

RAMSEY PROPERTIES OF RANDOM DISCRETE STRUCTURES

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ABSTRACT. We study thresholds for Ramsey properties of random discrete structures. In particular, we determine the threshold for Rado's theorem for solutions of partition regular systems of equations in random subsets of the integers and we prove the 1-statement of the conjectured threshold for Ramsey's theorem for random hypergraphs. Those results were conjectured by Rödl and Ruciński and similar results were obtained independently by Conlon and Gowers.

1. INTRODUCTION

Ramsey theory is an important branch of combinatorics. Roughly speaking, a Ramsey type result asserts that for some given configuration F and some integer r the existence of a configuration G such that any partition (or coloring) of G into r classes has the property that a copy of F is completely contained in one of the r partition classes. For example, one of the first results of this type can be found in the work of Hilbert [15], where it was shown that for every ℓ and for every finite partition of the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ there exists a partition class which contains an affine cube of dimension ℓ , i.e., a set of the form $\{x_0 + \sum_{i=1}^{\ell} \varepsilon_i x_i : \varepsilon_i \in \{0, 1\}\}$ for some $x_0, x_1, \dots, x_{\ell} \in \mathbb{N}$. Classical results of that type include the work of Schur [34], van der Waerden [37], Rado [25], Ramsey [26], Erdős and Szekeres [4], Hales and Jewett [14], Graham, Leeb, and Rothschild [12], and others (see, e.g., [13] for more details).

Applications of probabilistic arguments to obtain bounds in Ramsey theory have a long tradition. On the other hand, the study of Ramsey type properties of random structures was initiated only more recently by Łuczak, Ruciński, and Voigt [20] and further studied by Rödl and Ruciński with their collaborators [8, 10, 11, 27, 28, 29, 30, 31, 32] (for more related results by others see [9, 18, 19, 21, 22, 23, 24]). The aim of this paper is to establish a general result which yields Ramsey type results for random discrete structures (see Theorem 2.5). As a consequence, combined with the work from [30] we establish the threshold for Rado's theorem for random subsets of the integers (see Theorem 1.1) and we obtain the 1-statement for the conjectured threshold of Ramsey's theorem for random hypergraphs (see Theorem 1.2). Similar results were obtained independently by Conlon and Gowers [2].

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1.1. Random subsets of the integers. Ramsey type results for the integers embody the following pattern. For every finite coloring of \mathbb{N} there exist integers x_1, \dots, x_k all of the same color, which satisfy some prescribed condition. For the condition $x_1 + x_2 = x_3$ such a result was proved by Schur [34] and for x_1, \dots, x_k forming a non-trivial arithmetic progression of length k this is the result of van der Waerden [37]. In 1933 Rado [26] published a far-reaching generalization of these results. For an $\ell \times k$ matrix $A = (a_{ij})$ of integers consider the system $\mathcal{L}(A)$ of homogeneous linear equations

$$\sum_{j=1}^k a_{ij}x_j = 0 \quad \text{for } 1 \leq i \leq \ell.$$

We say that a matrix A is *partition regular* if for any finite coloring of \mathbb{N} there is always a solution (x_1, \dots, x_k) of $\mathcal{L}(A)$ with all x_i having the same color. Rado characterized partition regular matrices and it follows directly from that characterization that $k \geq \text{rank}(A) + 2$ is a necessary condition (see, e.g., [13] for details). We note that the single equation $x_1 + x_2 - x_3 = 0$ is partition regular due to Schur's theorem while the same follows for $x_1 + x_2 - 2x_3 = 0$ by van der Waerden's theorem. On the other hand, the equation $x_1 + x_2 - 3x_3 = 0$ fails to have that property.

We say a partition regular matrix A is *irredundant* if there exists a solution (x_1, \dots, x_k) of $\mathcal{L}(A)$ such that $x_i \neq x_j$ for all $1 \leq i < j \leq k$ and otherwise we say A is redundant. It is easy to show that for every redundant $\ell \times k$ matrix A there exists an irredundant $\ell' \times k'$ matrix A' for some $\ell' < \ell$ and $k' < k$ with the same family of solutions (viewed as sets). More precisely, $(y_1, \dots, y_{k'})$ is a solution of $\mathcal{L}(A')$ if and only if there exists a solution (x_1, \dots, x_k) for $\mathcal{L}(A)$ with

$$\{x_1, \dots, x_k\} = \{y_1, \dots, y_{k'}\}$$

(see, e.g., [30, Section 1] for details). Due to this consideration it is natural to restrict to irredundant, partition regular matrices A .

We denote by $[n] = \{1, \dots, n\}$ the first n positive integers and for a subset $Z \subseteq [n]$, a positive integer $r \in \mathbb{N}$, and an irredundant, $\ell \times k$ integer matrix A we write

$$Z \rightarrow (A)_r$$

if for every coloring of Z with r colors, there exists a solution (x_1, \dots, x_k) of $\mathcal{L}(A)$ such that all x_i are distinct and contained in Z and have the same color. A standard compactness argument combined with Rado's theorem yields that for any $r \in \mathbb{N}$ and every partition regular matrix A we have $[n] \rightarrow (A)_r$ for every sufficiently large n . Our first main result determines the density required by random subsets of $[n]$ to satisfy the same property.

For $p \in (0, 1]$ let $[n]_p$ denote the binomial random subset of $[n]$ with integers from $[n]$ included independently, each with probability p . In other words, we consider the finite probability space on all subsets of $[n]$, where

$$\mathbb{P}([n]_p = Z) = p^{|Z|}(1-p)^{n-|Z|}$$

holds for all $Z \subseteq [n]$. In [11, 29, 30] the question when $[n]_p \rightarrow (A)_r$ holds with probability close to 1 was investigated. To characterize the sequences of probabilities $\mathbf{p} = (p_n)_{n \in \mathbb{N}}$ with that property we consider the following parameter introduced in [30].

Let A be an $\ell \times k$ integer matrix and let the columns be indexed by $[k]$. For a partition $W \dot{\cup} \overline{W} = [k]$ of the columns of A , we denote by $A_{\overline{W}}$ the matrix obtained from A by restricting to the columns indexed by \overline{W} . Let $\text{rank}(A_{\overline{W}})$ be the rank of $A_{\overline{W}}$, where $\text{rank}(A_{\overline{W}}) = 0$ for $\overline{W} = \emptyset$. We set

$$m_A = \max_{\substack{W \dot{\cup} \overline{W} = [k] \\ |\overline{W}| \geq 2}} \frac{|\overline{W}| - 1}{|\overline{W}| - 1 + \text{rank}(A_{\overline{W}}) - \text{rank}(A)}. \quad (1)$$

It can be shown by elementary linear algebra that $m_A = m_{A'}$, whenever $\mathcal{L}(A)$ and $\mathcal{L}(A')$ have the same set of solutions. Moreover, in [30, Proposition 2.2 (ii)] it was proved that for irredundant, partition regular matrices A the denominator of (1) is always at least 1.

For example, if A consists of the single equation $x_1 + x_2 - x_3 = 0$ considered by Schur, then $m_A = 2$ and if A corresponds to an irredundant, partition regular matrix with the property that the solutions of $\mathcal{L}(A)$ form an arithmetic progression of length k , then $m_A = k - 1$. Note that in both of these examples for $p = \Omega(n^{-1/m_A})$ the expected number of solutions of $\mathcal{L}(A)$ in $[n]_p$ is of the same order of magnitude as the expected size of $[n]_p$, while this fails to be true for $p = o(n^{-1/m_A})$.

The definition of m_A generalizes this property (in an hereditary way) for arbitrary irredundant, partition regular matrices A . In fact, one of the main results in [30] asserts that for every irredundant, partition regular matrix A there exists some $c > 0$ such that if $\mathbf{p} = (p_n)$ satisfies $p_n \leq cn^{-1/m_A}$, then

$$\lim_{n \rightarrow \infty} \mathbb{P}([n]_{p_n} \rightarrow (A)_2) = 0. \quad (2)$$

Note that by definition $\mathbb{P}([n]_{p_n} \rightarrow (A)_r) \leq \mathbb{P}([n]_{p_n} \rightarrow (A)_2)$ for every $r \geq 2$. Moreover, extending a result from [29] in [30] the complementing result for $p \geq Cn^{-1/m_A}$ for some sufficiently large $C > c$ was obtained for a special subclass of partition regular matrices, which we consider below.

We say an irredundant, partition regular $\ell \times k$ matrix A is *density regular* if any subset $Z \subseteq \mathbb{N}$ with positive upper density, i.e.,

$$\limsup_{n \rightarrow \infty} \frac{|Z \cap [n]|}{n} > 0,$$

contains a solution (x_1, \dots, x_k) of $\mathcal{L}(A)$ with all x_i distinct. For example, Szemerédi's famous theorem on arithmetic progressions [35] shows that if the solutions of $\mathcal{L}(A)$ form an arithmetic progression, then A is density regular. More generally, it was shown in [5] that an irredundant, partition regular matrix is density regular if and only if $(1, \dots, 1)$ is a solution of $\mathcal{L}(A)$.

