} : z_{\nu(\star)} \notin Z \text{ and } g_i'(Z \cup \{z_{\nu(\star)}\}) = -1\right\}\right| + |W_\star| \\
&= \Omega + |W_\star| .
\end{aligned}$$

This concludes the proof that $\nu(\star) + 1$ is secure and, as we said before, this contradiction to the choice of $\nu(\star)$ establishes Lemma 3.3. □

We remark that the case $I = \varnothing$ can be handled similarly but much easier. In this case one can ignore all parts of the proof addressing I and, moreover, there is no need to divide the set $D_\star = A_{\nu(\star)}$ into two halves. Therefore, one obtains the following well-known statement.

Corollary 3.4. *If $n \geqslant r \geqslant 2$ and $N \geqslant |\Gamma|^{n^{r-1}}$, then for every colouring $f\colon [N]^{(r)} \longrightarrow \Gamma$ there are a set $X \subseteq [N]$ of size n and a function $f'\colon X^{(r-1)} \longrightarrow \Gamma$ governing the restriction of f to $X^{(r)}$.* □

3.2. **Definite orderings.** Resuming the task of proving Theorem 1.4 we shall now study orderings $([N]^{(k)}, \prec)$ that are "almost canonical" in the sense that we can easily extract a sign vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in \{+1, -1\}^k$ and a permutation $\sigma \in \mathfrak{S}_k$ from them such that $\prec$ behaves in some important ways like the canonical ordering associated with (ε, σ). For transparency we shall assume $N \geqslant 2k + 1$ throughout this investigation.

Observe that in the canonical case, an entry $\varepsilon_i = +1$ in the sign vector indicates that we can increase a set $x \in [N]^{(k)}$ with respect to $\prec$ by increasing its element in the i^{th} position, whereas an entry $\varepsilon_i = -1$ signifies that it is the other way around. When $\prec$ is arbitrary and $i \in [k]$ we call a triple $S = (X, A, Y) \in [N]^{(i-1)} \times [N]^{(2)} \times [N]^{(k-i)}$ with

$X < A < Y$ an *i-decider*. Notice that i-deciders are in natural bijective correspondence with $(k+1)$-subsets of $[N]$. If $S = (X, A, Y)$ is such an i-decider and $A = \{c, d\}$ with $c < d$ we may compare the sets $S(c) = X \cup \{c\} \cup Y$ and $S(d) = X \cup \{d\} \cup Y$ with respect to $\lessdot$. If $S(c) \lessdot S(d)$ we set $\varepsilon_i(S) = +1$ and in case $S(d) \lessdot S(c)$ we put $\varepsilon_i(S) = -1$. So roughly speaking, $\varepsilon_i(S) = +1$ means that the i-decider S "thinks" that ε_i should have the value $+1$. If all i-deciders S agree and yield the same sign $\varepsilon_i(S)$ we say that $\lessdot$ is *i-definite* and denote the common value of all $\varepsilon_i(S)$ by ε_i. It may happen that $\lessdot$ is i-definite for every $i \in [k]$ and in this case we call the ordering $\lessdot$ *sign-definite* and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ is referred to as its *sign-vector*. Clearly, the canonical ordering associated with a pair $(\varepsilon, \sigma) \in \{+1, -1\}^k \times \mathfrak{S}_k$ is sign-definite and has sign-vector ε in this sense.

Next we explain the extraction of a permutation from an "almost canonical" ordering. Let $ij \in [k]^{(2)}$ with $i < j$ be an arbitrary pair of indices. Observe that for a permutation $\sigma \in \mathfrak{S}_k$, the number i occurs before j in the list $\sigma(1), \ldots, \sigma(k)$ if and only if $\sigma^{-1}(i) < \sigma^{-1}(j)$. So for the canonical ordering considered in Definition 1.1 $\sigma^{-1}(i) < \sigma^{-1}(j)$ indicates that i has in the following sense higher priority than j. If we take a set $x \in [N]^{(k)}$ and change both its i^{th} and its j^{th} element, while keeping the relative positions of the elements fixed, then whether we increase or decrease x with respect to $\lessdot$ is decided by ε_i and the direction in which we move the i^{th} element alone. Similarly $\sigma^{-1}(j) < \sigma^{-1}(i)$ means that j has higher priority than i. Now suppose that the ordering $\lessdot$ is not necessarily canonical but still sign-definite with sign vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$. By an *$ij$-decider* we mean a quintuple

$$P = (X, A, Y, B, Z) \in [N]^{(i-1)} \times [N]^{(2)} \times [N]^{(j-i-1)} \times [N]^{(2)} \times [N]^{(k-j)}$$

with $X < A < Y < B < Z$. So ij-deciders are in natural bijective correspondence with $(k+2)$-subsets of $[N]$. Given such an ij-decider $P = (X, A, Y, B, Z)$ and two elements $a \in A$, $b \in B$ we set $P(a, b) = X \cup \{a\} \cup Y \cup \{b\} \cup Z$. The restriction of $\lessdot$ to the four-element set

$$\langle P \rangle = \big\{P(a, b) \colon (a, b) \in A \times B\big\}$$

is largely determined by ε_i and ε_j. Notably, if we write $A = \{c_i, d_i\}$ and $B = \{c_j, d_j\}$ such that $\varepsilon_i c_i < \varepsilon_i d_i$ and $\varepsilon_j c_j < \varepsilon_j d_j$, then the $\lessdot$-minimum of $\langle P \rangle$ is $P(c_i, c_j)$ and the $\lessdot$-maximum is $P(d_i, d_j)$. So the only piece of information we are lacking is how $P(c_i, d_j)$ compares to $P(d_i, c_j)$. We let $\sigma_{ij}(P)$ denote

the number	provided that
$+1$	$P(c_i, d_j) \lessdot P(d_i, c_j)$,
-1	$P(d_i, c_j) \lessdot P(c_i, d_j)$.

So intuitively $\sigma_{ij}(P) = +1$ means that P "believes" i to have higher priority than j, while in the opposite case it is the other way around. If $\sigma_{ij}(P)$ stays constant as P varies over all ij-deciders we say that $\dashv$ is ij-definite and denote the common value of all $\sigma_{ij}(P)$ by σ_{ij}. Finally, if $\dashv$ is *ij-definite* for all pairs $ij \in [k]^{(2)}$ we call $\dashv$ a *permutation-definite* ordering. We remark that in the canonical case $\sigma_{ij} = +1$ is equivalent to $\sigma^{-1}(i) < \sigma^{-1}(j)$ meaning that the statement $\sigma_{ij} \cdot \big(\sigma^{-1}(j) - \sigma^{-1}(i)\big) > 0$ is always valid. Let us check that, conversely, in the permutation-definite case we can define a permutation σ having this property.

Lemma 3.5. *Suppose that $N \geqslant 2k$. If the ordering $([N]^{(k)}, \dashv)$ is both sign-definite and permutation-definite, then there exists a permutation $\sigma \in \mathfrak{S}_k$ such that*

$$\sigma_{ij} \cdot \big(\sigma^{-1}(j) - \sigma^{-1}(i)\big) > 0$$

holds whenever $1 \leqslant i < j \leqslant k$.

Proof. Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ be the sign-vector of $\dashv$ and write $\{2i-1, 2i\} = \{c_i, d_i\}$ such that $\varepsilon_i c_i < \varepsilon_i d_i$ holds for every $i \in [k]$. Consider the sets

$$x_i = \{c_i\} \cup \{d_j \colon j \neq i\}$$

and let $\sigma \in \mathfrak{S}_k$ be the permutation satisfying

$$x_{\sigma(1)} \dashv \dots \dashv x_{\sigma(k)} .$$

Whenever $1 \leqslant i < j \leqslant k$ the quintuple

$$P_{ij} = \big(\{d_r \colon 1 \leqslant r < i\}, \{2i-1, 2i\}, \{d_s \colon i < s < j\}, \{2j-1, 2j\}, \{d_t \colon j < t \leqslant k\}\big)$$

is an ij-decider and σ_{ij} is the sign $+1$ if and only if $x_i \dashv x_j$, which is in turn equivalent to $\sigma^{-1}(i) < \sigma^{-1}(j)$. In other words, irrespective of whether σ_{ij} is positive or negative the statement $\sigma_{ij} \cdot \big(\sigma^{-1}(j) - \sigma^{-1}(i)\big) > 0$ holds. □

The remainder of this section deals with the question how much a sign-definite and permutation-definite ordering $([N]^{(k)}, \dashv)$ with sign-vector ε and permutation σ (obtained by means of Lemma 3.5) needs to have in common with the canonical ordering associated with (ε, σ) in the sense of Definition 1.1. For instance, Lemma 3.7 below asserts that under these circumstances there exists a dense subset $W \subseteq [N]$ such that $(W^{(k)}, \dashv)$ is ordered in this canonical way.

