HOMOMORPHISM THRESHOLDS FOR ODD CYCLES

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Abstract. The interplay of minimum degree conditions and structural properties of large graphs with forbidden subgraphs is a central topic in extremal graph theory. For a given graph F we define the homomorphism threshold as the infimum over all $\alpha \in [0,1]$ such that every n-vertex F-free graph G with minimum degree at least αn has a homomorphic image H of bounded order (i.e. independent of n), which is F-free as well. Without the restriction of H being F-free we recover the definition of the chromatic threshold, which was determined for every graph F by Allen et al. [*Adv. Math.* 235 (2013), 261–295]. The homomorphism threshold is less understood and we address the problem for odd cycles.

§1. Introduction

Many questions in extremal graph theory concern the interplay of minimum degree conditions and structural properties of large graphs with forbidden subgraphs (see, e.g., [3, 4, 20]). For a family of graphs $\mathscr{F}$ and $\alpha \in [0,1]$ we consider the class $\mathscr{G}_{\mathscr{F}}(\alpha)$ of $\mathscr{F}$-free graphs G with minimum degree at least $\alpha|V(G)|$, i.e.,

$$\mathscr{G}_{\mathscr{F}}(\alpha) = \left\{G \colon \delta(G) \geqslant \alpha|V(G)| \text{ and } F \nsubseteq G \text{ for all } F \in \mathscr{F}\right\},$$

and for $\mathscr{F} = \{F\}$ we simply write $\mathscr{G}_F(\alpha)$. Clearly, $\mathscr{G}_{\mathscr{F}}(0)$ contains all $\mathscr{F}$-free graphs and as α increases the membership in $\mathscr{G}_{\mathscr{F}}(\alpha)$ becomes more restrictive. When α is bigger than the *Turán density* $\pi(\mathscr{F})$, then $\mathscr{G}_{\mathscr{F}}(\alpha)$ contains only finitely many different isomorphism types. We are interested in structural properties of members of $G \in \mathscr{G}_{\mathscr{F}}(\alpha)$ as α moves from $\pi(\mathscr{F})$ to 0, where structural properties are captured by the existence of (graph) homomorphims $G \xrightarrow{\text{hom}} H$ for some 'small' graph H.

We begin the discussion with the case of requiring bounded chromatic number, i.e., when H is allowed to be a clique of bounded size (independent of G). In that direction, for $\mathcal{F} = \{K_3\}$, Erdős, Simonovits, and Hajnal [7, page 325] showed that for every $\varepsilon > 0$ there exists a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ with members from $\mathscr{G}_{K_3}(\frac{1}{3} - \varepsilon)$ with unbounded chromatic number, i.e., $\chi(G_n) \to \infty$ as $n \to \infty$. In the other direction, Erdős and Simonovits conjectured that such a sequence does not exist with members from $\mathscr{G}_{K_3}(\frac{1}{3} + \varepsilon)$. Moving

2010 *Mathematics Subject Classification.* 05C35 (05C07, 05C15, 05D40).

Key words and phrases. extremal graph theory, odd cycles, homomorphism threshold.

The second author is supported by ERC Consolidator Grant 724903.

away from the triangle to arbitrary graphs F (or more generally to families of graphs $\mathscr{F}$) this leads to the concept of the *chromatic threshold* defined by

$$\delta_\chi(\mathscr{F}) = \inf\big\{\alpha \in [0,1] \colon \text{there is } K = K(\mathscr{F},\alpha) \text{ such that } \chi(G) \leqslant K \text{ for every } G \in \mathscr{G}_{\mathscr{F}}(\alpha)\big\}$$

and we simply write $\delta_\chi(F)$ for $\delta_\chi(\{F\})$. The work of Erdős, Simonovits, and Hajnal then asserts $\delta_\chi(K_3) \geqslant 1/3$ and Erdős and Simonovits asked for a matching upper bound. Such an upper bound was provided by Thomassen [18] and, therefore, we have

$$\delta_\chi(K_3) = \frac{1}{3}\,. \tag{1.1}$$

Addressing another conjecture of Erdős and Simonovits from [7] concerning the chromatic threshold of C_5, it was also shown by Thomassen [19] that for all odd cycles of length at least 5 the chromatic threshold is zero, i.e., $\delta_\chi(C_{2k-1}) = 0$ for all $k \geqslant 3$. For larger cliques (1.1) generalises to $\delta_\chi(K_k) = \frac{2k-5}{2k-3}$ for all $k \geqslant 3$ (see [8, 16]). Extending earlier work of Łuczak and Thomassé [14] and of Lyle [15], eventually Allen, Böttcher, Griffiths, Kohayakawa, and Morris [1] resolved the general problem and determined the chromatic threshold $\delta_\chi(\mathscr{F})$ for every finite family of graphs $\mathscr{F}$.

In the definition of the chromatic threshold $\delta_\chi(\mathscr{F})$ we are concerned with the existence of a small homomorphic image H for every $G \in \mathscr{G}_{\mathscr{F}}(\alpha)$ with $\alpha > \delta_\chi(\mathscr{F})$. However, since we allowed H to be a clique, the homomorphic image is not required to be $\mathscr{F}$-free itself. Adding this additional restriction leads to the following definition, where H is required to be $\mathscr{F}$-free as well.

Definition 1.1. For a family of graphs $\mathscr{F}$ we define its *homomorphism threshold*

$$\begin{aligned}\delta_{\hom}(\mathscr{F}) = \inf\big\{\alpha \in [0,1] \colon &\text{there is an } \mathscr{F}\text{-free graph } H = H(\mathscr{F},\alpha) \\ &\text{such that } G \xrightarrow{\hom} H \text{ for every } G \in \mathscr{G}_{\mathscr{F}}(\alpha)\big\}\,.\end{aligned}$$

If $\mathscr{F} = \{F\}$ consists of a single graph only, then we again simply write $\delta_{\hom}(F)$.

It follows directly from the definition that

$$\pi(\mathscr{F}) \geqslant \delta_{\hom}(\mathscr{F}) \geqslant \delta_\chi(\mathscr{F})$$

and that $\delta_{\hom}(\mathscr{F}) = 0$ for all families $\mathscr{F}$ containing a bipartite graph. Łuczak [13] was the first to study the homomorphism threshold and strengthened (1.1) by showing that $\delta_{\hom}(K_3) = \delta_\chi(K_3) = 1/3$. This was extended to larger cliques by Goddard and Lyle [8] and Nikiforov [16] (see also [17]) and for every $k \geqslant 3$ we have

$$\delta_{\hom}(K_k) = \delta_\chi(K_k) = \frac{2k-5}{2k-3}\,. \tag{1.2}$$

A first step of generalising Łuczak's result by viewing K_3 as the shortest odd cycle, was recently undertaken by Letzter and Snyder [12] by showing

$$\delta_{\hom}(C_5) \leqslant \frac{1}{5} \qquad \text{and} \qquad \delta_{\hom}(\{C_3, C_5\}) = \frac{1}{5}\,.$$

We further generalise this result to (families of) cycles of arbitrary odd length and present the following result.

Theorem 1.2. *For every integer $k \geqslant 3$ we have*

- *(i)* $\delta_{\hom}(C_{2k-1}) \leqslant \frac{1}{2k-1}$ *and*
- *(ii)* $\delta_{\hom}(\mathscr{C}_{2k-1}) = \frac{1}{2k-1}$, *where the family $\mathscr{C}_{2k-1} = \{C_3, C_5, \dots, C_{2k-1}\}$ consists of all odd cycles of length at most $2k-1$.*

Note that for $k = 2$ part *(ii)* of Theorem 1.2 would include part *(i)* and this is Łuczak's theorem [13]. For $k = 3$ Theorem 1.2 was obtained by Letzter and Snyder [12]. We remark that our approach substantially differs from the work of Łuczak and of Letzter and Snyder. For example, Łuczak's proof relied on Szemerédi's regularity lemma, which is not required here. Moreover, the proof of Letzter and Snyder is based on a careful case analysis, which yields explicit graphs $H = H(C_5, \alpha)$ for every $\alpha > 1/5$ (see Section 5 for more details).

The lower bound in part *(ii)* of Theorem 1.2 is given by a sequence of generalised Andrásfai graphs, which we discuss in Section 2. For the proof of the upper bound of part *(i)* we exclude relatively long odd cycles in C_{2k-1}-free graphs with high minimum degree and we specify and prove such a result in Section 3. The proofs of both upper bounds in Theorem 1.2 then follow in Section 4.

§2. Generalised Andrásfai graphs

In this section we establish the lower bound of part *(ii)* of Theorem 1.2, which will be given by a sequence of so-called *generalised Andrásfai graphs*. For $k = 2$ those graphs already appeared in the work of Erdős [5] and were also considered by Andrásfai [2,3].