Complementing (2), Rödl and Ruciński showed in [30] that for every irredundant, density regular matrix A and every integer $r \geq 2$ there exists $C > 0$ such that if $\mathbf{p} = (p_n)$ satisfies $p_n \geq Cn^{-1/m_A}$, then

$$\lim_{n \rightarrow \infty} \mathbb{P}([n]_{p_n} \rightarrow (A)_r) = 1 \quad (3)$$

(see also [33] for a new proof). For the special case when solutions of $\mathcal{L}(A)$ form an arithmetic progression the same result appeared already in [29].

In other words, combining (2) and (3) it follows that $p_n = n^{-1/m_A}$ is the *threshold* for the property $[n]_{p_n} \rightarrow (A)_r$ for irredundant, density regular matrices A . It was conjectured in [30] that this extends to all irredundant, partition regular matrices A . For the special case when A consists only of the equation $x_1 + x_2 - x_3 = 0$

(considered by Schur) and $r = 2$ this was verified in [11]. Our first main result addresses the general case.

Theorem 1.1. *Let A be an irredundant, partition regular integer matrix and let $r \in \mathbb{N}$. There exist constants $0 < c < C$ such that for any sequence of probabilities $\mathbf{p} = (p_n)_{n \in \mathbb{N}}$ we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}([n]_{p_n} \rightarrow (A)_r) = \begin{cases} 1, & \text{if } p_n \geq Cn^{-1/m_A} \\ 0, & \text{if } p_n \leq cn^{-1/m_A}. \end{cases}$$

Due to (2) it suffices to show the 1-statement in the theorem above. This statement will follow from a more general result, Theorem 2.5, and we deduce the 1-statement of Theorem 1.1 in Section 3.

1.2. Ramsey properties for random hypergraphs. The second main result concerns partition properties of random hypergraphs. An ℓ -uniform hypergraph H is a pair (V, E) , where the vertex set V is some finite set and the edge set $E \subseteq [V]^\ell$ is a subfamily of the ℓ -element subsets of V . As usual we call 2-uniform hypergraphs simply graphs. For some hypergraph H we denote by $V(H)$ and $E(H)$ its vertex set and its edge set and we denote by $v(H)$ and $e(H)$ the cardinalities of those sets. For an integer n we denote by $K_n^{(\ell)}$ the complete ℓ -uniform hypergraph on n vertices, i.e., $v(K_n^{(\ell)}) = n$ and $e(K_n^{(\ell)}) = \binom{n}{\ell}$. For a subset $U \subseteq V(H)$ we denote by $E(U)$ the edges of H contained in U and we set $e(U) = |E(U)|$. Moreover, we write $H[U]$ for the subhypergraph induced on U , i.e., $H[U] = (U, E(U))$.

Ramsey's theorem [26] asserts that for every ℓ -uniform hypergraph F and every $r \in \mathbb{N}$ we have

$$K_n^{(\ell)} \rightarrow (F)_r$$

for sufficiently large n , i.e., every r -coloring of the edges of $K_n^{(\ell)}$ yields a monochromatic copy of F . More generally, for ℓ -uniform hypergraphs F and G and $r \in \mathbb{N}$ we write $G \rightarrow (F)_r$ if for every partition $E^1 \dot{\cup} \dots \dot{\cup} E^r = E(G)$ there exists some $s \in [r]$ and an injective mapping $\varphi: V(F) \rightarrow V(G)$ such that $\varphi(e) \in E^s$ for every $e \in E(F)$, i.e., there exists a monochromatic copy of F .

Similarly as in the context of Rado's theorem we are interested in random versions of Ramsey's theorem. Here we study the binomial model $G^{(\ell)}(n, p)$ of ℓ -uniform hypergraphs, where edges of the complete hypergraph $K_n^{(\ell)}$ are included independently with probability p . More formally, we consider the finite probability space with ground set $E(K_n^{(\ell)})$ where for any ℓ -uniform hypergraph H with vertex set $V(K_n^{(\ell)})$ we have

$$\mathbb{P}\left(G^{(\ell)}(n, p) = H\right) = p^{e(H)}(1-p)^{\binom{n}{\ell}-e(H)}.$$

For a fixed ℓ -uniform hypergraph F and $r \in \mathbb{N}$ we are interested in the asymptotic growth of the smallest sequence of probabilities $\mathbf{p} = (p_n)_{n \in \mathbb{N}}$ such that $G^{(\ell)}(n, p_n) \rightarrow (F)_r$ holds *asymptotically almost surely (a.a.s.)*, i.e.,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(G^{(\ell)}(n, p_n) \rightarrow (F)_r\right) = 1. \quad (4)$$

This question was first studied in [20] and there it was shown that (4) holds for $F = K_3$ being a graph triangle, $r = 2$, and $p = p_n \geq C/\sqrt{n}$ for some sufficiently large C (as noted in [20] this also follows implicitly from an earlier result in [6]). The result from [20] was generalized for the same condition on p to an arbitrary

number of colors by Rödl and Ruciński in [28]. Finally in [29] the same authors solved the problem for arbitrary graphs F and any number of colors $r \in \mathbb{N}$ by showing that (4) is valid as long as $p \geq Cn^{-1/m_F}$ for some $C = C(F, r)$, where (in general for an ℓ -uniform hypergraph F with $e(F) \geq 1$) we set

$$m_F = \max_{\substack{F' \subseteq F \\ e(F') \geq 1}} d(F') \quad \text{with} \quad d(F') = \begin{cases} \frac{e(F')-1}{v(F')-\ell}, & \text{if } v(F') > \ell \\ 1/\ell, & \text{if } v(F') = \ell. \end{cases} \quad (5)$$

It follows from the definition of m_F that if $p = \Omega(n^{-1/m(F)})$ then a.a.s. the number of copies of every subhypergraph $F' \subseteq F$ in the random hypergraph $G^{(\ell)}(n, p)$ has at least the same order of magnitude as the number of edges. This property seems to be a necessary condition for (4) to hold. This belief was indeed verified for graphs in [27], where it was shown that for “most” graphs F there exists some $c > 0$ such that for any $\mathbf{p} = (p_n)_{n \in \mathbb{N}}$ with $p_n \leq cn^{-1/m_F}$ we have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(G^{(2)}(n, p_n) \rightarrow (F)_2 \right) = 0$$

Here “most” means all graphs F with the exception of forests consisting of stars and paths of length three, which show a slightly different behavior (see [17, Chapter 8] for details).

Our second main result, Theorem 1.2, establishes the 1-statement for the conjectured threshold of Ramsey’s theorem in random ℓ -uniform hypergraphs. We believe that the matching 0-statement also holds for “most” hypergraphs F , but we will not study this here.

Theorem 1.2. *Let F be an ℓ -uniform hypergraph with maximum degree at least 2 and let $r \in \mathbb{N}$. There exists a constant $C > 0$ such that for any sequence of probabilities $\mathbf{p} = (p_n)_{n \in \mathbb{N}}$ satisfying $p_n \geq Cn^{-1/m_F}$ we have*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(G^{(\ell)}(n, p_n) \rightarrow (F)_r \right) = 1.$$

Theorem 1.2 was conjectured by Rödl and Ruciński [31, Conjecture 1.23]. In [31] and in [32] such a result was already established for the special cases when $F = K_4^{(3)}$ and for ℓ -partite, ℓ -uniform hypergraphs F . Theorem 1.2 follows from the more general result presented in Section 2 and we present the reduction in Section 3.

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2. MAIN TECHNICAL RESULT

In this section we introduce a general environment allowing us to prove Theorem 1.1 and Theorem 1.2 along the same lines. We note that the earlier results of Rödl and Ruciński in [29, 31] were based on applications of the regularity lemma for graphs and 3-uniform hypergraphs [7, 36]. Due to the somewhat technical nature of the regularity lemma for hypergraphs, proving even special cases of Theorem 1.2 for $\ell \geq 3$ presented several technical difficulties. Although the approach taken here uses some ideas from [29], we will, similarly as in [32], avoid the use of the regularity lemma. The approach here can be viewed as a refinement of the work in [33], where

related extremal and Turán-type problems for random subsets of the integers and random hypergraphs were studied.

2.1. Statement of main result. It will be convenient to consider sequences of k -uniform hypergraphs $\mathbf{H} = (H_n)_{n \in \mathbb{N}}$. In the context of Theorem 1.1 for a given irredundant, partition regular $\ell \times k$ matrix, one may think of the vertex set $V(H_n)$ to be $[n]$ and the edges being the solutions (x_1, \dots, x_k) of $\mathcal{L}(A)$ with $x_i \neq x_j$ for $1 \leq i < j \leq k$. In view of Theorem 1.2, for a given ℓ -uniform hypergraph F with k edges we may think of $V(H_n)$ being the edge set of $K_n^{(\ell)}$ and every edge of $E(H_n)$ corresponds to the edge set of a copy of F in $K_n^{(\ell)}$.

The two main assumptions allowing us to apply the main result, Theorem 2.5, are (r, ζ) -Ramsey (cf. Definition 2.1) and (K, \mathbf{p}) -boundedness (cf. Definition 2.4). Roughly speaking, \mathbf{H} will be (r, ζ) -Ramsey if a “quantitative Ramsey-type result” for the original structure holds, which guarantees not just one, but many monochromatic copies. For Rado’s theorem such a strengthening was deduced from Deuber’s theorem in [5] and for Ramsey’s theorem it follows directly from Ramsey’s original argument and was first observed by Erdős [3]. The (K, \mathbf{p}) -boundedness will impose a lower bound on \mathbf{p} and we will verify this condition for Theorem 1.1 and Theorem 1.2 in Section 3.