The strategy we use for this purpose is the following. Suppose that $x, y \in [N]^{(k)}$ are written in the form $x = \{x_1, \dots, x_k\}$ and $y = \{y_1, \dots, y_k\}$, where $x_1 < \dots < x_k$ and $y_1 < \dots < y_k$. Our goal is to prove that in many cases we have

$$x \dashv y \iff (\varepsilon_{\sigma(1)} x_{\sigma(1)}, \dots, \varepsilon_{\sigma(k)} x_{\sigma(k)}) <_{\text{lex}} (\varepsilon_{\sigma(1)} y_{\sigma(1)}, \dots, \varepsilon_{\sigma(1)} y_{\sigma(k)}) . \tag{3.2}$$

We shall say that a pair of k-sets (x, y) is (ε, σ)-*sound* if it satisfies the equivalence (3.2). Moreover, we set $\Delta(x, y) = \{i \in [k] \colon x_i \neq y_i\}$. The plan towards Lemma 3.7 is to construct a sequence of sets $[N] = W_1 \supseteq \cdots \supseteq W_k$ such that every pair (x, y) of k-sets with $x, y \subseteq W_t$ and $|\Delta(x, y)| \leqslant t$ is (ε, σ)-sound. The statement that follows will assist us in the step of an induction on t.

Lemma 3.6. *For $L \subseteq \mathbb{Z}$ let $(L^{(k)}, \dashv)$ be an ordering that is sign-definite with sign-vector ε and permutation-definite with permutation σ. Suppose $t \in [k-1]$ has the property that every pair (x, y) of k-sets with $x, y \subseteq L$ and $|\Delta(x, y)| \leqslant t$ is (ε, σ)-sound.*

Let $I \subseteq [k]$ be a set of $t + 1$ indices and let $i \in [k]$ be the least index with $\sigma(i) \in I$. Further, let $x = \{x_1, \ldots, x_k\}$ and $y = \{y_1, \ldots, y_k\}$ be two k-element subsets of L such that $\Delta(x, y) = I$ and $x_{\sigma(i)} < y_{\sigma(i)}$. If a denotes the least member of L larger than $x_{\sigma(i)}$ and the conditions

- $a < y_{\sigma(i)}$,
- $\sigma(i) = \min(I) \Longrightarrow a < \min\{x_{\sigma(i)+1}, y_{\sigma(i)+1}\}$,
- *and* $\sigma(i) = \max(I) \Longrightarrow a > \max\{x_{\sigma(i)-1}, y_{\sigma(i)-1}\}$

hold, then the pair (x, y) is (ε, σ)-sound.

Proof. Without loss of generality we may assume $\varepsilon_{\sigma(i)} = +1$. We are to prove that $x \dashv y$ and in all cases this will be accomplished by finding a k-set $z \subseteq L$ such that the pairs (x, z) and (z, y) are (ε, σ)-sound and $x \dashv z \dashv y$ holds.

Suppose first that $\min(I) < \sigma(i) < \max(I)$. Due to $a \in (x_{\sigma(i)}, y_{\sigma(i)}) \subseteq (x_{\sigma(i)-1}, y_{\sigma(i)+1})$ the set $z = \{x_1, \ldots, x_{\sigma(i)-1}, a, y_{\sigma(i)+1}, \ldots, y_k\}$ has k elements that have just been enumerated in increasing order. Moreover $\sigma(i) \neq \min(I)$ yields $|\Delta(x, z)| \leqslant t$ and, therefore, (x, z) is indeed (ε, σ)-sound, which implies $x \dashv z$. Similarly, $\sigma(i) \neq \max(I)$ leads to $z \dashv y$.

Next we suppose that $\sigma(i) = \min(I)$, which owing to the second bullet entails

$$a < \min\{x_{\sigma(i)+1}, y_{\sigma(i)+1}\}\,.$$

Let $j \in [k]$ be the second smallest index with $\sigma(j) \in I$.

If	and	then we set $z =$
$x_{\sigma(j)} < y_{\sigma(j)}$	$\sigma(j) < \max(I)$	$\{x_1, \ldots, x_{\sigma(i)-1}, a, x_{\sigma(i)+1}, \ldots, x_{\sigma(j)}, y_{\sigma(j)+1}, \ldots, y_k\}$,
$x_{\sigma(j)} < y_{\sigma(j)}$	$\sigma(j) = \max(I)$	$\{x_1, \ldots, x_{\sigma(i)-1}, a, x_{\sigma(i)+1}, \ldots, x_{\sigma(j)-1}, y_{\sigma(j)}, \ldots, y_k\}$,
$x_{\sigma(j)} > y_{\sigma(j)}$	$\sigma(j) < \max(I)$	$\{x_1, \ldots, x_{\sigma(i)-1}, a, y_{\sigma(i)+1}, \ldots, y_{\sigma(j)}, x_{\sigma(j)+1}, \ldots, x_k\}$,
$x_{\sigma(j)} > y_{\sigma(j)}$	$\sigma(j) = \max(I)$	$\{x_1, \ldots, x_{\sigma(i)-1}, a, y_{\sigma(i)+1}, \ldots, y_{\sigma(j)-1}, x_{\sigma(j)}, \ldots, x_k\}$.

In all four cases we have listed the elements of z in increasing order. Furthermore, in almost all cases we have $|\Delta(x, z)| \leqslant t$ and, consequently, $x \dashv z$. In fact, the only exception occurs if we are in the second case and $t = 2$. But if this happens, then

$$P = \big(\{x_1, \dots, x_{\sigma(i)-1}\}, \{x_{\sigma(i)}, a\}, \{x_{\sigma(i)+1}, \dots, x_{\sigma(j)-1}\}, \{x_{\sigma(j)}, y_{\sigma(j)}\}, \{y_{\sigma(j)+1}, \dots, y_k\}\big)$$

is an $\sigma(i)\sigma(j)$-decider. Since the ordering $\strictif$ is permutation-definite, we know that it orders the four-element set $\langle P \rangle$ "correctly" and, as x, z belong to this set, $x \strictif z$ holds in this case as well.

Similarly, in almost all cases we have $\Delta(y, z) \leqslant t$ and $y \strictif z$, the only exception occurring if we are in the fourth case and $t = 2$. Under these circumstances

$$Q = \big(\{x_1, \dots, x_{\sigma(i)-1}\}, \{a, y_{\sigma(i)}\}, \{y_{\sigma(i)+1}, \dots, y_{\sigma(j)-1}\}, \{x_{\sigma(j)}, y_{\sigma(j)}\}, \{x_{\sigma(j)+1}, \dots, x_k\}\big)$$

is an $\sigma(i)\sigma(j)$-decider and, as before, $y, z \in \langle Q \rangle$ implies $y \strictif z$.

This concludes our discussion of the case $\sigma(i) = \min(I)$ and it remains to deal with the case $\sigma(i) = \max(I)$. Here one can either perform a similar argument, or one argues that this case reduces to the previous one by reversion the ordering of the ground set. That is, one considers the ordering $\big((-L)^{(k)}, \strictif^\star\big)$ defined by

$$x \strictif^\star y \iff (-x) \strictif (-y)\,,$$

where $-L$ means $\{-\ell \colon \ell \in L\}$ and $-x$, $-y$ are defined similarly. If one replaces $\strictif$ by $\strictif^\star$, then $I^\star = (k+1) - I$ assumes the rôle of I, minima correspond to maxima and the second and third bullet in the assumption of our lemma are exchanged. □

We proceed with the existence of "dense" canonical subsets promised earlier.

Lemma 3.7. *If $N \geqslant 2k$ and the ordering $([N]^{(k)}, \strictif)$ is both sign-definite and permutation-definite, then there exists a set $W \subseteq [N]$ of size $|W| \geqslant 2^{1-k}N$ such that $\strictif$ is canonical on $W^{(k)}$.*

Proof. For every $t \in [k]$ we set

$$W_t = \big\{n \in [N] \colon n \equiv 1 \pmod{2^{t-1}}\big\}\,.$$

Clearly $|W_k| \geqslant 2^{1-k}N$, so it suffices to prove that $\strictif$ is canonical on $W_k^{(k)}$.

Denote the sign-vector of $\strictif$ by $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ and let $\sigma \in \mathfrak{S}_k$ be the permutation obtained from $\strictif$ by means of Lemma 3.5. Let us prove by induction on $t \in [k]$ that if two k-element subsets $x, y \subseteq W_t$ satisfy $|\Delta(x, y)| \leqslant t$, then (x, y) is (ε, σ)-sound.

In the base case $t = 1$ we have $W_1 = [N]$ and everything follows from our assumption that $\strictif$ be sign-definite. In the induction step from $t \in [k-1]$ to $t+1$ we appeal to Lemma 3.6. Since no two elements of W_{t+1} occur in consecutive positions of W_t, the bulleted assumptions are satisfied and thus there are no problems with the induction step.

Since $|\Delta(x, y)| \leqslant k$ holds for all k-element subsets of W_k, the case $t = k$ of our claim proves the lemma. □

Next we characterise the canonical orderings on $([N]^{(k)}, \dashv)$ for $N \geqslant 2k + 1$.

Lemma 3.8. *If $N \geqslant 2k + 1$ and the ordering $([N]^{(k)}, \dashv)$ is canonical, then there exists a pair $(\varepsilon, \sigma) \in \{-1, +1\}^k \times \mathfrak{S}_k$ such that $\dashv$ is the ordering associated with (ε, σ).*

Proof. Let ε and σ denote the sign vector and the permutation of $\dashv$ as introduced in §3.2. We shall show that $\dashv$ coincides with the canonical ordering associated to (ε, σ).