Definition 2.1. For every integer $k \geqslant 2$ we define the class $\mathscr{A}_k$ of *Andrásfai graphs* consisting of all graphs $G = (V, E)$, where V is a finite subset of the unit circle $\mathbb{R}/\mathbb{Z}$ and two vertices are adjacent if and only if their distance in $\mathbb{R}/\mathbb{Z}$ is bigger than $\frac{k-1}{2k-1}$, i.e., the neighbourhood of any vertex $v \in V \subseteq \mathbb{R}/\mathbb{Z}$ is given by the set $V \cap \big(v + \big(\frac{k-1}{2k-1}, \frac{k}{2k-1}\big)\big)$, where

$$v + \left(\tfrac{k-1}{2k-1}, \tfrac{k}{2k-1}\right) = \left\{v + x \colon x \in \left(\tfrac{k-1}{2k-1}, \tfrac{k}{2k-1}\right)\right\} \subseteq \mathbb{R}/\mathbb{Z}\,.$$

Moreover, for integers $k \geqslant 2$ and $r \geqslant 1$ the *Andrásfai graph* $A_{k,r}$ is isomorphic to a graph from $\mathscr{A}_k$ having the corners of a regular $((2k-1)(r-1)+2)$-gon as its vertices.

We remark that one can show that every graph $G \in \mathscr{A}_k$ is homomorphic to $A_{k,r}$ for sufficiently large r. The following properties of Andrásfai graphs are well-known and we include the proof for completeness.

Proposition 2.2. *For all integers $k \geqslant 2$ and $r \geqslant 1$ the following properties hold*

(*a*) *$A_{k,r}$ is r-regular,*
(*b*) *$A_{k,r}$ is $\mathscr{C}_{2k-1}$-free,*
(*c*) *if $r \geqslant 2$ then any two vertices of $A_{k,r}$ are contained in a cycle of length $2k+1$, and*
(*d*) *if $A_{k,r} \xrightarrow{\text{hom}} H$ for some graph H with $|V(H)| < |V(A_{k,r})|$, then H contains an odd cycle of length at most $2k-1$.*

In particular, it follows from (a), $|V(A_{k,r})| = (2k-1)(r-1)+2$, (b), and (d) that $\delta_{\text{hom}}(\mathscr{C}_{2k-1}) \geqslant \frac{1}{2k-1}$. As r can be chosen arbitrarily big.

Proof. For given integers $k \geqslant 2$ and $r \geqslant 1$ set

$$n = |V(A_{k,r})| = (2k-1)(r-1)+2$$

and let $v_0, \dots, v_{n-1}$ be the vertices of $A_{k,r}$ in cyclic order, i.e., we assume $v_i \equiv i/n \in \mathbb{R}/\mathbb{Z}$ for every $i = 0, \dots, n-1$. By definition of $A_{k,r}$ the neighbourhood of v_0 is contained in the open interval $\left(\frac{k-1}{2k-1}, \frac{k}{2k-1}\right) \subseteq \mathbb{R}/\mathbb{Z}$. Consequently,

$$N(v_0) = \{v_i \colon i = (k-1)(r-1)+1, \dots, k(r-1)+1\} \tag{2.1}$$

and part (*a*) follows by symmetry.

For part (*b*) we observe that for any closed walk $u_1 \dots u_\ell u_1$ of length ℓ in $A_{k,r}$ we have $(u_\ell - u_1) + \sum_{i=1}^{\ell-1}(u_i - u_{i+1}) = 0$ and owing to the definition of $A_{k,r}$ each term of that sum lies in $\left(\frac{k-1}{2k-1}, \frac{k}{2k-1}\right) \subseteq \mathbb{R}/\mathbb{Z}$. However, for every integer $j = 2, \dots, k$ we have

$$(j-1) \leqslant (2j-1)\frac{k-1}{2k-1} < (2j-1)\frac{k}{2k-1} \leqslant j\,.$$

Consequently, $(u_\ell - u_1) + \sum_{i=1}^{\ell-1}(u_i - u_{i+1}) \in (j-1, j)$. Since $0 \notin (j-1, j)$, no walk in $A_{k,r}$ of length $2j-1$ for $j \leqslant k$ can be closed and part (*b*) follows.

For part (*c*) we show below that starting in $u_0 = v_0$ and always choosing the closest clockwise neighbour in $A_{k,r}$, i.e., setting

$$u_j \equiv u_{j-1} + \frac{(k-1)(r-1)+1}{n} \equiv j\frac{(k-1)(r-1)+1}{n} \in \mathbb{R}/\mathbb{Z}\,, \tag{2.2}$$

defines a Hamiltonian cycle $C = u_0 \dots u_{n-1}u_0$ with the property that

$$u_1\,, \quad u_{(2k-1)+1}\,, \quad u_{2(2k-1)+1}\,, \quad \dots\,, \quad u_{(r-1)(2k-1)+1} = u_{n-1}$$

are the r neighbours of $u_0 = v_0$ in $A_{k,r}$. In other words, every $(2k-1)$-th vertex on the subpath $u_1 \dots u_{n-1}$ of the Hamiltonian cycle C is a neighbour of u_0. Considering the C_{2k+1}'s created by the chords between u_0 and its neighbours $u_{(2k-1)+1}, \dots, u_{(r-2)(2k-1)+1}$ shows that $u_0 = v_0$ lies on a cycle of length $2k+1$ with every other vertex of $A_{k,\ell}$, which by symmetry verifies part *(c)*.

It is left to show that the C defined above, has the desired properties , i.e. is Hamiltonian with the stated distribution of $N(v_0)$. It follows from the definition of C that $u_{n-1}u_0$ and u_iu_{i+1} are edges of $A_{k,r}$ for every $i = 0, \dots, n-2$ and, hence, C is a closed walk of length n. However, since

$$n = (2k-1)(r-1) + 2 = 2\big((k-1)(r-1)+1\big) + (r-1)$$

and $(k-1)(r-1)+1$ are relatively prime, it follows that C is indeed a Hamiltonian cycle. Moreover, we observe for $s = 0, \dots, r-1$ that

$$\begin{aligned} u_{s(2k-1)+1} &\overset{(2.2)}{\equiv} (s(2k-1)+1)\frac{(k-1)(r-1)+1}{n} \\ &\equiv (s(2k-1)+1)\frac{(k-1)(r-1)+1}{(2k-1)(r-1)+2} \\ &\equiv \frac{(k-1)(r-1)+1+s}{(2k-1)(r-1)+2} + s(k-1) \\ &\equiv \frac{(k-1)(r-1)+1+s}{n} \equiv v_{(k-1)(r-1)+1+s} \overset{(2.1)}{\in} N(v_0)\,, \end{aligned}$$

which shows the stated distribution of $N(v_0)$ on C.

Finally, assertion *(d)* is a direct consequence of part *(c)*. Suppose $\varphi\colon A_{k,r} \to H$ is a graph homomorphism with $|V(H)| < n$. Then there are two vertices x, $y \in V(A_{k,r})$ such that $\varphi(x) = \varphi(y)$. In particular $xy \notin E(A_{k,r})$ and in view of *(c)* the vertex $\varphi(x) = \varphi(y)$ must be contained in a closed odd walk of length at most $2k-1$ in H and, consequently, H contains an odd cycle of length at most $2k-1$. □

§3. Dense graphs without odd cycles

In this section we collect a few observation on local properties of graphs with high minimum degree and without an odd cycle of given length.

The main result of this section is the proof of Proposition 3.5, which gives some structural information on such graphs by excluding long odd cycles and pairs of odd cycles connected by a path of length 4.

We remark that in the following lemmas and in Proposition 3.5 the additional εn term in the minimum degree condition could be replaced by some polynomial in k. However, since we do not strive for the optimal condition in these auxiliary results, we chose to state them with the same assumption as in Theorem 1.2. We also remark that by the *length* of a path or more generally the length of a walk, we refer to the number of edges, where each edge is counted with its multiplicity. In particular, we denote by P_r the path on $r+1$ vertices.

Lemma 3.1. *Let $k \geqslant 2$, $\varepsilon > 0$, and let $G = (V, E)$ be a C_{2k-1}-free graph satisfying $|V| = n \geqslant 4k/\varepsilon$ and $\delta(G) \geqslant \big(\frac{1}{2k-1} + \varepsilon\big)n$.*

(i) For every vertex $v \in V$ we have $d(M) := 2e(M)/|M| < 2k$ for all $M \subseteq N(v)$.

(ii) For every two vertices $v, u \in V$, i f there is an odd v-u-path of length at most $2k-3$ in G, then u and v have less than $5k^2$ common neighbours in G.

In the proof of Lemma 3.1 we will use the following consequence of the Erdős-Gallai theorem on paths [6], also stated in *(ii)*, as well as Theorem 1 of *(ii)*.

Theorem 3.2. (Erdős & Gallai 1959)

(i) Let G be an n-vertex graph. If $e(G) \geqslant \frac{1}{2}kn$, then G contains a path with k vertices.