2.1.1. The Ramsey property. We consider the following abstract version of the Ramsey property and its quantitative strengthening, which requires not only one monochromatic copy in every coloring, but a fraction of all copies to be monochromatic.

Definition 2.1. Let $H = (V, E)$ be a k -uniform hypergraph and $r \in \mathbb{N}$. We say H is r -Ramsey if for every partition $V^1 \dot{\cup} \dots \dot{\cup} V^r$ of V there exists an $s \in [r]$ such that $e(V^s) \neq 0$.

For a subset $U \subseteq V$ and $\zeta > 0$, we say the induced subhypergraph $H[U]$ is (r, ζ) -Ramsey if for every partition $U^1 \dot{\cup} \dots \dot{\cup} U^r$ of U there exists an $s \in [r]$ such that $e(U^s) \geq \zeta |E|$. For the special case $V = U$ we simply say the hypergraph H is (r, ζ) -Ramsey.

For a sequence $\mathbf{H} = (H_n)_{n \in \mathbb{N}}$ of k -uniform hypergraphs, we say \mathbf{H} is (r, ζ) -Ramsey if for all but finitely many n the hypergraph H_n is (r, ζ) -Ramsey.

We note that in the definition of (r, ζ) -Ramsey the number of required monochromatic edges is given in terms of the global number $e(H)$ of the edges of H and not in terms of $e(U)$. The next observation follows directly from Definition 2.1.

Fact 2.2. Let r_1, \dots, r_ℓ be positive integers, let $\zeta > 0$, let $H = (V, E)$ be a k -uniform hypergraph, and let $U^1 \dot{\cup} \dots \dot{\cup} U^\ell$ be a partition of $U \subseteq V$. If $H[U]$ is $(\sum_{j=1}^\ell r_j, \zeta)$ -Ramsey, then there exists a $j \in [\ell]$ such that $H[U^j]$ is (r_j, ζ) -Ramsey. \square

Roughly speaking, our proof is based on an inductive argument on the number $i \in [k]$ of monochromatic vertices in every edge and on the number of colors of the Ramsey property r . In the induction step we will consider induced subhypergraphs, with “weaker” Ramsey properties. For the quantification of this process we will use the following recursive function.

Definition 2.3. We define the function $R: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ recursively by setting

$$R(1, r) = 1, \quad R(i, 1) = 1,$$

and

$$R(i+1, r+1) = R(i, r+1) + (r+1)R(i+1, r)$$

for every $i, r \in \mathbb{N}$.

2.1.2. *The boundedness property.* For the induction based on the number $i \in [k]$ of monochromatic vertices in every edge we need the following notion. For a k -uniform hypergraph $H = (V, E)$, $i \in [k-1]$, $v \in V$, and $U \subseteq V$ we denote by $\deg_i(v, U)$ the number of edges of H containing v and having at least i vertices in $U \setminus \{v\}$, i.e.,

$$\deg_i(v, U) = |\{e \in E : v \in e \text{ and } |(e \setminus \{v\}) \cap U| \geq i\}|. \quad (6)$$

In the proof it will be important that not “too” many partial monochromatic copies are attached to only a few vertices. For that we need some control on the expected number of $\sum_{v \in V} \deg_i(v, U)$ where $U = V_q$ is a binomial random subset of V . More formally, for $q = [0, 1]$ we consider

$$\mu_i(H, q) = \mathbb{E} \left[\sum_{v \in V} \deg_i^2(v, V_q) \right]. \quad (7)$$

The boundedness property ensures that $\mu_i(H_n, q)$ is dominated (up to a constant factor) by the contribution of those pairs of edges (e, e') , which share one vertex v and each edge intersects $U_q \setminus \{v\}$ separately in at least i vertices.

Definition 2.4. Let $\mathbf{H} = (H_n)_{n \in \mathbb{N}}$ be a sequence of k -uniform hypergraphs, let $\mathbf{p} = (p_n)_{n \in \mathbb{N}}$ be a sequence of probabilities, and let $K \geq 1$. We say \mathbf{H} is (K, \mathbf{p}) -bounded if the following is true.

For every $i \in [k-1]$, there exists n_0 such that for every $n \geq n_0$ and $q \geq p_n$ we have

$$\mu_i(H_n, q) \leq Kq^{2i} \frac{|E(H_n)|^2}{|V(H_n)|}. \quad (8)$$

Note that in fact for small functions \mathbf{p} the inequality (8) may fail. Consider the case when \mathbf{H} corresponds to sequence of hypergraphs for the Ramsey property when monochromatic triangles are guaranteed. In this case $V_n = \binom{n}{2}$ and E_n consists of the edges set of all triangles in the complete graph K_n . In this case, $k = 3$ and the asymptotic growth of $\mu_2(H_n, p)$ coincides with the maximum of the following two quantities

- (a) the expected number of triangles in K_n with at least two edges present in $G(n, p)$, i.e., the expected number of path with two edges in $G(n, p)$
- (b) the expected number of cycles of length four in $G(n, p)$.

Note that if $p \gg n^{-1/2}$, then the quantity in (b) dominates the one in (a), while the reverse is true if $p \ll n^{-1/2}$. For (8) to be true we require that the quantity from (b) dominates $\mu_2(H_n, p)$ and in this example this happens if

$$p = \Omega(n^{-1/2}) = \Omega(n^{-1/m_{K_3}})$$

(see (5)). In fact, we will see in the proof of Theorem 1.2 (see Section 3) that in this context the sequence \mathbf{H} is (K, \mathbf{p}) -bounded for some $K \geq 1$ if $p_n = \Omega(n^{-1/m_F})$. Similarly, in the context of Theorem 1.1 it will turn out that \mathbf{H} is (K, \mathbf{p}) -bounded for some $K \geq 1$ if $p_n = \Omega(n^{-1/m_A})$ (see (1)).

2.1.3. *Main technical result.* The results stated in the introduction are consequences of the following theorem.

Theorem 2.5. *Let $\mathbf{H} = (H_n)_{n \in \mathbb{N}}$ be a sequence of k -uniform hypergraphs, let $\mathbf{p} = (p_n)_{n \in \mathbb{N}}$ be a sequence of probabilities satisfying $p_n^k |E(H_n)| \rightarrow \infty$ as $n \rightarrow \infty$, and let $\zeta > 0$, $K \geq 1$, and $r \in \mathbb{N}$.*

If \mathbf{H} is $(R(k, r), \zeta)$ -Ramsey and (K, \mathbf{p}) -bounded, then there exists a $C \geq 1$ such that for $q_n \geq Cp_n$ a.a.s. the binomial random subset V_{n, q_n} of $V(H_n)$ induces an r -Ramsey hypergraph.

We remark that typically satisfying the (K, \mathbf{p}) -boundedness will be the more restrictive assumption on \mathbf{p} compared to $p_n^k |E(H_n)| \rightarrow \infty$. For example, if \mathbf{H} is given by the Ramsey property when monochromatic graph triangles are guaranteed, then $p_n^k |E(H_n)| \rightarrow \infty$ if $p \gg n^{-1}$, while (as discussed above) \mathbf{H} is (K, \mathbf{p}) -bounded only if $p_n = \Omega(n^{-1/2})$. The proof of Theorem 2.5 is based on induction and for the induction we will strengthen the statement (see Lemma 2.7 below).

For a k -uniform hypergraph $H = (V, E)$, subsets $W \subseteq U \subseteq V$, and any integer $i \in \{0, \dots, k\}$ we consider those edges of $H[U]$ which have at least i vertices in W and we denote this family by

$$E_U^i(W) = \{e \in E(U) : |e \cap W| \geq i\}.$$

Note that

$$E_U^0(W) = E(U) \quad \text{and} \quad E_U^k(W) = E(W)$$

for every $W \subseteq U$.

The next technical definition is crucial to our induction scheme.

Definition 2.6. *Let $H = (V, E)$ be a k -uniform hypergraph and $W \subseteq U \subseteq V$. Let $i \in [k]$, $r \in \mathbb{N}$, $\xi > 0$ and $q \in (0, 1]$. We say $H[W]$ is (i, r, ξ, q, U) -Ramsey if for every partition $W^1 \dot{\cup} \dots \dot{\cup} W^r$ of W there exists an $s \in [r]$ such that*

$$|E_U^i(W^s)| \geq \xi q^i |E|.$$

The next lemma states that under some fairly general assumptions $(R(i, r), \zeta)$ -Ramseyhood of $H[U]$ implies (with probability close to 1) that $H[U_q]$ is (i, r, ξ, q, U) -Ramsey.

Lemma 2.7. *Let $\mathbf{H} = (H_n = (V_n, E_n))_{n \in \mathbb{N}}$ be a sequence of k -uniform hypergraphs, let $\mathbf{p} = (p_n)_{n \in \mathbb{N}}$ be a sequence of probabilities satisfying $p_n \rightarrow 0$ and $p_n^k |E_n| \rightarrow \infty$ as $n \rightarrow \infty$, and let $K \geq 1$. Suppose \mathbf{H} is (K, \mathbf{p}) -bounded.*

For every $i \in [k]$, $r \in \mathbb{N}$, and $\zeta > 0$ there exist $\xi > 0$, $b > 0$, $C \geq 1$, and n_0 such that for every $n \geq n_0$, and every $q \geq Cp_n$ the following holds.