Otherwise there exists a pair (x, y) of subsets of $[N]$ that fails to be (ε, σ)-sound. Assume that such a pair (x, y) is chosen with $t + 1 = |\Delta(x, y)|$ minimum. Since $\dashv$ is sign-definite we know that $t \in [k-1]$. Set $I = \Delta(x, y)$ and let $i \in [k]$ be the smallest index with $\sigma(i) \in I$. By symmetry we may assume that $x_{\sigma(i)} < y_{\sigma(i)}$. Due to $|I| \geqslant 2$ it cannot be the case that $\sigma(i)$ is both the minimum and the maximum of I and as in the proof of Lemma 3.6 it suffices to study the case $\sigma(i) \neq \max(I)$.

First Case: $\max(x \cup y) < N$

Define a strictly increasing map $\varphi\colon x \cup y \longrightarrow [N]$ by $\varphi(z) = z$ for $z \leqslant x_{\sigma(i)}$ and $\varphi(z) = z + 1$ for $z > x_{\sigma(i)}$. Now by Lemma 3.6 the pair $\big(\varphi[x], \varphi[y]\big)$ is (ε, σ)-sound. Moreover, the canonicity of $\dashv$ tells us that $\big((x \cup y)^{(k)}, \dashv\big)$ and $\big(\varphi[x \cup y]^{(k)}, \dashv\big)$ are isomorphic via φ. For these reasons, the pair (x, y) is (ε, σ)-sound as well.

Second Case: $\max(x \cup y) = N$

Let $\eta\colon x \cup y \longrightarrow [|x \cup y|]$ be order-preserving. In view of $|x \cup y| \leqslant 2k < N$ we know from the first case that the pair $\big(\eta[x], \eta[y]\big)$ is (ε, σ)-sound and by canonicity so is (x, y). □

Remark 3.9. The results of this subsection easily yield the weaker upper bound

$$\mathrm{L}^{(k)}(n) \leqslant \mathrm{R}^{(k+2)}\left(2^{k-1}n, \tbinom{k+2}{2}!\right) \leqslant t_{k+1}(C_k n)\,.$$

To see this, let $([N]^{(k)}, \dashv)$ be any ordering, where $N = \mathrm{R}^{(k+2)}(2^{k-1}n, \binom{k+2}{2}!)$. For a set $A \in [N]^{(k+2)}$ there are $\binom{k+2}{2}!$ possibilities how the restriction of $\dashv$ to $A^{(k+2)}$ can look. So by the definition of Ramsey numbers there is a set $Z \subseteq [N]$ with $|Z| = 2^{k-1}n$ such that for all $A, B \subseteq Z$ with $|A| = |B| = k + 2$ the order-preserving map $\eta\colon A \longrightarrow B$ induces an order preserving map from $(A^{(k)}, \dashv)$ to $(B^{(k)}, \dashv)$. This implies, in particular, that $\dashv$ is sign-definite and permutation-definite on $Z^{(k)}$. Thus by Lemma 3.7 there exists a set $X \subseteq Z$ with $|X| \geqslant 2^{1-k}|Z| = n$ such that $\dashv$ orders $X^{(k)}$ canonically.

Now it remains to save one exponentiation, which requires a more careful reasoning.

3.3. **Further concepts.** Throughout this subsection, we fix a natural number $k \geqslant 2$. Let Γ denote the set of all orderings $\prec$ that can be imposed on $[k+2]^{(k)}$. We regard Γ as a set of colours having the size $|\Gamma| = \binom{k+2}{2}!$. As in Remark 3.9 we *associate* with every ordering $([N]^{(k)}, \prec\!\!\cdot)$ the colouring $f\colon [N]^{(k+2)} \longrightarrow \Gamma$ mapping every $A \in [N]^{(k+2)}$ to the unique ordering $\prec \in \Gamma$ with the property that $(A^{(k)}, \prec\!\!\cdot)$ and $([k+2]^{(k)}, \prec)$ are isomorphic via the order-preserving map from A to $[k+2]$. Roughly speaking, what Theorem 1.4 asserts is that colourings of this form are so special that one can prove a Ramsey theorem for them that requires just k exponentiations rather than the expected $k+1$ exponentiations.

For an integer $m \in [k+1]$ we shall say that an ordering $([N]^{(k)}, \prec\!\!\cdot)$ *is governed by m-sets* if there exists a function $f^m\colon [N]^{(m)} \longrightarrow \Gamma$ governing the function $f\colon [N]^{(k+2)} \longrightarrow \Gamma$ associated with $\prec\!\!\cdot$.

Definition 3.10. *If $([N]^{(k)}, \prec\!\!\cdot)$ is an ordering, $i, m \in [k]$, and $W \subseteq [N]$ we say that $\varepsilon_i^m\colon W^{(m)} \longrightarrow \Phi$ is an m-ary sign function for $(W^{(k)}, \prec\!\!\cdot)$ provided the following holds: If $S = (X, A, Y)$ is an i-decider with $X \cup A \cup Y \subseteq W$ and M denotes the set consisting of the m smallest elements of $X \cup A \cup Y$, then $\varepsilon_i(S) = \varepsilon_i^m(M)$.*

In other words, the existence of an m-ary sign function $\varepsilon_i^m\colon W^{(m)} \longrightarrow \Phi$ means that for i-deciders $S = (X, A, Y)$ with $X \cup A \cup Y \subseteq W$ the value of $\varepsilon_i(S)$ only depends on the first m elements of $X \cup A \cup Y$, while the remaining $k+1-m$ elements of $X \cup A \cup Y$ are destitute of any effect.

We shall frequently utilise the fact that if $m \in [2, k+1]$ and $([N]^{(k)}, \prec\!\!\cdot)$ is governed by m-sets, then for every $i \in [k]$ there exists an $(m-1)$-ary sign function $\varepsilon_i^{m-1}\colon [2, N]^{(m-1)} \longrightarrow \Phi$. This is because for an i-decider $S = (X, A, Y)$ with $X \cup A \cup Y \subseteq [2, N]$ the value of $\varepsilon_i(S)$ can be inferred from the colour $f(\{1\} \cup X \cup A \cup Y)$, where $f\colon [N]^{(k+2)} \longrightarrow \Gamma$ denotes the function associated with $\prec\!\!\cdot$.

Suppose that $m \in [2, k+1]$, $W \subseteq \mathbb{Z}$ and that $(W^{(k)}, \prec\!\!\cdot)$ denotes an ordering possessing for every $i \in [k]$ an $(m-1)$-ary sign function $\varepsilon_i^{m-1}\colon W^{(m-1)} \longrightarrow \Phi$. If $i < j$ from $[k]$ are two indices and $P = (X, A, Y, B, Z)$ is an ij-decider, we determine the colour $\sigma_{ij}^m(P)$ as follows: Let M denote the set consisting of the $m-1$ smallest elements of $X \cup A \cup Y \cup B \cup Z$, enumerate $A = \{c_i, d_i\}$ as well as $B = \{c_j, d_j\}$ in such a way that $\varepsilon_i^{m-1}(M)c_i < \varepsilon_i^{m-1}(M)d_i$ and $\varepsilon_j^{m-1}(M)c_j < \varepsilon_j^{m-1}(M)d_j$, and set

$\sigma_{ij}^m(P) =$	provided that
$+1$	$P(c_i, d_j) \prec\!\!\cdot\, P(d_i, c_j)$,
-1	$P(d_i, c_j) \prec\!\!\cdot\, P(c_i, d_j)$.

Definition 3.11. *If* $m \in [2, k+1]$, $ij \in [k]^{(2)}$, *and the ordering* $(W^{(k)}, \dashv)$ *possesses* $(m-1)$*-ary sign functions* $\varepsilon_1^{m-1}, \dots, \varepsilon_k^{m-1}$, *then we say that* $\sigma_{ij}^m \colon W^{(m)} \longrightarrow \Phi$ *is an* m*-ary permutation colouring if* $\sigma_{ij}^m(P) = \sigma_{ij}^m(M)$ *holds, whenever* $P = (X, A, Y, B, Z)$ *is an* ij*-decider and* M *denotes the set of the* m *smallest elements of* $X \cup A \cup Y \cup B \cup Z$.

It should be clear that if in addition to possessing the $(m-1)$-ary sign functions $\varepsilon_1^{m-1}, \dots, \varepsilon_k^{m-1}$ the ordering $(W^{(k)}, \dashv)$ is governed by m-sets, then all m-ary permutation colourings (σ_{ij}^m) with $ij \in [k]^{(2)}$ exist. Consequently, if $([N]^{(k)}, \dashv)$ is governed by m-sets, then for every pair $ij \in [k]^{(2)}$ there exists a permutation colouring $\sigma_{ij}^m \colon [2, N]^{(m)} \longrightarrow \Phi$.