(ii) Let $G = (A, B, E)$ be a bipartite graph with $|A| \geqslant |B| \geqslant k$. If $e(A, B) > (|A|+|B|)k$, then G contains an even path of length k.

Proof. Assertion *(i)* is a direct consequence of Theorem 3.2 *(i)*. Indeed , it implies that $d(M) \geqslant 2k$ yields a copy of P_{2k-3} in $M \subseteq N(v)$, which together with v would form a cycle C_{2k-1} in G.

For the proof of *(ii)* assume for a contradiction that $|N(v) \cap N(u)| \geqslant 5k^2$, and there is an odd v-u-path P of length at most $2k-3$. Let $A' = (N(v) \cap N(u)) \smallsetminus V(P)$, clearly, $|A'| \geqslant 4k^2$ so let $A \subseteq A'$ be a subset of A' with exaktly $4k^2$ vertices and $B = N(A) \smallsetminus (A \cup V(P))$. Since every vertex in A has at most $2k-2 < 2k$ neighbours in P we have

$$e(A, B) \geqslant |A| \cdot \delta(G) - 2e(A) - |A| \cdot 2k \overset{(i)}{>} |A|\left(\frac{1}{2k-1}n + \varepsilon n - 4k\right) \geqslant \frac{4k^2}{2k-1}n > 2k \cdot n\,.$$

Consequently, $|B| > 2k$ and Theorem 3.2 *(ii)* yields a P_{2k-2} in $G[A, B]$ and, hence, for every $\ell \in [k-2]$ there exists a $P_{2\ell}$ in $G[A, B]$ with end vertices in A. Together with the path P this yields a cycle C_{2k-1} in G, which is a contradiction to the assumption that G is C_{2k-1}-free. □

Lemma 3.1 yields the following corollary, which asserts that the first and the second neighbourhoods of a short odd cycle cover the "right" proportion of vertices.

Lemma 3.3. *Let $k \geqslant 2$, $\varepsilon > 0$, and let $G = (V, E)$ be a C_{2k-1}-free graph satisfying $|V| = n \geqslant 20k^3/\varepsilon$ and $\delta(G) \geqslant \big(\frac{1}{2k-1} + \varepsilon\big)n$. If $C = c_1 \dots c_\ell c_1$ is an odd cycle of length $\ell < 2k-1$ in G, then for every $i \in [\ell]$ there are subsets $M_i \subseteq N(c_i) \smallsetminus V(C)$, vertices $m_i \in M_i$, and subsets $L_i \subseteq N(m_i) \smallsetminus V(C)$ such that the sets $M_1, \dots, M_\ell, L_1, \dots, L_\ell$ are mutually disjoint and each of those sets contains at least $\frac{1}{2k-1}n$ vertices.*

Proof. Let $C = c_1 \dots c_\ell c_1$ be an odd cycle of length l in $G = (V, E)$, where $l < 2k - 1$. Since there is a path of odd length at most $\ell - 2 < 2k - 3$ between any two vertices of C, Lemma 3.1 *(ii)* tells us , that $|N(c_i) \cap N(c_j)| < 5k^2$ for all distinct $i, j \in [\ell]$. Consequently, we may discard up to at most $(\ell-1) \cdot 5k^2 + \ell < 10k^3$ vertices from the neighbourhoods $N(c_i)$ and obtain mutually disjoint sets $M_i \subseteq N(c_i) \smallsetminus V(C)$ of size at least

$$\delta(G) - 10k^3 \geqslant \frac{1}{2k-1}n + \varepsilon n - 10k^3 > \frac{1}{2k-1}n\,.$$

For every $i \in [\ell]$ fix an arbitrary vertex $m_i \in M_i$. Since there is a path of odd length at most $\ell - 2 < 2k - 3$ between any two vertices of C, there is a path of odd length at most $(\ell - 2) + 2 = \ell \leqslant 2k - 3$ between any two vertices m_i and m_j. Again we infer from Lemma 3.1 *(ii)* that $|N(m_i) \cap N(m_j)| < 5k^2$ for all distinct $i, j \in [\ell]$ and in the same way as before, we obtain mutually disjoint sets $L'_i \subseteq N(m_i) \smallsetminus V(C)$ of size at least $\delta(G) - 10k^3$.

Furthermore, since there also is a path of even length at most $\ell - 1 < 2k - 3$ between any two (not necessarily distinct) vertices of C, there is a path of odd length at most $(\ell - 1) + 1 = \ell \leqslant 2k - 3$ between any pair of vertices c_i and m_j. Again Lemma 3.1 *(ii)* implies that $|N(c_i) \cap N(m_j)| < 5k^2$ for all $i, j \in [\ell]$ and discarding at most $\ell \cdot 5k^2 < 10k^3$ vertices from each L'_i yields sets $L_i \subseteq N(m_i)$ such that $M_1, \dots, M_\ell, L_1, \dots, L_\ell$ are mutually disjoint and disjoint from $V(C)$. Moreover, the assumption $n \geqslant 20k^3/\varepsilon$ implies

$$|L_i| \geqslant |L'_i| - 10k^3 \geqslant \delta(G) - 20k^3 \geqslant \frac{1}{2k-1}n + \varepsilon n - 20k^3 \geqslant \frac{1}{2k-1}n\,,$$

which concludes the proof of the lemma. □

In the proof of part *(i)* of Theorem 1.2 it will be useful to exclude the graphs described in Definition 3.4 as subgraphs of a C_{2k-1}-free graph of sufficiently high minimum degree.

Definition 3.4. We denote by D_ℓ the graph on $2\ell + 3$ vertices that consist of two disjoint cycles of length ℓ and a path of length 4 joining these two cycles, which is internally disjoint to both cycles.

The following proposition excludes the appearance of some short odd cycles and D_ℓ's in the graphs G considered in Theorem 1.2.

Proposition 3.5. *Let $k \geqslant 2$, $\varepsilon > 0$, and $G = (V, E)$ be a C_{2k-1}-free graph satisfying $|V| = n \geqslant 20k^3/\varepsilon$ and $\delta(G) \geqslant \big(\frac{1}{2k-1} + \varepsilon\big)n$. Then*

(i) G is C_ℓ-free for every odd ℓ with $k \leqslant \ell \leqslant 2k - 1$.

(ii) G is D_ℓ-free for every odd ℓ with $\max\{3, 2k - 7\} \leqslant \ell \leqslant 2k - 1$.

Proof. Assertion *(i)* is a direct consequence of Lemma 3.3, as the mutually disjoint sets $M_1, \dots, M_\ell, L_1, \dots, L_\ell$ would not fit into $V(G)$.

For the proof of assertion *(ii)* we assume for a contradiction that $G = (V, E)$ contains a subgraph D_ℓ for some odd ℓ with $\max\{3, 2k - 7\} \leqslant \ell \leqslant 2k - 1$. Since the graph D_ℓ contain a cycle of length ℓ, we immediately infer from part *(i)*, that we may assume $\ell < k$. Consequently, $k > \ell \geqslant 2k - 7$ implies $k \leqslant 6$ and owing to $k > \ell \geqslant \max\{3, 2k - 7\}$ we see that the only remaining cases we have to consider are $(k, \ell) \in \{(4, 3), (5, 3), (6, 5)\}$. We discuss each of the cases below.

Case $k = 6$ and $\ell = 5$. Let $C = c_1 \dots c_5 c_1$ and $C' = c'_1 \dots c'_5 c'_1$ be the two cycles of length 5 appearing in $D_5 \subseteq G$ and suppose the path P of length 4 connects c_1 and c'_1. We observe that c'_5 is connected to every vertex of C by an odd path of length at most 9, as seen in Figure 3.1. In fact, $Q = c'_5 P$ connects c'_5 and c_1 by a path of length 5 and every other vertex of C can be reached by an even path of length at most 4 from c_1.

Furthermore, c'_5 is connected to every vertex in $N(C)$ by an odd path of length at most 9. For the vertices in $N(C) \smallsetminus N(c_1)$ we again follow the path Q and since c_2, c_3, c_4, and c_5 can be reached by an odd path of length at most 3 from c_1, as seen in Figure 3.1, every vertex in $N(C) \smallsetminus N(c_1)$ can be reached by an odd path of length at most $5 + 3 + 1 = 9$. For the vertices in $N(c_1)$ we utilise the path of length 4 from c'_5 to c'_1 in C'. Continuing then along P to c_1 shows that there are paths of length 9 connecting c'_5 with every vertex in $N(c_1)$.

As $9 = 2k - 3$, we infer from Lemma 3.1 *(ii)* that c'_5 has at most $10 \cdot (5k^2 + |Q|) < 10k^3$ neighbours in the sets $M_1, \dots, M_5, L_1, \dots, L_5$ given by Lemma 3.3 applied to C. However, since

$$\left|M_1 \cup \dots \cup M_5 \cup L_1 \cup \dots \cup L_5\right| \geqslant \frac{10}{11} n$$

this implies $\deg(c'_5) \leqslant \frac{n}{11} + 10k^3 < \frac{n}{11} + \varepsilon n$ by the assumption that $n > 20k^3/\varepsilon$, which contradicts the minimum degree assumption on G in this case.