If $U \subseteq V_n$ and $H_n[U]$ is $(R(i, r), \zeta)$ -Ramsey, then the binomial random subset U_q satisfies

$$\mathbb{P}(H[U_q] \text{ is } (i, r, \xi, q, U)\text{-Ramsey}) \geq 1 - 2^{-bq|V_n|}.$$

Theorem 2.5 follows from Lemma 2.7 applied with $i = k$ and $U = V_n$. Note that the property of being r -Ramsey is monotone and, hence, it suffices to verify Theorem 2.5 for $p_n = o(1)$.

2.2. **Probabilistic tools.** We will use Chernoff's inequality [1] in the following form.

Theorem 2.8 (Chernoff's inequality (see, e.g., [17, Corollary 2.3])). *Let $X \subseteq Y$ be finite sets and $p \in (0, 1]$. For every $0 < \varrho \leq 3/2$ we have*

$$\mathbb{P}(|X \cap Y_p| - p|X|) \geq \varrho p|X|) \leq 2 \exp(-\varrho^2 p|X|/3). \quad \square$$

We will also use Janson's inequality [16].

Theorem 2.9 (Janson's inequality (see, e.g. [17, Theorem 2.14])). *Let $H = (V, E)$ be a k -uniform hypergraph, $U \subseteq V$, $i \in \{2, \dots, k\}$, and $q \in (0, 1]$. For every edge $e \in E(U)$ fix some i -element subset $I(e) \subseteq e$ (in an arbitrary way) and set*

$$\mathbb{1}(e) = \begin{cases} 1, & \text{if } I(e) \subseteq U_q \\ 0, & \text{otherwise.} \end{cases}$$

For every $\varrho > 0$ the binomial random subset U_q satisfies

$$\begin{aligned} \mathbb{P}(|E_U^i(U_q)| \leq (1 - \varrho)q^i|E(U)|) &\leq \mathbb{P}\left(\sum_{e \in E(U)} \mathbb{1}(e) \leq (1 - \varrho)q^i|E(U)|\right) \\ &\leq \exp\left(-\frac{\varrho^2 q^{2i}|E(U)|^2}{2\bar{\Delta}_i}\right), \end{aligned}$$

where $\bar{\Delta}_i = \mathbb{E}[\sum \sum \{\mathbb{1}(e) \cdot \mathbb{1}(e') : e, e' \in E(U) \text{ and } I(e) \cap I(e') \neq \emptyset\}]$. \square

We note that $\bar{\Delta}_i$ can be bounded from above by $q\mu_{i-1}(H, q)$. In fact, it follows from the linearity of the expectation that

$$\begin{aligned} \bar{\Delta}_i &= \mathbb{E}\left[\sum \sum \{\mathbb{1}(e) \cdot \mathbb{1}(e') : e, e' \in E(U) \text{ and } I(e) \cap I(e') \neq \emptyset\}\right], \\ &\leq \mathbb{E}\left[\sum_{u \in U_q} \left|\{(e, e') : u \in I(e) \cap I(e'), I(e) \subseteq U_q, \text{ and } I(e') \subseteq U_q\}\right|\right] \\ &= \sum_{u \in U} q \mathbb{E}\left[\left|\{(e, e') : u \in I(e) \cap I(e'), (I(e) \setminus \{u\}) \subseteq U_q, \right.\right. \\ &\qquad \qquad \qquad \left.\left. \text{and } (I(e') \setminus \{u\}) \subseteq U_q\}\right|\right] \\ &\leq q \sum_{v \in V} \mathbb{E}[\deg_{i-1}^2(v, V_q)] \\ &= q\mu_{i-1}(H, q). \end{aligned} \tag{9}$$

We also use an approximate concentration result for (K, \mathbf{p}) -bounded hypergraphs. The boundedness of \mathbf{H} only bounds the expected value of the quantity $\sum_v \deg_i^2(v, V_p)$. In the proof of Lemma 2.7 we need an exponential upper tail bound for the deviation of $\sum_v \deg_i^2(v, V_p)$. Unfortunately, it is known that such a bound does not hold in general. However, it was shown by Rödl and Ruciński in [29] that at the price of deleting a few elements such a bound can be obtained and we will need the following variant of that observation.

Proposition 2.10 (Upper tail [29, Lemma 4]). *Let $\mathbf{H} = (H_n = (V_n, E_n))_{n \in \mathbb{N}}$ be a sequence of k -uniform hypergraphs, let $\mathbf{p} = (p_n)_{n \in \mathbb{N}}$ be a sequence of probabilities, and let $K \geq 1$. Suppose \mathbf{H} is (K, \mathbf{p}) -bounded.*

For every $i \in [k - 1]$ and every $\eta > 0$ there exist $b > 0$ and n_0 such that for every $n \geq n_0$ and every $q \geq p_n$ the binomial random subset $V_{n,q}$ has the following property

with probability at least $1 - 2^{-bq|V_n|+2+2\log_2(k)}$. There exists a set $|X| \subseteq V_{n,q}$ with $|X| \leq \eta q|V_n|$ such that

$$\sum_{v \in V_n} \deg_i^2(v, V_{n,q} \setminus X) \leq 16^k K q^{2i} \frac{|E_n|^2}{|V_n|}.$$

The proof follows the lines of [29, Lemma 4] (see also [33]).

Proof. Suppose \mathbf{H} is (K, \mathbf{p}) -bounded and $i \in [k-1]$ and $\eta > 0$ are given. We set

$$b = \frac{\eta}{4(k-1)^2} \quad (10)$$

and let n_0 be sufficiently large, so that (8) holds for every $n \geq n_0$ and $q \geq p_n$.

For every $j = i, \dots, 2(k-1)$ we consider the family \mathcal{S}_j defined as follows

$$\mathcal{S}_j = \left\{ (S, v, e, e') : S \subseteq V_n, v \in V_n, e, e' \in E_n \text{ such that } |S| = j, \right. \\ \left. v \in e \cap e', S \subseteq (e \cup e') \setminus \{v\}, |e \cap S| \geq i \text{ and } |e' \cap S| \geq i \right\}.$$

Let \mathcal{S}_j be the random variable denoting the number of elements (S, v, e, e') from \mathcal{S}_j with $S \in \binom{V_n, q}{j}$. By definition we have

$$\sum_{j=i}^{2k-2} \mathbb{E}[\mathcal{S}_j] \leq 2^{2(k-1)} \mu_i(H_n, q) = 4^{k-1} \mu_i(H_n, q)$$

and due to the (K, \mathbf{p}) -boundedness of \mathbf{H} we have

$$\max_{j=i, \dots, 2(k-1)} \mathbb{E}[\mathcal{S}_j] \leq \sum_{j=i}^{2k-2} \mathbb{E}[\mathcal{S}_j] \leq 4^{k-1} \mu_i(H_n, q) \leq 4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|}.$$

Let Z_j be the random variable denoting the number of sequences

$$((S_r, v_r, e_r, e'_r))_{r \in [z]} \in \mathcal{S}_j^z$$

of length

$$z = \left\lfloor \frac{\eta q |V_n|}{4(k-1)^2} \right\rfloor \quad (11)$$

which satisfy

- (i) the sets S_r are contained in $V_{n,q}$ and
- (ii) the sets S_r are mutually disjoint, i.e., $S_{r_1} \cap S_{r_2} = \emptyset$ for all $1 \leq r_1 < r_2 \leq z$.

Clearly, we have

$$\mathbb{E}[Z_j] \leq |\mathcal{S}_j|^z q^{jz} = (\mathbb{E}[\mathcal{S}_j])^z \leq \left(4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|} \right)^z. \quad (12)$$

On the other hand, we will show that if

$$\sum_{v \in V_n} \deg_i^2(v, V_{n,q} \setminus X) \geq 16^k K q^{2i} \frac{|E_n|^2}{|V_n|} \quad (13)$$

for every subset $X \subseteq V_{n,q}$ with $|X| \leq 4(k-1)^2 z \leq \eta q |V_n|$, then there exist at least

$$\left(16^k K q^{2i} \frac{|E_n|^2}{|V_n|} \right)^{2(k-1)z} \quad (14)$$

sequences $((S_r, v_r, e_r, e'_r))_{r \in [z]} \in (\mathcal{S}_i \cup \dots \cup \mathcal{S}_{2k-2})^{2(k-1)z}$ of length $2(k-1)z$ satisfying (i) and (ii) from above. Indeed, for any such sequence of length $y < 2(k-1)z$ we set $X = S_1 \dot{\cup} \dots \dot{\cup} S_y$. It follows from (13) that there are at least $16^k K q^{2i} |E_n|^2 / |V_n|$ ways to extend this sequence and hence (14) follows. For convenience we will fix exactly $16^k K q^{2i} |E_n|^2 / |V_n|$ choices for any element in the sequence.