3.4. **Plan.** Let us now pause to explain our strategy for proving Theorem 1.4. Given an arbitrary ordering $([N]^{(k)}, \dashv)$ with $N = t_k(n^{C_k})$ we need to find a set $X \subseteq [N]$ of size n such that $(X^{(k)}, \dashv)$ is ordered canonically. Following the same strategy as in Remark 3.9 it would suffice to find a set $Y \subseteq [N]$ of size $2^{k-1}n$ which is monochromatic with respect to the colouring $f \colon [N]^{(k+2)} \longrightarrow \Gamma$ associated with $\dashv$. So the problem is that we have only k exponentiations available but hope to synchronise sets of size $k+2$. What happens when we nevertheless perform the standard argument producing upper bounds on Ramsey numbers is that we obtain sets $[N] = Z_k \supseteq Z_{k-1} \supseteq \dots \supseteq Z_0$ such that for every $m \in [0, k]$ the ordering $(Z_m^{(k)}, \dashv)$ is governed by $(m+2)$-sets and the size of Z_m is, roughly, m times exponential. In particular, Z_0 has roughly the size $n^{C_k/2}$ and $(Z_0^{(k)}, \dashv)$ is governed by pairs. We may assume further that for every pair $ij \in [k]^{(2)}$ a permutation colouring $\sigma_{ij}^2 \colon Z_0^{(2)} \longrightarrow \Phi$ exists. Moreover, it would suffice to find a set $Y \subseteq Z_0$ that is monochromatic with respect to each σ_{ij}^2. This problem still sounds like we need an exponential rather than a polynomial dependence between $|Y|$ and $|Z_0|$, but Lemma 3.2 suggests the following alternative: if it turns out that the colourings σ_{ij}^2 are monotone, then we are done (for reasons explained in Lemma 3.14).

Now for some of the permutation colourings, like σ_{1j}^2 with $j \in [2, k]$, one can indeed prove that they are always monotone (see Lemma 3.12 below), but it seems unlikely that in the setting we have described so far one can prove that, for instance, σ_{57}^2 is always monotone. To address this issue we exploit that the colourings σ_{ij}^2 become known "gradually" in the course of the argument. When we have just constructed a set Z_m we already know a colouring σ_{ij}^{m+2} of its $(m+2)$-subsets such that, later, σ_{ij}^2 will govern its restriction to $Z_0^{(m+2)}$. Suppose that in the sequence of colourings $\sigma_{57}^k, \sigma_{57}^{k-1}, \dots, \sigma_{57}^2$ there is at least one that is monotone, say σ_{57}^{m+2}, where $m \geqslant 1$. What Lemma 3.3 tells us is that when selecting $Z_{m-1} \subseteq Z_m$ we can not only ensure that $(Z_{m-1}^{(k)}, \dashv)$ is governed by $(m+1)$-sets, but also that σ_{57}^{m+1} is still monotone. Iterating this observation we see that if some σ_{57}^m is monotone, then we can protect ourselves against ever loosing the monotonicity. Rather, we can arrange that all of $\sigma_{57}^m, \sigma_{57}^{m-1}, \dots, \sigma_{57}^2$ are monotone. We shall see in Lemma 3.13 that for $i \geqslant 2$ the permutation colouring σ_{ij}^i is automatically monotone. For these reasons, we can indeed find a set Z_0 of polynomial size on which all pair colourings σ_{ij}^2 are monotone, which essentially proves Theorem 1.4.

3.5. **Canonisation.** Coming to the details we begin by establishing the monotonicity of σ_{1j}^2 for $j \in [2, k]$.

Lemma 3.12. *If the ordering* $([N]^{(k)}, \dashv)$ *is sign-definite and governed by pairs, then for every* $j \in [2, k]$ *the colouring* σ_{1j}^2 *is increasing on* $[2, N-k]^{(2)}$.

Proof. Given three numbers $r < s < t$ from $[2, N-k]$ satisfying $\sigma_{1j}^2(rs) = +1$ we are to prove that $\sigma_{1j}^2(rt) = +1$. Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ be the sign vector of $\dashv$. We partition

$$[N-k+1, N] = Y \cup B \cup Z \quad \text{ such that } Y < B < Z$$

and $|Y| = j-2$, $|B| = 2$, $|Z| = k-j$ and we enumerate $B = \{c_j, d_j\}$ such that $\varepsilon_j c_j < \varepsilon_j d_j$. Notice that $P = (\varnothing, \{r, s\}, Y, B, Z)$ and $Q = (\varnothing, \{r, t\}, Y, B, Z)$ are $1j$-deciders, whence $\sigma_{1j}^2(rs) = \sigma_{1j}(P)$ and $\sigma_{1j}^2(rt) = \sigma_{1j}(Q)$. Thus our task is to derive $\sigma_{1j}(Q) = +1$ from $\sigma_{1j}(P) = +1$.

First Case: $\varepsilon_1 = +1$

Now $\sigma_{1j}(P) = +1$ is defined to mean $rYd_jZ \dashv sYc_jZ$, where rYd_jZ abbreviates $\{r\} \cup Y \cup \{d_j\} \cup Z$ and sYc_jZ is defined similarly. Moreover, $S = (\{s, t\}, Yc_jZ)$ is a 1-decider and $\varepsilon_1(S) = \varepsilon_1 = +1$ implies $sYc_jZ \dashv tYc_jZ$. So the transitivity of $\dashv$ leads to $rYd_jZ \dashv tYc_jZ$, for which reason we have indeed $\sigma_{1j}(Q) = +1$.

Second Case: $\varepsilon_1 = -1$

This time $\sigma_{ij}(P) = +1$ translates into $sYd_jZ \dashv rYc_jZ$ and working with the 1-decider $(\{s, t\}, Yd_jZ)$ we infer $tYd_jZ \dashv sYd_jZ$. Hence $tYd_jZ \dashv rYc_jZ$ and $\sigma_{1j}(Q) = +1$. □

Virtually the same proof also yields the following result.

Lemma 3.13. *If* $2 \leqslant i < j \leqslant k$ *and the ordering* $([N]^{(k)}, \dashv)$ *is governed by* i*-sets, then the restriction of* σ_{ij}^i *to* $[2, N-k]^{(i)}$ *is decreasing.*

Proof. Suppose that $X \subseteq [2, N-k]$ and $r, s \in [2, N-k]$ satisfy $|X| = i-1$, $X < r < s$, as well as $\sigma_{ij}^i(Xr) = -1$. We are to prove that $\sigma_{ij}^i(Xs) = -1$. To this end we set $t = N-k+1$ and take three sets $Y, Z, B \subseteq [N-k+2, N]$ with $Y < B < Z$ of the sizes $|Y| = j-i-1$, $|B| = 2$, and $|Z| = k-j$. Now $P = (X, \{r, t\}, Y, B, Z)$ and $Q = (X, \{s, t\}, Y, B, Z)$ are two ij-deciders, we have $\sigma_{ij}(P) = \sigma_{ij}^i(Xr) = -1$, and it remains to show $\sigma_{ij}(Q) = -1$. Recall that $\dashv$ being governed by i-sets implies, in particular, that there are $(i-1)$-ary sign functions ε_i^{i-1} and ε_j^{i-1} defined on $[2,N]^{(k-1)}$. So we can enumerate $B=\{c_j,d_j\}$ in such a way that $\varepsilon_j^{i-1}(X)c_j<\varepsilon_j^{i-1}(X)d_j$.

First Case: $\varepsilon_i^{i-1}(X)=+1$

Our assumption $\sigma_{ij}^i(P)=-1$ is defined to mean $XtYc_jZ \multimap XrYd_jZ$. Furthermore, the triple $S=(X,\{r,s\},Yd_jZ)$ is an i-decider and $\varepsilon_i(S)=\varepsilon_i^{i-1}(X)=+1$ implies that $XrYd_jZ \multimap XsYd_jZ$. So altogether we have $XtYc_jZ \multimap XsYd_jZ$ or, in other words, $\sigma_{ij}^i(Q)=-1$.

Second Case: $\varepsilon_i^{i-1}(X)=-1$

This time $\sigma_{ij}^i(P)=-1$ abbreviates the statement $XrYc_jZ \multimap XtYd_jZ$ and the case hypothesis $\varepsilon_i^{i-1}(X)=-1$ implies $XsYc_jZ \multimap XrYc_jZ$. For these reasons we have $XsYc_jZ \multimap XtYd_jZ$, i.e., $\sigma_{ij}^i(Q)=-1$. □

Next we elaborate on the reason why we care about the monotonicity of the permutation colourings.

Lemma 3.14. *There exists an absolute constant C'_k such that if $n\geqslant 2k$, $N\geqslant C'_k n^{k^2}$, the ordering $([N]^{(k)},\multimap)$ is governed by pairs, and all permutation colourings σ_{ij}^2 with $i\geqslant 2$ are monotone, then there is a set $Z\subseteq[N]$ of size n such that $(Z^{(k)},\multimap)$ is ordered canonically.*

Proof. Since the ordering $\multimap$ is governed by pairs, it has unary sign functions $\varepsilon_1,\dots,\varepsilon_k$ defined on $[2,N]^{(1)}$. In other words, every $x\in[2,N]$ has its own "opinion"

$$\varepsilon(x)=\big(\varepsilon_1(x),\dots,\varepsilon_k(x)\big)\in\Phi^k$$

what the sign vector should be, where $\Phi=\{-1,+1\}$ was introduced in (3.1). Due to the Schubfachprinzip, there are a set $Y\subseteq[2,N]$ of size $|Y|\geqslant 2^{-k}N$ as well as a vector $\varepsilon\in\Phi^k$ such that $\varepsilon(y)=\varepsilon$ holds for every $y\in Y$. Now $(Y^{(k)},\multimap)$ is sign-definite and by Lemma 3.12 the set $Y_\star\subseteq Y$ obtained from Y by deleting the smallest and the k largest elements has the following property: for every pair $ij\in[k]^{(2)}$ the restriction of σ_{ij}^2 to $Y_\star^{(2)}$ is monotone. Notice that $|Y_\star|\geqslant 2^{-k-1}N$.