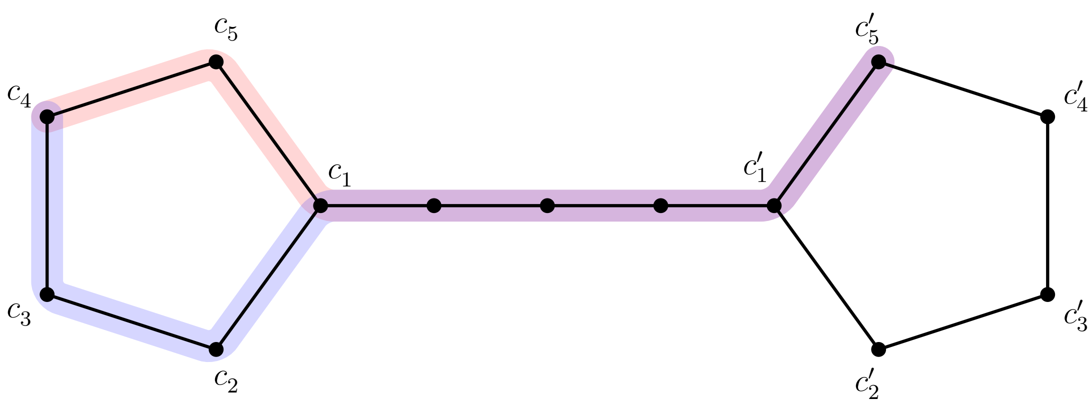

FIGURE 3.1. An odd path of length 7 from c'_5 to c_4 in red and an even path of length 8 from c'_5 to c_4 in blue as used in the proof of case $k = 6$ and $\ell = 5$.

Case $k = 5$ and $\ell = 3$. Let $C = c_1c_2c_3c_1$ and $C' = c'_1c'_2c'_3c'_1$ be the two triangles of $D_3 \subseteq G$ and suppose the path of length 4 connects c_1 and c'_1. Moreover, Lemma 3.3 applied with C yields vertices m_1, m_2, m_3 and vertex sets M_1, M_2, M_3 and L_1, L_2, L_3. It is easy to check that c'_2 and c'_3 can reach each c_i and m_i for every $i \in [3]$ by an odd path of length at most $7 = 2k - 3$, as seen in Figure 3.2 on the left . In view of Lemma 3.1 *(ii)*, and since $|N(c'_2)|, |N(c'_3)| \geqslant \delta(G) \geqslant n/9$ it follows that

$$\left|M_1 \cup M_2 \cup M_3 \cup L_1 \cup L_2 \cup L_3 \cup N(c'_2) \cup N(c'_3)\right| \geqslant \frac{8}{9}n\,.$$

Consequently, we infer from $|N(c'_1)| \geqslant \delta(G) \geqslant n/9 + \varepsilon n > n/9 + 40k^2$ that the vertex c'_1 must have at least $5k^2$ common neighbours with one of the eight vertices c_1, c_2, c_3, m_1, m_2, m_3, c'_2, c'_3. Since c'_1 can be connected by an odd path of length at most 7 to all of these eight vertices but c_1, we infer that c_1 and c'_1 have $5k^2$ common neighbours and we can fix such a neighbour disjoint from m_1, m_2, m_3, C and C'. In other words, we found a graph D'_3 consisting of C, C', and a path of length 2 between c_1 and c'_1. Consequently, c'_2 and c'_3 are connected to each c_i and each m_i for every $i \in [3]$ by an odd path of length at most 5. Hence, we can fix a neighbour m'_2 of c'_2, which can be connected to each c_i and each m_i for $i \in [3]$ and to c'_2 and c'_3 by an odd path of length at most 7, as seen in Figure 3.2 on the right. In other words, any two of the 9 vertices from $c_1, c_2, c_3, m_1, m_2, m_3, c'_2, c'_3$ and m'_2 are connected by an odd path of length at most 7 and thus have fewer than $5k^2$ common neighbours by Lemma 3.1 *(ii)*. However, since $\varepsilon n > 40k^2$ the minimum degree assumption implies that at least one of the former 8 vertices must have at least $5k^2$ common neighbours with m'_2.

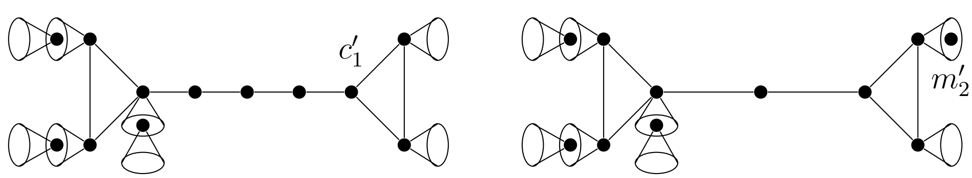

FIGURE 3.2. On the left the graph D_3 of the case $k = 5$ and $\ell = 3$ where the vertex c'_1 does not have enough neighbours, and on the right the graph D'_3 where the vertex m'_2 does not have enough neighbours.

Case $k = 4$ and $\ell = 3$. Again we consider the two triangles $C = c_1c_2c_3c_1$ and $C' = c'_1c'_2c'_3c'_1$ of $D_3 \subseteq G$ and assume c_1 and c'_1 are connected by a path $c_1p_1p_2p_3c'_1$ of length 4. We consider the vertices m_1, m_2, m_3 and sets M_1, M_2, M_3 , L_1, L_2, L_3 and M'_1, M'_2, M'_3 given by Lemma 3.3 applied with C and with C'.

Note that there can only be one edge between a vertex of C and a vertex of C', namely $c_1c'_1$, otherwise there is a C_7 in D_3. Therefore, if there are vertices c_i and c'_j

with $i, j \in [3]$ such that they have at least two common neighbours, G contains a graph D'_3 consisting of C, C' and a path of length 2 between c_i and c'_j. By symmetry, we may assume $i = j = 1$. However, in this case we see that c'_2 is connected to c_1, c_2, c_3 and m_1, m_2, m_3 by an odd path of length at most 5, as seen in Figure 3.3 on the right . Since

$$\left|M_1 \cup M_2 \cup M_3 \cup L_1 \cup L_2 \cup L_3\right| \geqslant \frac{6}{7}n\,,$$

the minimum degree assumption yields at least $(\varepsilon n - 4)/6 \geqslant 5k^2$ common neighbours of c'_2 and one of the vertices of $\{c_1, c_2, c_3, m_1, m_2, m_3\}$, which is a contradiction to Lemma 3.1 *(ii)*.

Assuming that no two vertices of C and C' have more than one common neighbour, we notice that p_1 can be connected to all three vertices of C and to all three vertices of C' by an odd path of length at most $5 = 2k - 3$, as seen in Figure 3.3 on the left. Which implies that

$$\left|M_1 \cup M_2 \cup M_3 \cup M'_1 \cup M'_2 \cup M'_3\right| \geqslant \frac{6}{7}n - 9\,.$$

Consequently, the minimum degree assumption yields at least $(\varepsilon n - 9 - 9)/6 \geqslant 5k^2$ common neighbours of p_1 and one of the vertices of C_1 or C'_1, which is a contradiction to Lemma 3.1 *(ii)*. □

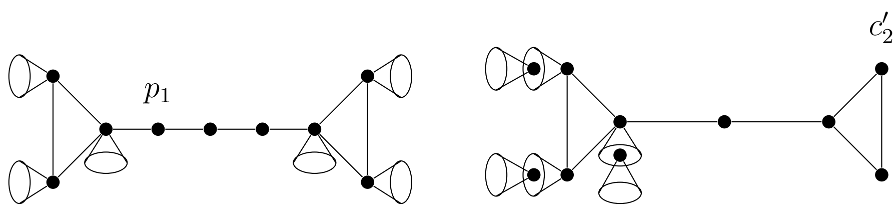

FIGURE 3.3. On the left the graph D_3 of the case $k = 4$ and $\ell = 3$ where the vertex p_1 does not have enough neighbours, and on the right the graph D'_3 where the vertex c'_2 does not have enough neighbours.

§4. UPPER BOUNDS FOR THEOREM 1.2

Proof of Theorem 1.2. We first prove assertion *(i)* of Theorem 1.2. Given a sufficiently large C_{2k-1}-free n-vertex graph $G = (V, E)$ with $\delta(G) \geqslant (\frac{1}{2k-1} + \varepsilon)n$ for $k \geqslant 3$ and $\varepsilon > 0$, it suffices to show that there exists a C_{2k-1}-free graph H with $|V(H)| \leqslant K = K(k, \varepsilon)$ and $G \xrightarrow{\text{hom}} H$. The required graph $H(C_{2k-1}, \alpha)$ for Definition 1.1 can then be taken to be the disjoint union of all non-isomorphic C_{2k-1}-free graphs on K vertices.