Moreover, for each such sequence there exists some index $j \in \{i, \dots, 2(k-1)\}$ and a subsequence of length z with members only from \mathcal{S}_j . Since we fixed exactly $16^k K q^{2i} |E_n|^2 / |V_n|$ choices for every element in the sequence, and since any given sequence of length z can appear in at most

$$\binom{2(k-1)z}{z} \leq 4^{kz}$$

different places of the long sequence, it follows that every such subsequence was contained in at most

$$4^{kz} \left(16^k K q^{2i} \frac{|E_n|^2}{|V_n|} \right)^{2(k-1)z-z}$$

of the original sequences. Consequently, there are at least

$$\frac{1}{4^{kz}} \left(16^k K q^{2i} \frac{|E_n|^2}{|V_n|} \right)^z = \left(4^k K q^{2i} \frac{|E_n|^2}{|V_n|} \right)^z$$

subsequences $((S_r, v_r, e_r, e'_r))_{r \in [z]} \in (\mathcal{S}_i \cup \dots \cup \mathcal{S}_{2k-2})^z$ of length z which satisfy (i) and (ii) from above and in addition for every such sequence there exists some index $j \in \{i, \dots, 2(k-1)\}$ such that all members of the sequence are from \mathcal{S}_j only.

However, the index j may differ for different of those subsequences. Let $j_0 \in \{i, \dots, 2(k-1)\}$ be the index, which corresponds to most subsequences. It follows that there are at least

$$\frac{(4^k K q^{2i} |E_n|^2 / |V_n|)^z}{2(k-1) - i + 1} \geq \frac{1}{k} \left(2 \cdot 4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|} \right)^z$$

sequences $((S_r, v_r, e_r, e'_r))_{r \in [z]} \in \mathcal{S}_{j_0}^z$ of length z satisfying (i) and (ii) and all its members being from \mathcal{S}_{j_0} .

Summarizing the above, from assumption (13) we deduced that there exists some index $j_0 \in \{i, \dots, 2(k-1)\}$ such that

$$Z_{j_0} \geq \frac{1}{k} \left(2 \cdot 4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|} \right)^z.$$

In view of (12), Markov's inequality bounds the probability of this event by

$$\begin{aligned} \mathbb{P} \left(\exists j_0 \in \{i, \dots, 2k-2\}: Z_{j_0} \geq \frac{2^z}{k} \left(4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|} \right)^z \right) \\ \leq \sum_{j=i}^{2k-2} \mathbb{P} \left(Z_j \geq \frac{2^z}{k} \left(4^{k-1} K q^{2i} \frac{|E_n|^2}{|V_n|} \right)^z \right) \stackrel{(12)}{\leq} \sum_{j=i}^{2k-2} \mathbb{P} \left(Z_j \geq \frac{2^z}{k} \mathbb{E}[Z_j] \right) \\ \leq 2k^2 \cdot 2^{-z} \stackrel{(10), (11)}{\leq} 2^{-(bq|V_n|-1)+1+2\log_2 k}, \end{aligned}$$

which concludes the proof of Proposition 2.10. \square

The next lemma, also due to Rödl and Ruciński from [29], states that if a binomial random subset enjoys a monotone property with very high probability, then a slightly larger random subset will have a "robust" variant of this property with

similar probability. Here we say an event holds with very high probability if the probability of failing is exponentially small in the expected size of V_q .

Proposition 2.11 ([29, Lemma 3]). *Let U be a set and let \mathcal{P} be a family of subsets of U closed under supersets. For all $\delta \in (0, 1)$ and $b > 0$ satisfying $\delta(3 - \log_2 \delta) \leq b$ and $q \in (0, 1]$ the following holds. If*

$$\mathbb{P}(U_{(1-\delta)q} \in \mathcal{P}) \geq 1 - 2^{-bq|U|}$$

then

$$\mathbb{P}(\forall X \subseteq U_q \text{ with } |X| \leq \delta q|U|/2 \text{ we have } (U_q \setminus X) \in \mathcal{P}) \geq 1 - 2^{-\delta^2 q|U|/20}. \quad \square$$

2.3. Proof of main result. We start with a simple observation.

Fact 2.12. *Let $H = (V, E)$ be a k -uniform hypergraph, let $U \subseteq V$, and let $\zeta > 0$ and $K \geq 1$. If $\sum_{v \in V} \deg_{k-1}^2(v, V) \leq K|E|^2/|V|$ and $e(U) \geq \zeta|E|$, then the set*

$$Y = \left\{ u \in U : \deg_{k-1}(u, U) \geq \frac{\zeta}{2} \frac{|E|}{|V|} \right\}$$

satisfies

$$|Y| \geq \frac{\zeta^2}{4K} |V|.$$

Proof. Due to the definition of Y we have

$$\sum_{y \in Y} \deg_{k-1}(y, U) \geq e(U) - \frac{\zeta}{2} \frac{|E|}{|V|} |U| \geq \frac{\zeta}{2} |E|.$$

Hence, it follows from the Cauchy-Schwarz inequality

$$\begin{aligned} \frac{\zeta^2}{4} |E|^2 &\leq \left(\sum_{y \in Y} \deg_{k-1}(y, U) \right)^2 \leq |Y| \sum_{y \in Y} \deg_{k-1}^2(y, U) \\ &\leq |Y| \sum_{v \in V} \deg_{k-1}^2(v, V) \leq |Y| \cdot K \frac{|E|^2}{|V|}, \end{aligned}$$

which yields the claim. \square

Proof of Lemma 2.7. Let $\mathbf{H} = (H_n = (V_n, E_n))_{n \in \mathbb{N}}$ be a sequence of k -uniform hypergraphs, let $\mathbf{p} = (p_n)_{n \in \mathbb{N}}$ be a sequence of probabilities such that $p_n \rightarrow 0$ and $p_n^k |E_n| \rightarrow \infty$ and \mathbf{H} is (K, \mathbf{p}) -bounded for some $K \geq 1$. We prove Lemma 2.7 by induction on $i + r$.

Induction start for $i = 1$ and $r \in \mathbb{N}$. In this case we need to show that for given r , and ζ there exist ξ, b, C , and n_0 so that for $n \geq n_0$ the $(R(1, r), \zeta)$ -Ramseyhood of $H_n[U]$ implies $(1, r, \xi, q, U)$ -Ramseyhood of $H[U_q]$ with very high probability. This will follow from Chernoff's inequality and Fact 2.12. In fact, let $\zeta > 0$. We set

$$\xi = \frac{\zeta^3}{16krK}, \quad b = \frac{\zeta^2}{49K}, \quad C = 1,$$

and let n be sufficiently large. Note that for sufficiently large n , the (K, \mathbf{p}) -boundedness of \mathbf{H} (applied for $q = 1$) yields

$$\mu_{k-1}(H_n, 1) = \sum_{v \in V} \deg_{k-1}^2(v, V) \leq K \frac{|E_n|^2}{|V_n|}.$$

For every $U \subseteq V_n$ satisfying $H_n[U]$ is $(R(1, r), \zeta)$ -Ramsey we have $e(U) \geq \zeta|E_n|$. Consequently, we infer from Fact 2.12 that

$$|Y| \geq \frac{\zeta^2}{4K} |V_n|. \quad (15)$$

Furthermore, due to the definition of Y we have for every $q \in (0, 1]$

$$|E_U^1(U_q)| \geq \frac{1}{k} |Y \cap U_q| \cdot \frac{\zeta}{2} \frac{|E_n|}{|V_n|}$$

and for every partition $U_q^1 \dot{\cup} \dots \dot{\cup} U_q^r$ of U_q there exists an $s \in [r]$ such that

$$|E_U^1(U_q^s)| \geq \frac{\zeta}{2kr} |Y \cap U_q| \frac{|E_n|}{|V_n|}.$$

Finally, it follows from Chernoff's inequality that $|Y \cap U_q| \geq q|Y|/2$ with probability at least $1 - 2\exp(-q|Y|/12)$. Hence, the choice of ξ and b and (15) yields

$$\mathbb{P}(H_n[U_q] \text{ is } (1, r, \xi, q, U)\text{-Ramsey}) \geq 1 - 2\exp(-q|Y|/12) \geq 1 - 2^{-bq|V_n|}$$

for sufficiently large n and $q \geq p_n$, since $q|V_n| \geq p_n|V_n| \geq p_n|E_n|^{1/k} \rightarrow \infty$.

Induction start for $i \geq 2$ and $r = 1$. This case follows from Janson's inequality. For $\zeta > 0$ we set

$$\xi = \frac{\zeta}{2}, \quad b = \frac{\zeta^2}{8K}, \quad C = 1.$$

Let n be sufficiently large and $q \geq p_n$ so that

$$\mu_{i-1}(H_n, q) \leq Kq^{2i-2}|E_n|^2/|V_n|.$$

For every $U \subseteq V_n$ for which $H_n[U]$ is $(R(i, 1), \zeta)$ -Ramsey we have $e(U) \geq \zeta|E_n|$. Consequently, (9) combined with Janson's inequality applied with $\varrho = 1/2$ yields

$$\mathbb{P}\left(|E_U^i(U_q)| \leq \frac{\zeta}{2} q^i |E_n|\right) \leq \exp\left(-\frac{\zeta^2 q^{2i-1} |E_n|^2}{8\mu_{i-1}(H_n, q)}\right) \leq 2^{-bq|V_n|},$$

which yields the lemma for $r = 1$.

Induction step. We will verify the lemma for $i+1 \geq 2$ and $r+1 \geq 2$ and suppose the lemma holds for i and $r+1$ and for $i+1$ and r . Let $\zeta > 0$ be given.