Now Lemma 3.2 applied to $I=[k]^{(2)}$ and the family $(\sigma_{ij}^2)_{ij\in I}$ yields a set $Z_\star\subseteq Y_\star$ of size $|Z_\star|=2^{k-1}n$ that is monochromatic with respect to every σ_{ij}^2. This means that $(Z_\star^{(k)},\multimap)$ is not only sign-definite but also permutation-definite, and Lemma 3.7 leads to the desired set Z. □

It remains to execute the plan laid out in §3.4.

Proof of Theorem 1.4. Given $k\geqslant 2$ we let C'_k denote the constant delivered by the previous lemma. Recall that Γ denotes the set of linear orderings of $[k+2]^{(k)}$. Now for an arbitrary number $n \geqslant 2k$ we define recursively

$$\begin{aligned} N_0 &= C'_k n^{k^2} + k + 1\,, \\ N_m &= (2|\Gamma|)^{N_{m-1}^{m+1}} \cdot (2k^2 N_{m-1}^m)^{k^2 N_{m-1}^m} + k + 1 \text{ for } m \in [k-2]\,, \\ N_{k-1} &= |\Gamma|^{N_{k-2}^{k-2}}\,, \\ \text{and finally } N_k &= |\Gamma|^{N_{k-1}^{k-1}}\,. \end{aligned}$$

An easy induction on m discloses that for every $m \in [0, k]$ there exists an absolute constant $C_{k,m}$ such that $N_m \leqslant t_m\big(C_{k,m} n^{2k^2}\big)$ holds for all $n \geqslant 2k$. In particular, there exists an absolute constant C_k such that independently of n we have $N_k \leqslant t_k(n^{C_k})$. So it suffices to show that for every ordering $([N_k]^{(k)}, \dashv)$ there exists a set $Z \subseteq [N_k]$ of size n such that $\dashv$ orders $Z^{(k)}$ canonically. Two successive applications of Corollary 3.4 yield a set $Z_{k-2} \subseteq [N]$ of size $|Z_{k-2}| = N_{k-2}$ such that $(Z_{k-2}^{(k)}, \dashv)$ is governed by k-sets.

Claim 3.15. *For every $m \in [0, k-2]$ there exists a set $Z_m \subseteq [N_k]$ of size $|Z_m| = N_m - k - 1$ such that $(Z_m^{(k)}, \dashv)$ is governed by $(m+2)$-sets and for every $ij \in [m+2, k]^{(2)}$ there is a monotone permutation colouring $\sigma_{ij}^{m+2} \colon Z_m^{(m+2)} \longrightarrow \Phi$.*

Proof of Claim 3.15. We argue by decreasing induction on m, the base case $m = k-2$ already being known. Now suppose that $m \in [k-2]$ and that we already have obtained a set $Z_m \subseteq [N_k]$ with the desired property.

We plan to apply Lemma 3.3 to N_{m-1}, $m+2$, $m+2$, and Z_m here in place of n, r, s, and X there. Instead of f there we take the function $f^{m+2} \colon Z_m^{(m+2)} \longrightarrow \Gamma$ governing the function $f \colon [N_k]^{(k+2)} \longrightarrow \Gamma$ associated with $\dashv$. Finally, we take the index set $I = [m+2, k]^{(2)}$, which clearly satisfies $|I| \leqslant k^2$, and the family $(\sigma_{ij}^{m+2})_{ij \in I}$ of monotone permutation colourings $\sigma_{ij}^{m+2} \colon Z_m^{(m+2)} \longrightarrow \Phi$ provided by the induction hypothesis. Since

$$N_m - k - 1 \geqslant (2|\Gamma|)^{N_{m-1}^{m+1}} \cdot (2|I| N_{m-1}^m)^{|I| N_{m-1}^m}\,,$$

the intended application of Lemma 3.3 is justified and we can obtain a set $Z'_{m-1} \subseteq Z_m$ of size $|Z'_{m-1}| = N_{m-1}$ such that $\big({Z'}_{m-1}^{(k)}, \dashv\big)$ is governed by $(m+1)$-sets and the permutation colourings $(\sigma_{ij}^{m+1})_{ij \in I}$ are monotone. The set Z_{m-1} arising from Z'_{m-1} by cutting off the smallest and the k largest elements has all desired properties, because by Lemma 3.13 the permutation colourings $\sigma_{m+1,j}^{m+1}$ with $j \in [m+2, k]$ are monotone as well. □

Observe that the case $m = 0$ yields a set Z_0 of size $|Z_0| = N_0 - (k+1) = C'_k n^{k^2}$. By our choice of C'_k according to Lemma 3.14 there exists a set $Z \subseteq Z_0$ of size $|Z| = n$ such that $(Z^{(k)}, \dashv)$ is ordered canonically. This concludes the proof of Theorem 1.4. □

§4. Shelah's upper bound on Erdős-Rado numbers

In this section we show that the results in §3.1 can also be used for reproving Shelah's theorem that k-uniform Erdős-Rado numbers grow (at most) $k-1$ times exponentially. More precisely, we shall show $\mathrm{ER}^{(k)}(n) \leqslant t_{k-1}(C_k n^{6k})$, where, let us recall, an estimate of the form $\mathrm{ER}^{(k)}(n) \leqslant N$ means that for every equivalence relation $\equiv$ on $[N]^{(k)}$ there exist an n-element subset $X \subseteq [N]$ and a set $I \subseteq [k]$ such that $\equiv$ and $\equiv_I$ agree on $X^{(k)}$. Here, for two sets of positive integers $x = \{x_1, \dots, x_k\}$ and $y = \{y_1, \dots, y_k\}$ with $x_1 < \dots < x_k$ and $y_1 < \dots < y_k$ and for a subset $I \subseteq [k]$ the statement $x \equiv_I y$ is defined to mean that $x_i = y_i$ holds for all $i \in I$. Throughout the entire section we fix an integer $k \geqslant 2$. Moreover, Γ will always denote the set of all equivalence relations on $[k+2]^{(k)}$.

4.1. **Preparation.** Beginning as in Remark 3.9 we *associate* with every equivalence relation $([N]^{(k)}, \equiv)$ the colouring $f\colon [N]^{(k+2)} \longrightarrow \Gamma$ mapping every set $A \in [N]^{(k+2)}$ to the unique equivalence relation $\sim \in \Gamma$ with the property that $(A^{(k)}, \equiv)$ and $([k+2]^{(k)}, \sim)$ are isomorphic via the order-preserving map from A to $[k+2]$. For $m \in [k+2]$ we say that $([N]^{(k)}, \equiv)$ is *governed by m-sets* if there exists a function $f^m\colon [N]^{(m)} \longrightarrow \Gamma$ which governs f. For instance, every equivalence relation is governed by $(k+2)$-sets. Next we associate with every equivalence relation some auxiliary colourings g_i using the set of colours $\Phi = \{-1, +1\}$ introduced in (3.1).

Definition 4.1. *Let $N \geqslant k+1$ be an integer and let $i \in [k]$ be an index. The i^{th} auxiliary function $g_i\colon [N]^{(k+1)} \longrightarrow \Phi$ of an equivalence relation $([N]^{(k)}, \equiv)$ is defined by*

$$g_i(z_1, \dots, z_{k+1}) = 1 \iff \{z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{k+1}\} \equiv \{z_1, \dots, z_i, z_{i+2}, \dots, z_{k+1}\}$$

whenever $1 \leqslant z_1 < \dots < z_{k+1} \leqslant N$.

Like the permutation colourings in the previous section, these functions will later be synchronised by means of an iterative application of Lemma 3.3. Notice that if an equivalence relation $([N]^{(k)}, \equiv)$ is governed by m-sets, then there exists a function $g_i^{m-1}\colon [2, N]^{(m-1)} \longrightarrow \Phi$ governing the restriction of g_i to $[2, N]^{(k+1)}$. Our next result asserts that at some stage of our argument these functions will automatically become monotone.