In particular, the constant K must be independent of n. Without loss of generality we may assume that $2/\varepsilon$ is an integer. In order to define K, consider the function $f\colon \mathbb{R} \to \mathbb{R}$

with $x \mapsto x2^x$ and set

$$m = \max\left\{\left\lceil\frac{2\ln(3/\varepsilon)}{\varepsilon^2}\right\rceil, 8k^2\right\} \qquad \text{and} \qquad K = \underbrace{f \circ f \circ \cdots \circ f}_{2k\text{-times}}\left((2/\varepsilon+1)^{\binom{m}{4k}}\right), \tag{4.1}$$

i.e., K is given by a $2(k+1)$-times iterated exponential function in $\mathrm{poly}(1/\varepsilon, k)$.

Considering a random m-element subsets $X \subseteq V$, it follows from the concentration of the hypergeometric distribution (see e.g. [11, inequality (2.6) and Theorem 2.10]) for any fixed vertex $v \in V$

$$\mathbb{P}\Big(|N(v) \cap X| \leqslant (\tfrac{1}{2k-1} + \varepsilon)m - t\Big) \leqslant \exp\Big(-\tfrac{t^2}{2m}\Big),$$

for every $t > 0$. Since our choice of m in (4.1) yields $m/2k > 4k$ it follows with $t = \varepsilon m$, that there exists a set X of size m, such that all but at most $\varepsilon n/3$ vertices of G have at least $4k$ neighbours in X. We fix such a set $X = \{x_1, \ldots, x_m\}$ and set

$$Y = \big\{v \in V : |N(v) \cap X| \geqslant 4k\big\}.$$

For every $y \in Y$ fix a set $X(y)$ of exactly $4k$ neighbours of y in X in an arbitrary way. We partition Y into $\binom{m}{4k}$ sets, where two vertices y, $y' \in Y$ belong to the same partition class if $X(y) = X(y')$. Removing all the classes with fewer than $8k/\varepsilon$ vertices from this partition yields a partition $\mathcal{Q}$ of a subset of Y of size

$$\Big|\bigcup \mathcal{Q}\Big| \geqslant |Y| - \binom{m}{4k}\frac{8k}{\varepsilon} \geqslant \Big(n - \frac{\varepsilon}{3}n\Big) - \binom{m}{4k}\frac{8k}{\varepsilon} > n - \frac{\varepsilon}{2}n, \tag{4.2}$$

where the last inequality holds for sufficiently large n. For convenience we may index the partition classes of $\mathcal{Q}$ by a suitable set $I = [M]$ with $M \leqslant \binom{m}{4k}$, i.e., $\mathcal{Q} = (Q_i)_{i \in I}$.

Next we define a partition $\mathcal{R}$ of the whole vertex set V, based on the neighbourhoods with respect to the partition classes of $\mathcal{Q}$. More precisely we assign to each vertex $v \in V$ a vector $\mu(v) = (\mu_i(v))_{i \in I}$, where $\mu_i(v)$ equals the proportion of vertices in Q_i that are neighbours of v “rounded down” to the next integer multiple of $\varepsilon/2$, i.e.

$$\mu_i(v) = \left\lfloor \frac{|N(v) \cap Q_i|}{|Q_i|} \cdot \frac{2}{\varepsilon}\right\rfloor \cdot \frac{\varepsilon}{2}. \tag{4.3}$$

In particular, since every class from $\mathcal{Q}$ has at least $8k/\varepsilon$ vertices, we have

$$\big|N(v) \cap Q_i\big| \geqslant 4k \tag{4.4}$$

for every $v \in V$ with $\mu_i(v) > 0$.

We now define the partition $\mathcal{R}$. The classes of $\mathcal{R}$ are given by the equivalence classes of the relation $\mu_i(v) = \mu_i(v')$ for every $i \in I$. Owing to the discretisation of $\mu_i(v)$ the partition $\mathcal{R}$ has at most

$$(2/\varepsilon + 1)^{|I|} \leqslant (2/\varepsilon + 1)^{\binom{m}{4k}}$$

parts. Furthermore, we note

$$\begin{aligned}\sum_{i\in I}\mu_i(v)|Q_i| &\geqslant d(v) - \Big|V \smallsetminus \bigcup \mathcal{Q}\Big| - \sum_{i\in I}\frac{\varepsilon}{2}|Q_i| \\ &\overset{(4.2)}{>} \left(\frac{1}{2k-1}+\varepsilon\right)n - \frac{\varepsilon}{2}n - \frac{\varepsilon}{2}n \\ &\geqslant \left(\frac{1}{2k-1}\right)n \end{aligned} \tag{4.5}$$

for every $v \in V$. For later reference we make the following observation.

Claim 4.1. *For every $i \in I$ no two distinct vertices $v, v' \in V$ with $\mu_i(v), \mu_i(v') > 0$ are joined by an odd v-v'-path of length at most $2k-5$ in G.*

Proof. Suppose for a contradiction, that for some $i \in I$ and $v \neq v'$ we have $\mu_i(v), \mu_i(v') > 0$ and there is an odd v-v'-path P of length at most $2k-5$ in G. Let q_i be a neighbour of v in Q_i and let q_i' be a neighbour of v' in Q_i, such that $q_i \neq q_i'$ and both not contained in P (see (4.4)). Consequently, there is a q_i-q_i'-path $P' \subseteq G$ of odd length $2k-1-2\ell$ for some $\ell \in [k-2]$.

Since all vertices of Q_i have $4k$ common neighbours in X, there is a set X' consisting of ℓ of these neighbours from $X \smallsetminus V(P')$. Similarly, there is a set $Q_i' \subseteq Q_i$ of $\ell-1$ vertices in $Q_i \smallsetminus (V(P') \cup X')$. Clearly, $X' \cup Q_i' \cup \{q_i, q_i'\}$ spans a q_i-q_i'-path P'' of length 2ℓ, which together with P' yields a copy of C_{2k-1} in G. This, however, contradicts the assumption that G is C_{2k-1}-free. □

Starting with the partition $\mathcal{R}^0 = \mathcal{R}$ we inductively refine this partition $2k$ times and obtain partitions $\mathcal{R}^0 \succcurlyeq \mathcal{R}^1 \succcurlyeq \dots \succcurlyeq \mathcal{R}^{2k}$. Given $\mathcal{R}^i$ we define $\mathcal{R}^{i+1}$ by subdividing every partition class such that vertices remain in the same class if and only if they have neighbours in the same classes of R^i. More precisely, two vertices v, v' from some partition class of $\mathcal{R}^i$ stay in the same class in $\mathcal{R}^{i+1}$ if and only if for every class R^i_j from $\mathcal{R}^i$ we have

$$N(v) \cap R^i_j \neq \varnothing \quad \Longleftrightarrow \quad N(v') \cap R^i_j \neq \varnothing\,.$$

Owing to this inductive process and our choice of K in (4.1) the partition $\mathcal{R}^{2k}$ consists of at most K classes. Since $k \geqslant 3$, claim 4.1 implies that the classes of $\mathcal{R}^0$ are independent sets in G and, therefore, also the classes of $\mathcal{R}^{2k}$ are independent. Hence, we may define the reduced graph H of $\mathcal{R}^{2k}$, where each class $\mathcal{R}^{2k}$ is a vertex of H and two vertices are adjecent, if the corresponding partition classes induce at least one crossing edge in G. Obviously, we have

$$G \xrightarrow{\text{hom}} H \qquad \text{and} \qquad |V(H)| \leqslant K \tag{4.6}$$

and it is left to show that H is also C_{2k-1}-free (see Claim 4.4). For the proof of this property we first collect a few observations concerning the interplay of odd paths in H and walks in G (see Claims 4.2 and 4.3).

Denote by $\mathcal{R}^i(v)$ the unique class of the partition $\mathcal{R}^i$ which contains the vertex $v \in V$. Similarly, for $j \geqslant i$ let $\mathcal{R}^i(R)$ be the unique class of the partition $\mathcal{R}^i$ which is a superset of $R \in \mathcal{R}^j$.

Claim 4.2. *If there is a walk $W_H = h_1h_2 \dots h_s$ in H for some integer $s \leqslant 2k$, then there are vertices $w_i \in \mathcal{R}^{2k-i+1}(h_i) \subseteq \mathcal{R}^0(h_i)$ for every $i \in [s]$ such that $W = w_1w_2 \dots w_s$ is a walk in G. Moreover, w_1 can be chosen arbitrarily in $h_1 = \mathcal{R}^{2k}(h_1)$.*

Proof. We shall locate the walk W in an inductive manner and note that for $s = 1$ it is trivial.