Outline. We will expose the random set U_q in L rounds, where $L = L(i, r+1, \zeta)$ is some carefully chosen constant. Suppose U_{q_1} is the outcome of the first round and let $U_{q_1}^1 \dot{\cup} \dots \dot{\cup} U_{q_1}^{r+1}$ be an arbitrary partition of U_{q_1} . Due to the induction assumption applied for i and $r+1$ we will infer that there must be some color $s_1 \in [r+1]$ such that $|E_{U_{q_1}}^i(U_{q_1}^{s_1})| = \Omega(q_1^i |E|)$.

We consider the set $W_1 \subseteq U$ of vertices such that every vertex $w \in W_1$ is contained in at least $\Omega(q_1^i |E|/|V|)$ edges from $E_{U_{q_1}}^i(U_{q_1}^{s_1})$. Note that if for some later round, say in the ℓ -th round for $\ell > 1$, some $w \in W_1$ appears in U_{q_ℓ} and w will be also colored with the same color s_1 , i.e., $w \in U_{q_\ell}^{s_1}$, then this will create edges in $E_U^{i+1}(U_{q_1}^{s_1} \cup U_{q_\ell}^{s_1}) \subseteq E_U^{i+1}(U_q^{s_1})$. We will infer from the (K, \mathbf{p}) -boundedness of \mathbf{H} that W_1 is of linear order, i.e., $|W_1| = \Omega(|V|)$.

In the second round we would like to repeat the same argument and obtain some $s_2 \in [r+1]$ and a set W_2 . However, in order to obtain *new* vertices w with a similar property as in the first round in the second round we will have to ensure that $W_2 \setminus W_1$ is large. For that we will apply the induction assumption to $U \setminus W_1$. In fact, this is the reason for allowing an arbitrary subset $U \subseteq V$ in Lemma 2.7. As a result we will ensure that $|W_2 \setminus W_1| \geq \lambda|V|$ for some fixed $\lambda > 0$ (only depending

on K , $i+1$, $r+1$, and ζ). In general we want to repeat this argument to obtain sets $W_1, \dots, W_{\ell-1}$ such that $|W_j \setminus \bigcup_{i=1}^{j-1} W_i| \geq \lambda|V|$ for every $j \in [\ell-1]$ and integers $s_1, \dots, s_{\ell-1} \in [r+1]$ such that $|E_U^i(U_{q_j}^{s_j})| = \Omega(q_j^i|E|)$ for every $j \in [\ell-1]$.

However, recall that we can only apply the induction assumption in the ℓ -th round as stated above for i and $r+1$ to $\hat{U} = U \setminus (W_1 \cup \dots \cup W_{\ell-1})$ if $H[\hat{U}]$ is still $(R(i, r+1), \zeta)$ -Ramsey. Hence, we have to stop the procedure described above when $H[\hat{U}]$ fails to be $(R(i, r+1), \zeta)$ -Ramsey. Clearly, after at most $1/\lambda < L$ rounds we arrive at the situation that $H[\hat{U}]$ is not $(R(i, r+1), \zeta)$ -Ramsey and then we will argue as follows.

Since $H[U]$ was $(R(i+1, r+1), \zeta)$ -Ramsey, by Fact 2.2 and the definition of $R(\cdot, \cdot)$ we then have that $H[W_1 \cup \dots \cup W_\ell]$ must be $((r+1) \cdot R(i+1, r), \zeta)$ -Ramsey. Consequently, there must be some $t \in [r+1]$ such that for

$$W^t = \bigcup_{j: s_j=t} W_j$$

we have that $H[W^t]$ is $(R(i+1, r), \zeta)$ -Ramsey. In other words, we are ready to apply the induction assumption with $i+1$ and r to W^t . By definition of W^t every vertex of W^t is contained in $\Omega(q^i|E|/|V|)$ edges from $E_U^i(\bigcup_{j=1}^{\ell-1} U_{q_j}^t)$ and, therefore, if a substantial fraction of the vertices $U_{q_\ell} \cap W^t$ will be assigned the color t , then we have $|E_U^{i+1}(\bigcup_{j=1}^{\ell} U_{q_j}^t)| = \Omega(q^{i+1}|E|)$, which is what we have to show. If, on the other hand, the number of vertices of color t in $U_{q_\ell} \cap W^t$ is negligible, then the induction assumption applied for $i+1$ and r to W^t will yield that $E_U^{i+1}(U_{q_\ell}^s)$ is large for some $s \in [r+1] \setminus \{t\}$.

In the proof we have to ensure that the error probabilities in the later rounds will counter the number of $(r+1)$ -colorings of the earlier rounds. This will require that all statements in the proof have to hold with very high probability. For that we will choose q_ℓ in such a way that q_ℓ is sufficiently larger than $\sum_{j=1}^{\ell-1} q_j$, but on the other hand, for every $\ell \in [L]$ we require that q_ℓ is of the same order of magnitude as q . We now give the details of this proof and first define all constants involved in the proof.

Constants. The number of rounds L will depend on the constant $\xi(i, r+1, \zeta)$, which is given by the induction assumption. More precisely, let

$$\xi' = \xi(i, r+1, \zeta), \quad b' = b(i, r+1, \zeta), \quad (16)$$

$$C' = C(i, r+1, \zeta), \quad \text{and} \quad n' = n_0(i, r+1, \zeta) \quad (17)$$

be given by the induction assumption applied for i , $r+1$, and ζ . We set

$$L = \left\lceil \frac{4^{2k+i+1}K}{(\xi')^2} + 1 \right\rceil. \quad (18)$$

Moreover, we will appeal to the induction assumption for $i+1$, r , and ζ and let

$$\xi^* = \xi(i+1, r, \zeta), \quad b^* = b(i+1, r, \zeta), \quad (19)$$

$$C^* = C(i+1, r, \zeta), \quad \text{and} \quad n^* = n_0(i+1, r, \zeta) \quad (20)$$

be the corresponding constants. Let $1/2 \geq \delta > 0$ be sufficiently small so that

$$\delta(3 - \log_2 \delta) \leq \min \left\{ \frac{b'}{2}, \frac{b^*}{2} \right\} \quad (21)$$

and set

$$\hat{b} = \frac{\delta^2 \zeta^2}{80K}. \quad (22)$$

Furthermore, we appeal to Proposition 2.10 with K , i , and

$$\eta = \frac{\delta \zeta^2}{8K} \quad (23)$$

and obtain

$$\bar{b} = b(K, i, \eta) \quad \text{and} \quad \bar{n} = n_0(K, i, \eta). \quad (24)$$

Next we set

$$b_{\min} = \min \left\{ \frac{\zeta^2}{25K}, \frac{\hat{b}}{2}, \frac{\bar{b}}{4} \right\} \quad \text{and} \quad B = \max \left\{ \frac{4 \log_2(r+1)}{b_{\min}}, \frac{32K}{\delta \zeta^2} \right\} + 1 \quad (25)$$

and finally let

$$\xi = \min \left\{ \frac{\delta \zeta^2 \xi' (1-\delta)^i}{32kK}, \xi^* (1-\delta)^{i+1} \right\} \cdot \left(\frac{B-1}{B^L-1} \right)^{i+1}, \quad (26)$$

$$b = \frac{b_{\min}}{3} \frac{B-1}{B^L-1}, \quad (27)$$

$$C = \frac{\max\{C', C^*\}}{1-\delta} \cdot \frac{B^L-1}{B-1}, \quad (28)$$

and let $n_0 \geq \max\{n', n^*, \bar{n}\}$ be sufficiently large such that the (K, \mathbf{p}) -boundedness of \mathbf{H} yields (8) for H_n for every $n \geq n_0$. Moreover, let n_0 be large enough so that

$$Cp_n \leq 1 \quad \text{and} \quad 2 + 2 \log_2(k) < \frac{1}{2} \bar{b} p_n |V_n|. \quad (29)$$

for all $n \geq n_0$. Note that such a choice of n_0 indeed exists since

$$p_n \rightarrow 0 \quad \text{and} \quad p_n |V_n| \geq (p_n^k |E_n|^k)^{1/k} \rightarrow \infty$$

by assumption of Lemma 2.7. Let $n \geq n_0$ and let $q \in (0, 1]$ be such that

$$Cp_n \leq q.$$

Finally, appealing to the assumptions of Lemma 2.7, let $U \subseteq V_n$ be such that

$$H_n[U] \text{ is } (R(i+1, r+1), \zeta)\text{-Ramsey.} \quad (30)$$

From now on we drop the subscript n for a simpler notation. We have to show that $H[U_q]$ is $(i+1, r+1, \xi, q, U)$ -Ramsey with very high probability.