Lemma 4.2. *If $N \geqslant 2k$, $m \in [k]$, and $([N]^{(k)}, \equiv)$ is governed by $(m+1)$-sets, then the function $g_m^m\colon [2, N]^{(m)} \longrightarrow \Phi$ is increasing on $[2, N-k]^{(m)}$.*

Proof. Let some integers $2 \leqslant a_1 < \dots < a_m < a'_m \leqslant N-k$ satisfying $g_m^m(a_1, \dots, a_m) = 1$ be given. We are to prove that $g_m^m(a_1, \dots, a_{m-1}, a'_m) = 1$ holds as well. Set $a_0 = 1$ and $a_{m+i} = N - k + i$ for every $i \in [k+1-m]$. Our assumption yields $g_m(a_1, \dots, a_{k+1}) = 1$, whence

$$\{a_1, \dots, a_m, a_{m+2}, \dots, a_{k+1}\} \equiv \{a_1, \dots, a_{m-1}, a_{m+1}, \dots, a_{k+1}\}\,.$$

This piece of information is among the facts encoded in $f(a_0, \dots, a_{k+1})$ and, as $([N]^{(k)}, \equiv)$ is governed by $(m+1)$-sets, the same information is known to

$$f(a_0, \dots, a_m, a'_m, a_{m+2}, \dots, a_{k+1})\,.$$

Consequently, we also have

$$\{a_1, \dots, a_m, a_{m+2}, \dots, a_{k+1}\} \equiv \{a_1, \dots, a_{m-1}, a'_m, a_{m+2}, \dots, a_{k+1}\}\,.$$

Both displayed equivalences have the same left side, so the transitivity of $\equiv$ leads to

$$\{a_1, \dots, a_{m-1}, a'_m, a_{m+2}, \dots, a_{k+1}\} \equiv \{a_1, \dots, a_{m-1}, a_{m+1}, \dots, a_{k+1}\}\,,$$

for which reason

$$g^m_m(a_1, \dots, a_{m-1}, a'_m) = g_m(a_1, \dots, a_{m-1}, a'_m, a_{m+1}, \dots, a_{k+1}) = 1\,. \qquad \square$$

Let us observe that for canonical equivalence relations $\equiv_I$ with $I \subseteq [k]$ the auxiliary functions $g_1, \dots, g_k$ are constant. Moreover, if $\varepsilon_1, \dots, \varepsilon_k \in \Phi$ denote the values these functions always attain, then $I = \{i \in [k] \colon \varepsilon_i = -1\}$. In the converse direction, if we have an equivalence relation $([N]^{(k)}, \equiv)$ with the property that such colours $\varepsilon_1, \dots, \varepsilon_k \in \Phi$ exist, then we may define $I = \{i \in [k] \colon \varepsilon_i = -1\}$, but it does not follow immediately that $\equiv$ coincides with the canonical equivalence relation $\equiv_I$. This is why we require an additional purging argument.

Definition 4.3. *For $i \in [k]$ an equivalence relation $([N]^{(k)}, \equiv)$ is said to be i-purged if for all integers $a_1, \dots, a_i, b_1, \dots, b_i, c_{i+1}, \dots, c_{k+1}$ satisfying*

- *(i)* $1 \leqslant a_1 < \dots < a_i$ *and* $1 \leqslant b_1 < \dots < b_i$,
- *(ii)* $\max(a_i, b_i) < c_{i+1} < \dots < c_{k+1} \leqslant N$,
- *(iii)* $g_i(a_1, \dots, a_i, c_{i+1}, \dots, c_{k+1}) = g_i(b_1, \dots, b_i, c_{i+1}, \dots, c_{k+1}) = -1$,
- *(iv)* *and* $\{a_1, \dots, a_i, c_{i+1}, \dots, c_k\} \equiv \{b_1, \dots, b_i, c_{i+1}, \dots, c_k\}$

we have $a_i = b_i$.

Lemma 4.4. *Suppose that the equivalence relation $([N]^{(k)}, \equiv)$ is i-purged for every $i \in [k]$. If the auxiliary functions $g_1, \dots, g_k$ are constant, then $([N-1]^{(k)}, \equiv)$ is canonical.*

Proof. Let $\varepsilon_1, \dots, \varepsilon_k \in \Phi$ be the values which $g_1, \dots, g_k$ always attain and set

$$I = \big\{i \in [k] \colon \varepsilon_i = -1\big\}\,.$$

It suffices to prove that for any two sets $a = \{a_1, \dots, a_k\}$ and $b = \{b_1, \dots, b_k\}$ from $[N-1]^{(k)}$ whose elements have just been enumerated in increasing order the statement

$$\{a_1, \dots, a_k\} \equiv \{b_1, \dots, b_k\} \iff \forall i \in I \;\; a_i = b_i \tag{4.1}$$

is valid. To this end we argue by induction on the number $\mu = |\{i \in [k] \colon a_i \neq b_i\}|$ counting the indices where a and b disagree. In the base case $\mu = 0$ we have $a = b$ and both sides of the equivalence (4.1) are true. Now suppose that $\mu \in [k]$ has the property that (4.1) holds for all pairs of sets from $[N]^{(k)}$ disagreeing less than μ times. For the induction step we assume that $a, b \in [N]^{(k)}$ disagree with respect to exactly μ indices.

Beginning with the backwards implication we assume $a_i = b_i$ for all $i \in I$ and intend to derive $a \equiv b$. Notice that the smallest index $i(\star) \in [k]$ satisfying $a_{i(\star)} \neq b_{i(\star)}$ belongs to $[k] \smallsetminus I$, whence $\varepsilon_{i(\star)} = 1$. By symmetry we may suppose that $a_{i(\star)} < b_{i(\star)}$. The induction hypothesis implies that the set $c = b \cup \{a_{i(\star)}\} \smallsetminus \{b_{i(\star)}\}$ is equivalent to a with respect to $\equiv$. Moreover, the definition of

$$g_{i(\star)}(b_1, \dots, b_{i(\star)-1}, a_{i(\star)}, b_{i(\star)}, \dots, b_k) = \varepsilon_{i(\star)} = 1$$

yields $b \equiv c$, so altogether we have indeed $a \equiv c \equiv b$.

Now suppose conversely that $a \equiv b$, where $a, b \in [N-1]^{(k)}$. Due to $\mu > 0$ there exists a largest integer $i(\star) \in [k]$ such that $a_{i(\star)} \neq b_{i(\star)}$. Applying the assumption that $\equiv$ be $i(\star)$-purged to

- $a_1 < \dots < a_{i(\star)}$, $b_1 < \dots < b_{i(\star)}$,
- and $a_{i(\star)+1} = b_{i(\star)+1}, \dots, a_k = b_k, N$ playing the rôles of $c_{i(\star)+1}, \dots, c_{k+1}$

we infer $\varepsilon_{i(\star)} = +1$, i.e., $i(\star) \notin I$. By symmetry we may again assume $a_{i(\star)} < b_{i(\star)}$. Now

$$g_{i(\star)}(a_1, \dots, a_{i(\star)}, b_{i(\star)}, a_{i(\star)+1}, \dots, a_k) = \varepsilon_{i(\star)} = 1$$

implies that the set $c = a \cup \{b_{i(\star)}\} \smallsetminus \{a_{i(\star)}\}$ is equivalent to a with respect to $\equiv$. Since b and c disagree only $\mu - 1$ times, the induction hypothesis yields $b_i = c_i$ for all $i \in I$. Together with $i(\star) \notin I$ this shows that we have indeed $a_i = b_i$ for all $i \in I$. This completes the induction step and, therefore, the proof of (4.1). In particular, $\equiv$ is canonical. □

The next result shows that from a quantitative point of view purging is accompanied by a polynomial dependence of the involved constants.

Lemma 4.5. *Let $m \in [k]$, $n \geqslant k$, and $N \geqslant (2kn)^{2k}$. If an equivalence relation $([N]^{(k)}, \equiv)$ is governed by $(m+1)$-sets, then there exists a set $Z \subseteq [N]$ of size n such that $(Z^{(k)}, \equiv)$ is m-purged.*

Proof. The potential counterexamples to $([2, N]^{(k)}, \equiv)$ being m-purged can be regarded as $(k + m + 1)$-tuples of integers satisfying the conditions *(i)*–*(iv)* from Definition 4.3 and $a_m \neq b_m$. By symmetry it suffices to worry about those counterexamples for which $a_m < b_m$. Bearing this in mind we define Ψ to be the set of all $(k + m)$-tuples

$$(a_1, \dots, a_m, b_1, \dots, b_m, c_{m+1}, \dots, c_k) \in [2, N]^{k+m}$$

such that

- $a_1 < \dots < a_m$, $b_1 < \dots < b_m$,
- $a_m < b_m$,
- $\max(a_m, b_m) < c_{m+1} < \dots < c_k$,
- $g^m_m(b_1, \dots, b_m) = -1$,
- and $\{a_1, \dots, a_m, c_{m+1}, \dots, c_k\} \equiv \{b_1, \dots, b_m, c_{m+1}, \dots, c_k\}$.

The key observation is that two members of Ψ cannot differ in their $(2m)^{\text{th}}$ entry alone.