For $s \geqslant 2$ let a walk $W' = w_1w_2 \dots w_{s-1}$ satisfying $w_i \in \mathcal{R}^{2k-i+1}(h_i)$ for every $i \in [s-1]$ be given. The walk W_H in H guarantees an edge between $\mathcal{R}^{2k}(h_{s-1})$ and $\mathcal{R}^{2k}(h_s)$ and, hence, there is an edge between $\mathcal{R}^{2k-(s-1)+1}(h_{s-1})$ and $\mathcal{R}^{2k-(s-1)+1}(h_s)$. Consequently, the construction of the refinements shows that $w_{s-1} \in \mathcal{R}^{2k-(s-1)+1}(h_{s-1})$ must have a neighbour $w_s \in \mathcal{R}^{2k-s+1}(h_s)$ and the walk $W = W'w_s = w_1 \dots w_{s-1}w_s$ has the desired properties. □

Even if we assume in Claim 4.2 that W_H is a path in H and, in particular, $h_i \neq h_j$ for all distinct $i, j \in [s]$, it may happen that $\mathcal{R}^0(h_i) = \mathcal{R}^0(h_j)$ and, hence, we cannot guarantee $w_i \neq w_j$. In other words, even if we apply Claim 4.2 to a path in H, the promised walk W might not be a path. However, combined with Proposition 3.5 we can get the following improvement.

Claim 4.3. *If there is an odd path $P_H = h_1 \dots h_{s+1}$ of length $s \leqslant 2k-1$ in H, then there are vertices $v_1 \in \mathcal{R}^0(h_1)$ and $v_{s+1} \in \mathcal{R}^0(h_{s+1})$ such that there is an odd path of length at most s between them.*

Proof. Consider a walk $W = w_1w_2 \dots w_{s+1}$ in G with $w_i \in \mathcal{R}^0(h_i)$ given by Claim 4.2. If this walk does not contain an odd w_1-w_{s+1}-path already, then W must contain an odd cycle. Below we shall show that this leads to a contradiction and, hence, W contains an odd w_1-w_{s+1}-path.

Considering an odd cycle $C = c_1 \dots c_\ell c_1$ contained in $W \subseteq G$, such that

$$c_1 = w_{i_1},\ c_2 = w_{i_2},\ \dots,\ c_\ell = w_{i_\ell}, \qquad \text{and} \qquad c_1 = w_{i_{\ell+1}} = w_{i_1}$$

for some set of indices satisfying $1 \leqslant i_1 < i_2 < \dots < i_\ell < i_{\ell+1} \leqslant s+1$. To find such a path, consider the walk W, delete an even w_1-w_{s+1}-path. Now the remaining edges of W are a family of closed walks, take an odd closed walk $W' = w'_1w'_2 \dots w'_{s'+1} = w_{j_1}w_{j_2} \dots w_{j_{s'+1}}$, where $w'_1 = w'_{s'+1}$. If W' is not a cycle, there is a smaller closed walk $W'' \subset W'$, such that the edges of W' without the edges of W'' are also a closed walk, both retaining the order of the vertices from W' and therefore from W. One of $\{W', W''\}$ needs to be odd. Iterating this process eventually gives rise to the odd cycle C.

In view of Proposition 3.5 *(i)* we must have $3 \leqslant \ell < k$. Consequently, $\ell \leqslant 2k-5$ and since ℓ is odd, it follows from Claim 4.1 that there is no path of length ℓ between any two vertices from $\mathcal{R}^0(c_1) = \mathcal{R}^0(h_{i_1}) = \mathcal{R}^0(h_{i_{\ell+1}})$. Moreover, Claim 4.1 tells us that the ℓ classes $\mathcal{R}^0(c_1) = \mathcal{R}^0(h_{i_1}) = \mathcal{R}^0(h_{i_{\ell+1}}), \dots, \mathcal{R}^0(c_\ell) = \mathcal{R}^0(h_{i_\ell})$ from $\mathcal{R}^0$ are distinct, since otherwise the cycle C would contain an odd path of length at most $2k-7$ between two vertices of some class in $\mathcal{R}^0$.

Since P_H is a path in H, we have $h_{i_1} \neq h_{i_{\ell+1}}$ and the cycle C avoids at least one of the sets h_{i_1} or $h_{i_{\ell+1}}$. Without loss of generality we may assume C avoids h_{i_1} and we fix an arbitrary vertex $c'_1 \in h_{i_1}$.

We are going to locate a second cycle of length ℓ in G that starts and ends in c'_1. By construction this cycle is going to visit the same partition classes of $\mathcal{R}^0$ as C. For that we shall repeat the argument from Claim 4.2 starting with $h_{i_1} \dots h_{i_\ell} h_{i_{\ell+1}}$ even though this is not necessarily a subpath of P_H. However, since $h_{i_1} \dots h_{i_\ell} h_{i_{\ell+1}}$ appear in that order in P_H, we can repeat the reasoning of Claim 4.2 starting with the vertex $c'_1 \in h_{i_1}$. Continuing in an inductive manner, for $j \in [\ell]$ we have to consider the two cases $i_{j+1} = i_j + 1$ and $i_{j+1} > i_j + 1$.

In the first case, we can indeed proceed as in the proof of Claim 4.2, since this means that $h_{i_j} h_{i_{j+1}}$ is an edge of P_H. The second case , by construction of C, only occurs, when $w_{i_j} = w_{i_{j+1}-1}$ and

$$\mathcal{R}^{2k-i_j+1}(h_{i_j}) \subseteq \mathcal{R}^{2k-(i_{j+1}-1)+1}(h_{i_{j+1}-1}) \,.$$

Owing to the fact that $w_{i_{j+1}-1} w_{i_{j+1}}$ is an edge of W and that $w_{i_{j+1}-1} \in \mathcal{R}^{2k-i_j+1}(h_{i_j})$ and $w_{i_{j+1}-1} \in \mathcal{R}^{2k-(i_{j+1}-1)+1}(h_{i_{j+1}-1})$, we infer from the construction of the refinements that $w_{i_j} = w_{i_{j+1}-1}$ also has a neighbour in $\mathcal{R}^{2k-i_{j+1}+1}(h_{i_{j+1}})$, which concludes the induction step.

Therefore, we obtain another walk $C' = c'_1 \dots c'_\ell c'_{\ell+1}$ where $c'_j \in \mathcal{R}^0(h_{i_j}) = R^0(c_j)$. Recalling that the ℓ classes $\mathcal{R}^0(h_{i_1}), \dots, \mathcal{R}^0(h_{i_\ell})$ are pairwise distinct, this implies that C' is either a path or a cycle of odd length $\ell \leqslant 2k-5$. Moreover, since $\mathcal{R}^0(h_{i_1}) = \mathcal{R}^0(h_{i_{\ell+1}})$ we infer from Claim 4.1 that C' cannot be a path and, hence, it must be an odd cycle of length $\ell \leqslant 2k-5$. By construction c'_1 avoids C, and hence C' and C are disjoint, as otherwise we would have an odd path of length ℓ connecting c_1 and c'_1 in $\mathcal{R}^0(c_1)$, which would contradict Claim 4.1 again.

Consequently, C and C' form a copy of D_ℓ since c_1 and c_1' are connected by a path of length 4 whose three internal vertices avoid C and C' (and the middle vertex is from X). Owing to Proposition 3.5 *(ii)* we have $\ell \leqslant 2k-9$, but in D_ℓ there exists an odd path of length $\ell+4 \leqslant 2k-5$ between c_i and c_i' for every $i = 2, \ldots, \ell$, which again contradicts Claim 4.1. □

After these preparations we are now ready to conclude the proof of part *(i)* of Theorem 1.2.

Claim 4.4. *The graph H is C_{2k-1}-free.*

Proof. Assume for a contradiction that there is a cycle $C_H = h_1 \ldots h_{2k-1}h_1$ of length $2k-1$ in H. We recall that the vertices of H are partition classes of $\mathcal{R}^{2k}$ and for a simpler notation we set for any vertex h_x of C_H

$$\mu_i(h_x) := \mu_i(v)\,,$$

where v is an arbitrary vertex from $\mathcal{R}^0(h_x)$ and the definition of $\mathcal{R} = \mathcal{R}^0$ shows that the definition of $\mu_i(h_x)$ is indeed independent of the choice of $v \in \mathcal{R}^0(h_x)$.