As mentioned above we will expose U_q in L rounds, where the elements in the ℓ -th round will be included with probability q_ℓ . For that let q_1 be the solution of the equation

$$1 - q = \prod_{\ell=1}^L (1 - B^{\ell-1} q_1)$$

and set

$$q_\ell = B^{\ell-1} q_1$$

for every $\ell = 2, \dots, L$. We have

$$q_1 \geq \frac{q}{\sum_{\ell=1}^L B^{\ell-1}} = q \frac{B-1}{B^L-1}, \quad (31)$$

Due to the choice of C in (28) and $q \geq Cp_n$ we have

$$q_L > \dots > q_1 \geq \frac{\max\{C', C^*\}}{1 - \delta} \cdot p_n. \quad (32)$$

The choice of B in (25) yields for every $\ell = 2, \dots, L$

$$\begin{aligned} \sum_{j=1}^{\ell-1} q_j &= q_1 \sum_{j=0}^{\ell-2} B^j = q_1 \frac{B^{\ell-1} - 1}{B - 1} \\ &\stackrel{(25)}{\leq} \min \left\{ \frac{b_{\min}}{4 \log_2(r+1)}, \frac{\delta \zeta^2}{32K} \right\} B^{\ell-1} q_1 \leq \frac{\delta \zeta^2}{32K} q_\ell. \end{aligned} \quad (33)$$

For later reference we note that due to the choice of constants in (16), (17), (19), (20), and (32) the following statements hold by induction assumption. For every subset $S \subseteq U$ and $\ell \in [L]$ we have

$$\begin{aligned} H[S] &\text{ is } ((R(i, r+1), \zeta)\text{-Ramsey}) \\ &\Rightarrow \mathbb{P}(H[S_{(1-\delta)q_\ell}]) \text{ is } (i, r+1, \xi', (1-\delta)q_\ell, S)\text{-Ramsey} \geq 1 - 2^{-b'(1-\delta)q_\ell|S|} \end{aligned} \quad (34)$$

and

$$\begin{aligned} H[S] &\text{ is } ((R(i+1, r), \zeta)\text{-Ramsey}) \\ &\Rightarrow \mathbb{P}(H[S_{(1-\delta)q_\ell}]) \text{ is } (i+1, r, \xi^*, (1-\delta)q_\ell, S)\text{-Ramsey} \geq 1 - 2^{-b^*(1-\delta)q_\ell|S|}. \end{aligned} \quad (35)$$

Details of the induction step. For our analysis we require some notation. Recall that the random subsets of the L rounds are denoted by U_{q_1}, \dots, U_{q_L} . Let $U_q = \bigcup_{\ell \in [L]} U_{q_\ell}$. Moreover, we let $\chi_\ell: U_{q_\ell} \rightarrow [r+1]$ be a partition of U_{q_ℓ} and we denote the partition classes by $U_{q_\ell}^1 \dot{\cup} \dots \dot{\cup} U_{q_\ell}^{r+1}$, i.e., for every $s \in [r+1]$ and $\ell \in [L]$

$$U_{q_\ell}^s = \chi_\ell^{-1}(s).$$

Since the sets U_{q_j} and $U_{q_{j'}}$ may not be disjoint we will require that the partitions χ_j and $\chi_{j'}$ are consistent, i.e., those functions agree on $U_{q_j} \cap U_{q_{j'}}$.

In the proof those vertices of U which are contained in many edges in $E_U^i(U_{q_\ell}^s)$ play a crucial role (recall the vertices of W_1, W_2, \dots from the outline of the proof). For that we define for every $\ell \in [L]$ and $s \in [r+1]$ the set

$$W_\ell^s = \left\{ u \in U : \deg_{i,U}(u, U_{q_\ell}^s) \geq \frac{\xi'(1-\delta)^i}{2} q_\ell^i \frac{|E|}{|V|} \right\}, \quad (36)$$

where

$$\deg_{i,U}(u, U_{q_\ell}^s) = \left| \{ e \in E(U) : u \in e \text{ and } |(e \setminus \{u\}) \cap U_{q_\ell}^s| \geq i \} \right| \quad (37)$$

is the degree of the vertex u in the edge set $E_U^i(U_{q_\ell}^s)$. It follows directly from the definitions in (6) and (37) that

$$\deg_{i,U}(u, U_{q_\ell}^s) \leq \deg_{i,V}(u, U_{q_\ell}^s) \leq \deg_{i,V}(u, U_{q_\ell}) = \deg_i(u, U_{q_\ell}). \quad (38)$$

Finally, for $\ell \in [L]$ we denote by W_ℓ the set of vertices with large degree in some partition class, i.e.,

$$W_\ell = \bigcup_{s \in [r+1]} W_\ell^s.$$

The following claim, roughly speaking, says that given subsets $U_{q_1}, \dots, U_{q_{\ell-1}}$ of U and consistent partitions $\chi_j: U_{q_j} \rightarrow [r+1]$ for $j \in [\ell-1]$ the random set U_{q_ℓ} satisfies the following with very high probability:

For any $(r+1)$ -partition of U_{q_ℓ} either W_ℓ contains $\Omega(|V|)$ new elements disjoint from $W_1, \dots, W_{\ell-1}$ (see (ii.a) below) or there exists some $s \in [r+1]$ such that $E_U^{i+1}(\bigcup_{j=1}^\ell U_{q_j}^s)$ will be large (see (ii.b) below).

Claim 1. *Let $\ell \in [L]$, let subsets $U_{q_1}, \dots, U_{q_{\ell-1}}$ of U satisfy*

$$\left| \bigcup_{j=1}^{\ell-1} U_{q_j} \right| \leq 2 \sum_{j=1}^{\ell-1} q_j |U| \quad (39)$$

and let consistent $(r+1)$ -partitions $\chi_j: U_{q_j} \rightarrow [r+1]$ for $j \in [\ell-1]$ be given.

With probability at least

$$1 - 2^{-b_{\min q_\ell} |V|}$$

the random set U_{q_ℓ} satisfies the following:

$$(i) \quad |U_{q_\ell}| \leq 2q_\ell |U|$$

and for every partition $\chi_\ell: U_{q_\ell} \rightarrow [r+1]$ one of the following holds

(ii.a)

$$\left| W_\ell \setminus \bigcup_{j=1}^{\ell-1} W_j \right| \geq \frac{(\xi')^2}{4^{2k+i+1} K} |V|,$$

(ii.b) there exists an $s \in [r+1]$ such that

$$\left| E_U^{i+1} \left(\bigcup_{j=1}^\ell U_{q_j}^s \right) \right| \geq \xi q^{i+1} |E|.$$

We first deduce Lemma 2.7 from Claim 1. Let \mathcal{A} denote the event that $H[U_q]$ is $(i+1, r+1, \xi, q, U)$ -Ramsey and for given $\mathbf{u}_q(\ell-1) = (U_{q_1}, \dots, U_{q_{\ell-1}})$ and for given $\chi(\ell-1) = (\chi_1, \dots, \chi_{\ell-1})$ with $\chi_j: U_{q_j} \rightarrow [r+1]$ being consistent for $j = 1, \dots, \ell-1$, let $\mathcal{B}_{\chi(\ell-1)}$ be the event that the conclusion of Claim 1 holds. In other words, Claim 1 states that for the randomly chosen set U_{q_ℓ} we have

$$\mathbb{P}(\mathcal{B}_{\chi(\ell-1)} \mid \mathbf{u}_q(\ell-1)) \geq 1 - 2^{-b_{\min q_\ell} |V|} \quad (40)$$

for any choice of $\mathbf{u}_q(\ell-1)$ and any consistent family of partitions $\chi(\ell-1)$. Note that $\mathbf{u}_q(0)$ and $\chi(0)$ are vectors of length 0. For $\ell = 1$ we set

$$\mathbb{P}(\mathcal{B}_{\chi(0)} \mid \mathbf{u}_q(0)) = \mathbb{P}(\mathcal{B}_{\chi(0)}),$$

where $\mathcal{B}_{\chi(0)}$ denotes the event that

$$(i) \quad |U_{q_1}| \leq 2q_1 |U|$$

and for every partition $\chi_1: U_{q_1} \rightarrow [r+1]$ one of the following holds

$$(ii.a) \quad |W_1| \geq (\xi')^2 |V| / (4^{2k+i+1} K),$$

$$(ii.b) \quad \text{there exists an } s \in [r+1] \text{ such that } |E_U^{i+1}(U_{q_1}^s)| \geq \xi q^{i+1} |E|.$$

Again Claim 1 states that

$$\mathbb{P}(\mathcal{B}_{\chi(0)}) \geq 1 - 2^{-b_{\min q_1} |V|}.$$

Note that if $\mathcal{B}_{\chi(\ell)}$ holds for every $\ell \in [L]$, then alternative (ii.a) cannot always occur since

$$\frac{4^{2k+i+1} K}{(\xi')^2} \stackrel{(18)}{<} L.$$

Hence, if $\mathcal{B}_{\chi(\ell)}$ holds for every $\ell \in [L]$, then conclusion (ii.b) in Claim 1 must hold for some $\ell \in [L]$. Consequently, for every partition of $\bigcup_{j=1}^\ell U_{q_j}$ into $r+1$ classes there exists some $s \in [r+1]$ such that $|E_U^{i+1}(\bigcup_{j=1}^\ell U_{q_j}^s)| \geq \xi q^{i+1} |E|$. In other words, since $\bigcup_{j=1}^\ell U_{q_j} \subseteq U_q$, the hypergraph $H[U_q]$ is $(i+1, r+1, \xi, q, U)$ -Ramsey

and event \mathcal{A} occurs. Below we will verify that this happens with a sufficiently high probability

$$\mathbb{P}(\neg \mathcal{A}) \leq \sum_{\ell=1}^L \sum_{\mathbf{u}_q(\ell-1)} \sum_{\chi(\ell-1)} \mathbb{P}(\neg \mathcal{B}_{\chi(\ell-1)} \mid \mathbf{u}_q(\ell-1)) \mathbb{P}(\mathbf{u}_q(\ell-1)),$$

where the middle sum runs over all choices of $\mathbf{u}_q(\ell-1) = (U_{q_1}, \dots, U_{q_{\ell-1}})$ satisfying (39) and the inner sum runs over all partitions $\chi(\ell-1)$ of $\mathbf{u}_q(\ell-1)$. Note that there are at most $(r+1)^{2|V| \sum_{j=1}^{\ell-1} q_j}$ such partitions of $\mathbf{u}_q(\ell-1)$. Therefore, (40) yields

$$\mathbb{P}(\neg \mathcal{A}) \leq \sum_{\ell=1}^L (r+1)^{2|V| \sum_{j=1}^{\ell-1} q_j} \cdot 2^{-b_{\min} q_{\ell} |V|}.$$