Claim 4.6. *If* $(a_1, \dots, a_m, b_1, \dots, b_m, c_{m+1}, \dots, c_k)$ *and* $(a_1, \dots, a_m, b_1, \dots, b'_m, c_{m+1}, \dots, c_k)$ *belong to* Ψ*, then* $b_m = b'_m$*.*

Proof. Assume contrariwise that $b_m < b'_m$. Due to

$$\{b_1, \dots, b'_m, c_{m+1}, \dots, c_k\} \equiv \{a_1, \dots, a_m, c_{m+1}, \dots, c_k\} \equiv \{b_1, \dots, b_m, c_{m+1}, \dots, c_k\}$$

we know $g_m(b_1, \dots, b_m, b'_m, c_{m+1}, \dots, c_k) = 1$, which contradicts $g^m_m(b_1, \dots, b_m) = -1$. □

Now we consider the partition

$$\Psi = \Psi_0 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} \Psi_{m-1}$$

defined by

$$\Psi_i = \big\{(a_1, \dots, a_m, b_1, \dots, b_m, c_{m+1}, \dots, c_k) \in \Psi \colon |\{a_1, \dots, a_m, b_1, \dots, b_{m-1}\}| = m + i\big\}$$

for every $i \in [0, m-1]$. As a consequence of Claim 4.6 every $(a_1, \dots, c_k) \in \Psi_i$ can be described uniquely by specifying

(1) the $(k + i)$-element subset $X = \{a_1, \dots, a_m, b_1, \dots, b_{m-1}, c_{m+1}, \dots, c_k\}$ of $[2, N]$

(2) and telling for each of $a_1, \dots, a_m, b_1, \dots, b_{m-1}$ to which of the $m + i$ members of $Y = \{a_1, \dots, a_m, b_1, \dots, b_{m-1}\}$ they are equal.

There are $\binom{N-1}{k+i}$ possibilities for the first decision. The set Y can be obtained from X by removing the $k - m$ largest elements and, in particular, Y is definable from X. Thus there are at most $(m + i)^{2m-1}$ possibilities for *(2)*. Altogether, theses considerations establish

$$|\Psi_i| \leqslant \binom{N-1}{k+i}(m+i)^{2m-1} \leqslant (N-1)^{k+i}(2m)^{2m-1}$$

In combination with

$$\sum_{Z\in[2,N]^{(n)}} |\Psi \cap Z^{k+m}| = \sum_{i=0}^{m-1} \binom{(N-1)-(k+i+1)}{n-(k+i+1)} |\Psi_i|$$
$$\leqslant \binom{N-1}{n} \sum_{i=0}^{m-1} \Big(\frac{n}{N-1}\Big)^{k+i+1} |\Psi_i|$$

this implies

$$\sum_{Z\in[2,N]^{(n)}} |\Psi \cap Z^{k+m}| \leqslant \binom{N-1}{n} \sum_{i=0}^{m-1} \frac{n^{k+i+1}}{N-1}(2m)^{2m-1}$$
$$\leqslant \frac{(2mn)^{2k}}{2(N-1)}\binom{N-1}{n} < \binom{N-1}{n}.$$

Consequently some set $Z \in [2, N]^{(n)}$ satisfies $\Psi \cap Z^{k+m} = \varnothing$. Due to the definition of Ψ every such set Z is m-purged. □

4.2. **Plan.** Suppose that we have an equivalence relation $([N]^{(k)}, \equiv)$, where N is, roughly speaking, $k-1$ times exponential in n, and that we want to find a subset $X \subseteq [N]$ of size n such that $(X^{(k)}, \equiv)$ is canonical. The material in the previous subsection suggests the following strategy. Letting $f\colon [N]^{(k+2)} \longrightarrow \Gamma$ denote the function associated with $\equiv$ we want to select appropriate subsets $[N] = Z_{k-1} \supseteq \cdots \supseteq Z_0$ such that for each $m \in [0, k-1]$ the size of Z_m is, roughly, m times exponential in n and the restriction of f to $Z_m^{(k+2)}$ is governed by $(m+3)$-sets. Lemma 4.2 tells us that if we construct such a sequence of sets $(Z_m)_{0\leqslant m\leqslant k-1}$ by means of an iterative applications of Lemma 3.3, then we can achieve that the restrictions of $g_2^2, \ldots, g_k^2$ to $Z_0^{(2)}$ are monotone. Now $|Z_0|$ depends polynomially on n, which severely restricts our possibilities as to how the argument can be continued. Certainly Lemma 3.2 allows us to synchronise $g_2, \ldots, g_k$. Moreover, synchronising g_1 will only cost an additional square root; essentially this is due to the formula $\mathrm{ER}^{(1)}(n) \leqslant n^2$. Finally k successive applications of Lemma 4.5 allow us to perform a complete purging, and by Lemma 4.4 we will thereby be done.

Now if one completely ignores the last exponent C_k for which one will end up proving $\mathrm{ER}^{(k)}(n) \leqslant t_{k-1}(n^{C_k})$, then it is entirely immaterial in which order one carries the various steps of the argument out. But from a quantitative point of view it seems recommendable to perform each purging step as early as possible. Illustrating this point by means of an example we give a brief sketch as to how one can prove $\mathrm{ER}^{(4)}(n) \leqslant t_3(Cn^{24})$.

- Start with an arbitrary equivalence relation $([N]^{(4)}, \equiv)$, where $N = t_3(Cn^{24})$, and let $f\colon [N]^{(6)} \longrightarrow \Gamma$ be its associated function.

- By Corollary 3.4 there is a set $Z_2 \subseteq Z_3 = [N]$ of size $t_2(C'n^{24})$ on which f is governed by quintuples. Now g_4^4 is monotone by Lemma 4.2.
- Before doing anything else we appeal to Lemma 4.5 and get a 4-purged set $P_2 \subseteq Z_2$ of size $t_2(C''n^{24})$, where C'' is extremely close to C'.
- Next we apply Lemma 3.3 and obtain $Z_1 \subseteq P_2$ of size $t_1(C'''n^{24})$ such that g_3^3, g_4^3 are monotone and f is governed by quadruples.
- Now Lemma 4.5 leads us to a 3-purged set $P_1 \subseteq Z_1$ of size $t_1(C''''n^{24})$.
- Only now we appeal to Lemma 3.3 again and take $Z_0 \subseteq P_1$ of size $C'''''n^8$ such that g_2^2, g_3^2, g_4^2 are monotone and f is governed by triples.

The next lemma tells us how to complete the argument once Z_0 is found.

Lemma 4.7. *Let $n \geqslant k$, $N \geqslant (k-1)!(2n)^{2k}$, and let $([N]^{(k)}, \equiv)$ be an equivalence relation which is governed by triples and i-purged for every $i \in [3,k]$. If the functions $g_2^2, \dots, g_k^2$ defined on $[2,N]^{(2)}$ are monotone, then there exists a set $X \subseteq [N]$ of size n such that $(X^{(k)}, \equiv)$ is canonical.*

Proof. Due to the monotonicity of $g_2^2, \dots, g_k^2$ and $N - 1 \geqslant (k-1)!(4n^2-1)^k$ Lemma 3.2 yields a subset $Y \subseteq [2,N]$ of size $|Y| = 4n^2 - 1$ such that the functions $g_2^2, \dots, g_k^2$ are constant on $Y^{(2)}$. Denote the $k-1$ largest elements of Y by $y_1 < \dots < y_{k-1}$ and define an equivalence relation $\sim$ on $Y \smallsetminus \{y_1, \dots, y_{k-1}\}$ by

$$a \sim b \iff \{a, y_1, \dots, y_{k-1}\} \equiv \{b, y_1, \dots, y_{k-1}\}\,.$$

In view of $n \geqslant k \geqslant 2$ we have $|Y \smallsetminus \{y_1, \dots, y_{k-1}\}| = 4n^2 - k \geqslant (2n-k+1)^2 + 1$ and, consequently, there exists a set $Z^- \subseteq Y \smallsetminus \{y_1, \dots, y_{k-1}\}$ of size $2n - k + 2$ which is either contained in an equivalence class of $\sim$ or consists of numbers that are mutually non-equivalent with respect to $\sim$. In both cases the set $Z = Z^- \cup \{y_1, \dots, y_{k-1}\}$ has the property that the restriction of g_1^2 to $Z^{(2)}$ is constant. Notice that the cardinality of Z is $2n+1$. Enumerating the elements of Z in increasing order we write $Z = \{z_1, \dots, z_{2n+1}\}$.

The remainder of the proof we shall show that

$$X_\star = \big\{z_1, z_3, \dots, z_{2n+1}\big\}$$

has the property that $(X_\star^{(k)}, \equiv)$ is 1-purged and 2-purged. In the light of Lemma 4.4 this will imply that the n-element set $X = X_\star \smallsetminus \{z_{2n+1}\}$ is as desired.

Starting with the former goal we let $\varepsilon_1 \in \Phi$ be the constant value that g_1^2 attains on $X_\star^{(2)}$. Because of Definition 4.3 *(iii)* we may suppose that $\varepsilon_1 = -1$. Now let

$$a_1 \leqslant b_1 < c_2 < \dots < c_{k+1}$$

be $k+2$ members of $X_\star$ satisfying $\{a_1, c_2, \dots, c_k\} \equiv \{b_1, c_2, \dots, c_k\}$. As the assumption $a_1 < b_1$ would yield $\varepsilon_1 = g_1(a_1, b_1, c_2, \dots, c_k) = 1$, we have indeed $a_1 = b_1$, as required.