By (4.5) we have

$$\sum_{x=1}^{2k-1}\sum_{i\in I}\mu_i(h_x)|Q_i| > n \geqslant \sum_{i\in I}|Q_i|$$

and, hence, there is some $i \in I$ such that

$$\sum_{x=1}^{2k-1}\mu_i(h_x) > 1\,. \tag{4.7}$$

In particular, there are at least two distinct vertices h_x and h_y of C_H such that $\mu_i(h_x) > 0$ and $\mu_i(h_y) > 0$. On the other hand, among three vertices of C_H two are connected by an odd path of length at most $2k-5$ in C_H, since the negation is only true for vertices with distance 2 on C_H. Therefore it follows from Claim 4.3 and Claim 4.1, that no other vertex h_z with $z \in [2k-1] \setminus \{x, y\}$ satisfies $\mu_i(h_z) > 0$. Consequently, we have $\mu_i(h_x) + \mu_i(h_y) > 1$, which means that any two vertices $v \in \mathcal{R}^0(h_x)$ and $u \in \mathcal{R}^0(h_y)$ have a common neighbour in Q_i. In fact, since $2/\varepsilon$ is assumed to be an integer, v and u have at least $2|Q_i|/\varepsilon > 4k$ joint neighbours. Moreover, again Claim 4.3 and Claim 4.1 imply that h_x and h_y are connected by a path of length $2k-3$ in C_H and that there is a path P of length $2k-3$ in G connecting some $v \in \mathcal{R}^0(h_x)$ and $u \in \mathcal{R}^0(h_y)$. Using one of the joint neighbours in Q_i outside P yields a copy of C_{2k-1} in G. This contradicts the C_{2k-1}-freeness of G and concludes the proof of Claim 4.4. □

Claim 4.4 together with (4.6) establishes the proof of part *(i)* of Theorem 1.2 and it remains to consider part *(ii)*, when G is assumed to be $\mathscr{C}_{2k-1}$-free.

In view of Proposition 2.2 it suffices to verify the upper bound of assertion *(ii)* of Theorem 1.2. Compared to the proof of part *(i)* of Theorem 1.2, we have the additional assumption that G is not only C_{2k-1}-free, but also contains no cycle C_ℓ for any odd $\ell < 2k-1$. Consequently, the graph H defined in the paragraph before (4.6) in the proof of part *(i)* satisfies (4.6) in this case as well and owing to Claim 4.4 it is C_{2k-1}-free. Hence, we only have to show that the C_ℓ-freeness of G for every odd $\ell \leqslant 2k-3$ can be carried over to H in this situation, which is rendered by the following claim.

Claim 4.5. *If G is $\mathscr{C}_{2k-1}$-free, then H is also $\mathscr{C}_{2k-1}$-free.*

Proof. Recall, that we assume $k \geqslant 3$. Suppose for a contradiction that H contains a cycle $C_H = h_1 \dots h_\ell h_1$ for some odd integer ℓ with $3 \leqslant \ell \leqslant 2k-1$. In fact, it follows from Claim 4.4 that $\ell \leqslant 2k-3$. Moreover, applying Claim 4.2 to C_H yields a walk W of length ℓ in G which starts and ends in $\mathcal{R}^0(h_1)$. Since G contains no odd cycle of length at most ℓ, the walk W contains an odd path of length at most ℓ connecting two vertices in $\mathcal{R}^0(h_1)$. Therefore, Claim 4.1 implies that $\ell = 2k-3$ and by symmetry we infer that for every $x \in [2k-3]$ there exists an odd path of length $2k-3$ between two vertices v_x, $u_x \in \mathcal{R}^0(h_x)$.

As in the proof of Claim 4.4 we infer from (4.5) that

$$\sum_{x=1}^{2k-3} \sum_{i \in I} \mu_i(h_x)|Q_i| > \frac{2k-3}{2k-1} n > \frac{1}{2} \sum_{i \in I} |Q_i| \,,$$

where we used $k \geqslant 3$ for the last inequality. Consequently, there is some index $i \in I$ such that $\sum_{x=1}^{2k-3} \mu_i(h_x) > 1/2$. Since for every distinct x, $y \in [2k-3]$ there exists an odd path of length at most $2k-5$ connecting a vertex from $\mathcal{R}^0(h_x)$ with a vertex from $\mathcal{R}^0(h_y)$ there is only one vertex of C_H such that $\mu_i(h_x) > 0$ and, hence, for that $x \in [2k-3]$ we have $\mu_i(h_x) > 1/2$. In particular, every two distinct vertices v, $u \in \mathcal{R}^0(h_x)$ have a common neighbour in Q_i and, since $2/\varepsilon$ is assumed to be an integer, v and u have at least $2|Q_i|/\varepsilon > 4k$ joint neighbours. Applying this observation to v_x and u_x leads to an odd cycle of length $2k-1$ in G, which is a contradiction and concludes the proof of Claim 4.5. □

This concludes the proof of Theorem 1.2. □

§5. Odd Tetrahedra

Letzter and Snyder [12] obtained a stronger version of Theorem 1.2 *(ii)* for $k = 3$, by showing that the homomorphic images can be chosen from the family of generalised Andrásfai graphs (see Definition 2.1). More precisely, it was shown, that $G \xrightarrow{\text{hom}} A_{3,r}$ for every $G \in \mathscr{G}_{\mathscr{C}_5}(\alpha)$ as long as $\alpha > \frac{r+1}{5r+2}$. However, it turns out that such an explicit form of the theorem does not extend to other values of $k \geqslant 2$. For $k = 2$ this was observed by Häggkvist [10], who showed that there exist appropriate (unbalanced) blow-ups of the Grötzsch graph in $\mathscr{G}_{C_3}(10/29)$ which are 4-chromatic, while $\chi(A_{2,r}) \leqslant 3$ for every $r \geqslant 1$.

In Proposition 5.5 below we provide a counterexample for a stronger version of Theorem 1.2 *(ii)* (like the one obtained in [12]) for every $k > 3$ by exhibiting graphs in $\mathscr{G}_{\mathscr{C}_{2k-1}}(\frac{1}{2k-1} + \varepsilon)$ for some $\varepsilon > 0$ that are not homomorphic to any generalised Andrásfai graph from $\mathscr{A}_k$ (see Definition 2.1).

Definition 5.1 ((2k + 1)-tetrahedra)**.** Given $k \geqslant 2$ we denote by $\mathscr{T}_k$ the set of graphs T consisting of

(i) one cycle C_T with three branch vertices a_T, b_T, and $c_T \in V(C_T)$,
(ii) a center vertex z_T, and
(iii) internally vertex disjoint paths (called spokes) P_{az}, P_{bz}, P_{cz} connecting the branch vertices with the center vertex.

Furthermore, we require that each cycle in T containing z_T and exactly two of the branch vertices must have length $2k + 1$, and the spokes have length at least 2.

Lemma 5.2. *For all integers $k \geqslant 2$ and $r \geqslant 1$ there is no $(2k + 1)$-tetrahedra $T \in \mathscr{T}_k$ that is homomorphic to the Andrásfai graph $A_{k,r}$.*

Proof. Let $T \in \mathscr{T}_k$ be given and let the three spokes consist of $\ell_a, \ell_b, \ell_c \geqslant 2$ edges, respectively. Suppose for a contradiction that $T \xrightarrow{\text{hom}} A_{k,r}$ and let φ be such a homomorphism. Since T contains an odd cycle we have $r \geqslant 2$ and let $C_A = u_0 \dots u_{(2k-1)(r-1)+1} u_0$ be the Hamiltonian cycle of $A_{k,r}$ such that $N(u_0) = \{u_{i(2k-1)+1} \colon i = 0, \dots, r-1\}$ (c.f. proof of Proposition 2.2 *(c)*).

Claim 5.3. *Let v, v' be two vertices of a $2k+1$ cycle C in T with distance $d \geqslant 2$ in C. If $\varphi(v) = u_0$, than $\varphi(v') \in \{u_{i(2k-1)+d}, u_{i(2k-1)+(2k+1-d)}\}$ for some integer $0 \leqslant i \leqslant r-2$.*

Proof. In C there are two paths between v and v' and let d and d' be their lengths. There cannot be a path of length $d - 2s$ or $d' - 2s$ with $s \geqslant 1$ between $\varphi(v)$ and $\varphi(v')$, since this path together with the embedding of the v-v'-path of other parity from C would form a closed odd walk of length less than $2k+1$, contradicting Proposition 2.2 *(b)*. Similarly, $\varphi(v')$ is not in the neighbourhood of $\varphi(v) = u_0$ in $A_{k,r}$, since $2 \leqslant d \leqslant k < d' \leqslant 2k-1$.

Consequently, $\varphi(v')$ will lie on a segment S between $u_{i(2k-1)+1}$ and $u_{(i+1)(2k-1)+1}$ on the Hamiltonian cycle C_A for some integer $0 \leqslant i \leqslant r-2$. The segment S, together with $u_0 = \varphi(v)$ forms a C_{2k+1}, and since there are only two vertices with distance d from $u_0 = \varphi(v)$ on this C_{2k+1}, an embedding of v' onto any other vertex gives rise to a v-v'-path of length $d-2s$ or $d'-2s$ with $s \geqslant 1$. Therefore, $\varphi(v') \in \{u_{i(2k-1)+d}, u_{i(2k-1)+(2k+1-d)}\}$ as claimed. □

Claim 5.4. *Let v, v', v'' be distinct vertices of a $2k+1$ cycle C in T. Let P' be the path from v to v' avoiding v'' on C and let P'' be the path from v to v'' avoiding v' on C. Suppose $d', d'' \geqslant 2$ are the lengths of P' and P''. If $\varphi(v) = u_0$, then $\varphi(v') = u_{i(2k-1)+d'}$ and $\varphi(v'') = u_{j(2k-1)+(2k+1-d'')}$, or $\varphi(v') = u_{i(2k-1)+(2k+1-d')}$ and $\varphi(v'') = u_{j(2k-1)+d''}$, for some integers $0 \leqslant i, j \leqslant r-1$.*

Proof. By Claim 5.3 it suffices to show, that $\varphi(v') = u_{i(2k-1)+d'}$ implies $\varphi(v'') \neq u_{j(2k-1)+d''}$ and $\varphi(v') = u_{i(2k-1)+(2k+1-d')}$ implies $\varphi(v'') \neq u_{j(2k-1)+(2k+1-d'')}$, for all $0 \leqslant i, j \leqslant r-1$.