It remains to show that $(X_\star^{(k)}, \equiv)$ is 2-purged as well. As before, we may suppose that the constant value ε_2 attained by g_2 on $X_\star^{(k+1)}$ is -1. Assume for the sake of contradiction that there exist $k+3$ members $a_1 < a_2 < b_2$ and $b_1 < b_2 < c_3 < \dots < c_{k+1}$ of X such that

$$\{a_1, a_2, c_3, \dots, c_k\} \equiv \{b_1, b_2, c_3, \dots, c_k\}\,. \tag{4.2}$$

Pick a number $q \in Z$ such that $b_2 < q < c_3$. Since f is governed by triples, we have

$$f(a_1, a_2, b_1, b_2, c_3, \dots, c_k) = f(a_1, a_2, b_1, q, c_3, \dots, c_k)\,.$$

So (4.2) entails

$$\{a_1, a_2, c_3, \dots, c_k\} \equiv \{b_1, q, c_3, \dots, c_k\}$$

and altogether we obtain

$$\{b_1, b_2, c_3, \dots, c_k\} \equiv \{b_1, q, c_3, \dots, c_k\}\,,$$

which leads us to the contradiction

$$\varepsilon_2 = g_2(b_1, b_2, q, c_3, \dots, c_k) = 1\,. \qquad \square$$

4.3. **Canonisation.** It remains to execute the plan discussed in the foregoing subsection. Given $n \geqslant k \geqslant 2$ we define positive integers $N_0, P_1, N_1, \dots, P_{k-2}, N_{k-2}, N_{k-1}$ according to the following recursive rules.

$$\begin{aligned}
N_0 &= (k-1)!(2n)^{2k} + (k+1)\,,\\
P_m &= (2|\Gamma|)^{N_{m-1}^{m+2}} \cdot (2kN_{m-1}^m)^{kN_{m-1}^m} \qquad \text{for } m \in [k-2]\,,\\
N_m &= (2kP_m)^{2k} + (k+1) \qquad \text{for } m \in [k-2]\,,\\
N_{k-1} &= |\Gamma|^{N_{k-2}^{k+1}}\,.
\end{aligned}$$

We want to prove that if $N \geqslant N_{k-1}$ then for every equivalence relation $([N]^{(k)}, \equiv)$ there exists a set $X \subseteq [N]$ of size n such that $(X^{(k)}, \equiv)$ is canonical. This will imply

$$\mathrm{ER}^{(k)}(n) \leqslant N_{k-1} \leqslant t_{k-1}(C_k n^{6k})\,, \tag{4.3}$$

where C_k depends only on k.

Let $f\colon [N]^{k+2} \longrightarrow \Gamma$ be the function associated to a given equivalence relation $([N]^{(k)}, \equiv)$, where $N \geqslant N_{k-1}$. An initial application of Corollary 3.4 leads to a set $Z'_{k-2} \subseteq [N]$ such that $(Z'^{(k)}_{k-2}, \equiv)$ is governed by $(k+1)$-sets and $|Z'_{k-2}| = N_{k-2}$. According to Lemma 4.2 the set $Z_{k-2} \subseteq Z'_{k-2}$ that arises when one removes the smallest and the k largest elements of Z'_{k-2} has the property that $g_k^k\colon Z_{k-2}^{(k)} \longrightarrow \Phi$ is monotone. Notice that $|Z_{k-2}| = N_{k-2} - (k+1)$.

Claim 4.8. *For every $m \in [0, k-2]$ there exists a set $Z_m \subseteq [N]$ of size $|Z_m| = N_m - (k+1)$ with the following properties:*

(*i*) $(Z_m^{(k)}, \equiv)$ *is governed by* $(m+3)$*-sets.*
(*ii*) *For every* $i \in [m+2, k]$ *the function* g_i^{m+2} *is monotone.*
(*iii*) *If* $i \in [m+3, k]$, *then* $(Z_m^{(k)}, \equiv)$ *is* i*-purged.*

Observe that the case $m = 0$ yields a set Z_0 of size $|Z_0| = (k-1)!(2n)^{2k}$. Thus Lemma 4.7 shows that Claim 4.8 implies (4.3).

Proof of Claim 4.8. We argue by decreasing induction on m. In the base case $m = k-2$ clause (*iii*) holds vacuously and the set Z_{k-2} defined above has the desired properties.

Now suppose that for some $m \in [k-2]$ we have already found a subset $Z_m \subseteq [N]$ of size $N_m - (k+1) = (2kP_m)^{2k}$ fulfilling (*i*), (*ii*), and (*iii*). Owing to Lemma 4.5 there exists a set $Q_m \subseteq Z_m$ of size $|Q_m| = P_m$ such that $(P_m^{(k)}, \equiv)$ is $(m+2)$-purged. Next we apply Lemma 3.3 to $f^{m+3} \colon P_m^{(m+3)} \longrightarrow \Gamma$ and the family $(g_i^{m+2})_{m+2 \leqslant i \leqslant k}$ of monotone colourings. This yields a set $Z'_{m-1} \subseteq P_m$ of size N_{m-1} such that $({Z'}_m^{(k)}, \equiv)$ is governed by $(m+2)$-sets and for every $i \in [m+2, k]$ the function g_i^{m+1} is monotone. Finally Lemma 4.2 tells us that the set $Z_{m-1} \subseteq Z'_{m-1}$ obtained by removing the smallest and the k largest elements from Z'_{m-1} has the desired properties. □

References

[1] M. Bucic, B. Sudakov, and T. Tran, *Erdős-Szekeres theorem for multidimensional arrays*, available at arXiv:1910.13318. Submitted. ↑1.2

[2] D. Conlon, J. Fox, and B. Sudakov, *Hypergraph Ramsey numbers*, J. Amer. Math. Soc. **23** (2010), no. 1, 247–266, DOI 10.1090/S0894-0347-09-00645-6. MR2552253 ↑1.2

[3] W. Deuber, R. L. Graham, H. J. Prömel, and B. Voigt, *A canonical partition theorem for equivalence relations on* $\mathbf{Z}^t$, J. Combin. Theory Ser. A **34** (1983), no. 3, 331–339, DOI 10.1016/0097-3165(83)90067-5. MR700039 ↑1.1

[4] P. Erdős, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. **53** (1947), 292–294, DOI 10.1090/S0002-9904-1947-08785-1. MR19911 ↑1.2

[5] P. Erdős and R. L. Graham, *Old and new problems and results in combinatorial number theory*, Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique], vol. 28, Université de Genève, L'Enseignement Mathématique, Geneva, 1980. MR592420 ↑1.1

[6] P. Erdős and A. Hajnal, *On Ramsey like theorems. Problems and results*, Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), Inst. Math. Appl., Southend-on-Sea, 1972, pp. 123–140. MR0337636 ↑1.2

[7] P. Erdős, A. Hajnal, and R. Rado, *Partition relations for cardinal numbers*, Acta Math. Acad. Sci. Hungar. **16** (1965), 93–196, DOI 10.1007/BF01886396. MR202613 ↑1.2, 2.1

[8] P. Erdős and R. Rado, *A combinatorial theorem*, J. London Math. Soc. **25** (1950), 249–255, DOI 10.1112/jlms/s1-25.4.249. MR37886 ↑1.1

[9] ———, *Combinatorial theorems on classifications of subsets of a given set*, Proc. London Math. Soc. (3) **2** (1952), 417–439, DOI 10.1112/plms/s3-2.1.417. MR65615 ↑1.1

[10] P. Erdős and G. Szekeres, *A combinatorial problem in geometry*, Compositio Math. **2** (1935), 463–470. MR1556929 ↑1.2

[11] J. Fox, 2019. Personal communication to V. Rödl and M. Sales. ↑1.2

[12] R. L. Graham, K. Leeb, and B. L. Rothschild, *Ramsey's theorem for a class of categories*, Proc. Nat. Acad. Sci. U.S.A. **69** (1972), 119–120, DOI 10.1073/pnas.69.1.119. MR306009 ↑1.1

[13] ———, *Ramsey's theorem for a class of categories*, Advances in Math. **8** (1972), 417–433, DOI 10.1016/0001-8708(72)90005-9. MR306010 ↑1.1

[14] R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey theory*, 2nd ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., New York, 1990. A Wiley-Interscience Publication. MR1044995 ↑1.2

[15] H. Lefmann and V. Rödl, *On Erdős-Rado numbers*, Combinatorica **15** (1995), no. 1, 85–104, DOI 10.1007/BF01294461. MR1325273 ↑1.2

[16] J. Nešetřil, H. J. Prömel, V. Rödl, and B. Voigt, *Canonizing ordering theorems for Hales-Jewett structures*, J. Combin. Theory Ser. A **40** (1985), no. 2, 394–408, DOI 10.1016/0097-3165(85)90098-6. MR814422 ↑1

[17] J. Nešetřil and V. Rödl, *Statistics of orderings*, Abh. Math. Semin. Univ. Hambg. **87** (2017), no. 2, 421–433, DOI 10.1007/s12188-016-0174-x. MR3696159 ↑1

[18] F. P. Ramsey, *On a Problem of Formal Logic*, Proceedings London Mathematical Society **30** (1930), no. 1, 264–286, DOI 10.1112/plms/s2-30.1.264. ↑1.1

[19] S. Shelah, *Finite canonization*, Comment. Math. Univ. Carolin. **37** (1996), no. 3, 445–456. MR1426909 ↑1.2

Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany

Email address: `christian.reiher@uni-hamburg.de`

Email address: `schacht@math.uni-hamburg.de`

Email address: `kevin.sames@uni-hamburg.de`

Department of Mathematics, Emory University, Atlanta, USA

Email address: `vrodl@emory.edu`

Email address: `marcelo.tadeu.sales@emory.edu`