In the first case, we may assume that $i \leqslant j$. Since $u_{j(2k-1)+2}$ is a neighbour of $u_{i(2k-1)+1}$, we may consider the path P starting with the path in C_A from $u_{i(2k-1)+d'}$ to $u_{i(2k-1)+1}$ together with the edge from $u_{i(2k-1)+1}u_{j(2k-1)+2}$ and then following C_A to $u_{j(2k-1)+d''}$. The path P consists of $(d'-1)+1+(d''-2) = d'+d''-2$ edges. Together with the embedding of the path between v' and v'' from C avoiding v, this yields a closed odd walk of length at most $2k-1$ in $A_{k,r}$, contradicting Proposition 2.2 *(b)*. A similar argument for the second case concludes the proof of the claim. □

Note that $i(2k-1)+d \neq i(2k-1)+(2k+1-d)$ for all integers $d, i \geqslant 0$. Since z_T lies in three C_{2k+1}, each also containing two of the vertices a_T, b_T, c_T, if $\varphi(z_T) = u_0$, then it follows from Claim 5.4, that not all three branch vertices can be embedded onto $A_{k,r}$. Consequently, there is no homomorphism from T to $A_{k,r}$ and Lemma 5.2 is proved. □

Suitable blow-ups of $(2k+1)$-tetrahedrons show that for every $k \geqslant 4$ there are graphs in $\mathscr{G}_{\mathscr{C}_{2k-1}}(\frac{1}{2k-1}+\varepsilon)$ for $\varepsilon > 0$ that are not homomorphic to $A_{k,r}$ for any $r \geqslant 1$.

Proposition 5.5. *For every integer $k \geqslant 4$ there is some $\varepsilon > 0$ and there are infinitely many graphs in $\mathscr{G}_{\mathscr{C}_{2k-1}}(\frac{1}{2k-1}+\varepsilon)$ that are not homomorphic to $A_{k,r}$ for any $r \geqslant 1$.*

Proof. Let $k \geqslant 4$ be fixed and consider the graph T^* witch is obtained from K_4, by replacing two independent edges by a path of length $2(k-3)+1$ and the other four edges are replaced by a path of length 3. In particular, $|V(T^*)| = 4k$ and $T^* \in \mathscr{T}_k$, as all the original triangles of K_4 are replaced a C_{2k+1}. Owing to Lemma 5.2, we know that T^* is not homomorphic to $A_{k,r}$ and the construction also ensures that T^* is $\mathscr{C}_{2k-1}$-free.

If $k \geqslant 4$ is even we consider the following blow-ups of T^*. For every integer $f \geqslant 1$ we consider T^{e}_f obtained from T^* where the four vertices of degree three and the inner vertices on the two long paths with distance $0 \pmod 4$ to one of the two end vertices of the path are replaced by independent sets of size $2f$, while all the other vertices are replaced by independent sets of size f. The graph T^{e}_f is $3f$ regular and has

$$2f \cdot 2(k-2) + f \cdot 2(k+2) = (6k-4) \cdot f$$

vertices. Consequently,

$$\frac{\delta(T_f^{\mathrm{e}})}{|V(T_f^{\mathrm{e}})|} = \frac{3}{6k-4} \geqslant \frac{3}{6k-3} + \varepsilon = \frac{1}{2k-1} + \varepsilon$$

for sufficiently small $\varepsilon > 0$. Moreover, since T_f^{e} is a blow-up of T^* it is also $\mathscr{C}_{2k-1}$-free and not embeddable into $A_{k,r}$, which shows proves Proposition 5.5 of even integers $k \geqslant 4$.

For odd integers $k \geqslant 5$ we also consider blow-ups of T^*. For some integer $f \geqslant 1$ let T_f^{o} be obtained from T^* by replacing the vertices of degree three and the inner vertices on the two long paths with distance 1 (mod 4) to one of the end vertices of the path by independent sets of size f and all the remaining vertices are kept unchanged. This blow-up has

$$f \cdot 2(k-1) + 2(k+1) = (f+1)(2k-2) + 4$$

vertices and minimum degree $f+1$. Consequently,

$$\frac{\delta(T_f^{\mathrm{o}})}{|V(T_f^{\mathrm{o}})|} = \frac{f+1}{(2k-2)(f+1)+4} \geqslant \frac{1}{2k-1} + \varepsilon$$

for sufficiently small $\varepsilon > 0$ and sufficiently large f. Again the blow-up T_f^{o} is $\mathscr{C}_{2k-1}$-free and not embeddable into $A_{k,r}$, which concludes the proof of Proposition 5.5 for odd integers $k \geqslant 5$. □

§6. Concluding remarks

Theorem 1.2 provides only an upper bound for $\delta_{\hom}(C_{2k-1})$ and at this point it is not clear if it is best possible. Proving a matching lower or just showing $\delta_{\hom}(C_{2k-1}) > 0$, would require to establish the existence of a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ with members from $\mathscr{G}_{C_{2k-1}}(\alpha)$ for some $\alpha > 0$ having no homomorphic C_{2k-1}-free image H of bounded size. However, without imposing H to be C_{2k-1}-free itself, no such sequence exists for $k \geqslant 3$, as was shown by Thomassen [19], as the chromatic threshold of odd cycles other than the triangle is 0, which makes the problem somewhat delicate and for the first open case we raise the following question.

Question 6.1. Is it true that $\delta_{\hom}(C_5) > 0$?

The affirmative answer to Question 6.1 would, in particular, show that there is a graph F with $\delta_{\hom}(F) > \delta_{\chi}(F)$. To our knowledge such a strict inequality is only known for families of graphs $\mathscr{F}$, like for $\mathscr{F} = \mathscr{C}_{2k-1}$ for $k \geqslant 3$.

The lack of lower bounds for families consisting of a single graph, may suggest the following natural variation of the homomorphic threshold

$$\delta'_{\text{hom}}(F) = \inf\big\{\alpha \in [0,1] \colon \text{there is an } \mathscr{F}\text{-free graph } H = H(\mathscr{F},\alpha) \\ \text{such that } G \xrightarrow{\text{hom}} H \text{ for every } G \in \mathscr{G}_F(\alpha)\big\}\,,$$

where $\mathscr{F}$ consists of all surjective homomorphic images of F. For odd cycles we have $\delta'_{\text{hom}}(C_{2k-1}) = \delta_{\text{hom}}(\mathscr{C}_{2k-1})$ and in view of Theorem 1.2 it seems possible that $\delta'_{\text{hom}}(F)$ is easier to determine.

In the proof of Theorem 1.2 we showed that every $G \in \mathscr{G}_{C_{2k-1}}(\frac{1}{2k-1}+\varepsilon)$ is homomorphic to a C_{2k-1}-free graph H on at most $K = K(k,\varepsilon)$ vertices, where K is given by a $2(k+1)$-times iterated exponential function in $\text{poly}(1/\varepsilon, k)$. We believe that this dependency is far from being optimal and maybe already $K = O(\text{poly}(1/\varepsilon, k))$ is sufficient.

In Proposition 3.5 *(i)* we observed that C_{2k-1}-free graphs G of high minimum degree are in addition also C_{2j-1}-free for some sufficiently large $j < k$ depending on the imposed minimum degree. A more careful analysis of the argument may yield the correct dependency between j and the minimum degree of G and, moreover, yield a stability version of such a result. However, for a shorter presentation we used the same minimum degree assumption as given by Theorem 1.2, which sufficed for our purposes. It would also be interesting to see, if the excluded cycles of shorter odd length can be also excluded for the homomorphic image H in the proof of Theorem 1.2.

Finally, we remark that the blow-ups of tetrahedra considered in Section 5 are not from $\mathscr{G}_{\mathscr{C}_{2k-1}}(\frac{1}{2k-2})$. This suggests the question whether for every $k \geqslant 4$ and every $G \in \mathscr{G}_{\mathscr{C}_{2k-1}}(\frac{1}{2k-2})$ there is some $r \geqslant 1$ such that $G \xrightarrow{\text{hom}} A_{k,r}$.

Acknowledgments. We we thank both referees for their detailed and helpful remarks.

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