Since $\sum_{j=1}^{\ell-1} q_j \leq \frac{b_{\min}}{4 \log_2(r+1)} q_{\ell}$ by (33) and $q_1 \leq q_{\ell}$ we have

$$\mathbb{P}(\neg \mathcal{A}) \leq L \cdot 2^{-b_{\min} q_1 |V|/2} \stackrel{(27), (31)}{\leq} 2^{-b q |V|}$$

where the last inequality holds for sufficiently large n . This concludes the proof of Lemma 2.7 and it is left to verify Claim 1. \square

Proof of Claim 1. Let $\ell \in [L]$, $U_{q_1}, \dots, U_{q_{\ell-1}}$ and partitions $\chi_1, \dots, \chi_{\ell-1}$ be given. Note that this defines the sets W_j^s for $j \in [\ell-1]$ and $s \in [r+1]$ as well. We first observe that property (i) of Claim 1 holds with high probability. Since (30) holds, we have $e(U) \geq \zeta |E|$ and, therefore, the (K, \mathbf{p}) -boundedness of \mathbf{H} combined with Fact 2.12 yields

$$|U| \geq \frac{\zeta^2}{4K} |V|.$$

Hence, Chernoff's inequality yields

$$\mathbb{P}(|U_{q_{\ell}}| \geq 2q_{\ell}|U|) \leq 2 \exp(-q_{\ell}|U|/3) \stackrel{(25)}{\leq} 2^{-2b_{\min} q_{\ell} |V|}. \quad (41)$$

for sufficiently large n .

For the rest of the proof we distinguish two cases depending on the structure of the complement of $\bigcup_{j=1}^{\ell-1} W_j$ in U . For that we set

$$\hat{U} = U \setminus \bigcup_{j=1}^{\ell-1} W_j.$$

Since $H[U]$ is $(R(i+1, r+1), \zeta)$ -Ramsey by assumption it follows from the definition of the function $R(\cdot, \cdot)$ in Definition 2.3 combined with Fact 2.2 that either $H[\hat{U}]$ is $(R(i, r+1), \zeta)$ -Ramsey or $H[U \setminus \hat{U}]$ is $((r+1)R(i+1, r), \zeta)$ -Ramsey. In the first case we will use the induction assumption for i and $r+1$ (see Case 1 below).

In the second case we recall that $U \setminus \hat{U} = \bigcup_{j \in [\ell-1]} W_j$ and $W_j = \bigcup_{s \in [r+1]} W_j^s$, which leads to a partition of $U \setminus \hat{U}$ into $r+1$ classes. Again, Fact 2.2 yields that one of those classes is $(R(i+1, r), \zeta)$ -Ramsey and for that set we can apply the induction assumption for $i+1$ and r in this case (see Case 2 below).

Case 1 ($H[\hat{U}]$ is $(R(i, r+1), \zeta)$ -Ramsey). In this case we show that conclusion (ii.a) of Claim 1 holds. Note that it follows from the Ramseyness of $H[\hat{U}]$ that $e(\hat{U}) \geq \zeta|E|$. Therefore, the (K, \mathbf{p}) -boundedness of \mathbf{H} combined with Fact 2.12 yields

$$|\hat{U}| \geq \frac{\zeta^2}{4K}|V|. \quad (42)$$

In this case we appeal to the induction assumption for i and $r+1$ and focus on the restriction to $H[\hat{U}]$ (cf. (34)). In fact, the induction assumption for i and $r+1$ yields

$$\begin{aligned} \mathbb{P}\left(H[\hat{U}_{(1-\delta)q_\ell}] \text{ is } (i, r+1, \xi', (1-\delta)q_\ell, \hat{U})\text{-Ramsey}\right) &\geq 1 - 2^{-b'(1-\delta)q_\ell|\hat{U}|} \\ &\geq 1 - 2^{-(b'/2)q_\ell|\hat{U}|}. \end{aligned}$$

Since being $(i, r+1, \xi', (1-\delta)q_\ell, \hat{U})$ -Ramsey is closed under supersets, in view of (21) we infer from Proposition 2.11 that with probability at least

$$1 - 2^{-\delta^2 q_\ell |\hat{U}|/20} \stackrel{(22), (42)}{\geq} 1 - 2^{-\hat{b}q_\ell |V|} \quad (43)$$

the random set \hat{U}_{q_ℓ} has the property that $H[\hat{U}_{q_\ell} \setminus X]$ is $(i, r+1, \xi', (1-\delta)q_\ell, \hat{U})$ -Ramsey for every $X \subseteq \hat{U}_{q_\ell}$ with $|X| \leq \delta q_\ell |\hat{U}|/2$. In other words, for every such set X and every partition $\hat{U}_{q_\ell}^1 \dot{\cup} \dots \dot{\cup} \hat{U}_{q_\ell}^{r+1}$ of \hat{U}_{q_ℓ} there exists an $s \in [r+1]$ such that

$$\left|E_{\hat{U}}^i(\hat{U}_{q_\ell}^s \setminus X)\right| \geq \xi'(1-\delta)^i q_\ell^i |E|. \quad (44)$$

Recalling the definition

$$E_{\hat{U}}^i(\hat{U}_{q_\ell}^s \setminus X) = \{e \in E(\hat{U}) : |e \cap (\hat{U}_{q_\ell}^s \setminus X)| \geq i\}.$$

and recalling that $i+1 \leq k$ and the definition in (37) we note that

$$\begin{aligned} \left|E_{\hat{U}}^i(\hat{U}_{q_\ell}^s \setminus X)\right| &\leq \sum_{u \in \hat{U}} \left|\{e \in E(\hat{U}) : u \in e \text{ and } (e \setminus \{u\}) \cap (\hat{U}_{q_\ell}^s \setminus X) \geq i\}\right| \\ &= \sum_{u \in \hat{U}} \deg_{i, \hat{U}}(u, \hat{U}_{q_\ell}^s \setminus X). \end{aligned}$$

Consequently, (44) implies

$$\sum_{u \in \hat{U}} \deg_{i, \hat{U}}(u, \hat{U}_{q_\ell}^s \setminus X) \geq \xi'(1-\delta)^i q_\ell^i |E|. \quad (45)$$

Moreover, due to Proposition 2.10 and the choice of constants in (24) with probability at least

$$1 - 2^{-\bar{b}q_\ell |V| + 2 + 2 \log_2(k)} \stackrel{(29), (32)}{\geq} 1 - 2^{-\bar{b}q_\ell |V|/2} \quad (46)$$

there exists a set $X \subseteq V_{q_\ell}$ of size at most

$$|X| \leq \eta q_\ell |V| \stackrel{(23), (42)}{\leq} \delta q_\ell |\hat{U}|/2$$

such that

$$\sum_{u \in \hat{U}} \deg_{i, \hat{U}}^2(u, \hat{U}_{q_\ell}^s \setminus X) \leq \sum_{v \in V} \deg_i^2(v, V_{q_\ell} \setminus X) \leq 16^k K q_\ell^{2i} \frac{|E|^2}{|V|}, \quad (47)$$

where the last inequality follows in a similar way as (38). Let

$$\hat{W}_\ell^s = \left\{ u \in \hat{U} : \deg_{i, \hat{U}}(u, \hat{U}_{q_\ell}^s) \geq \frac{\xi'(1-\delta)^i}{2} q_\ell^i \frac{|E|}{|V|} \right\}. \quad (48)$$

Note that by definition $\hat{W}_\ell^s \subseteq W_\ell$ and \hat{W}_ℓ^s is disjoint from $\bigcup_{j=1}^{\ell-1} W_j$.

Summarizing, due to (43) and (46) with probability at least

$$1 - 2^{-\hat{b}q_\ell|V|} - 2^{-\bar{b}q_\ell|V|/2}$$

the random set \hat{U}_{q_ℓ} satisfies properties (45) and (47) for every partition of \hat{U}_{q_ℓ} and we infer by the Cauchy-Schwarz inequality that

$$\begin{aligned} 16^k K q_\ell^{2i} \frac{|E|^2}{|V|} &\stackrel{(47)}{\geq} \sum_{u \in \hat{U}} \deg_{i, \hat{U}}^2(u, \hat{U}_{q_\ell}^s \setminus X) \geq \sum_{u \in \hat{W}_\ell^s} \deg_{i, \hat{U}}^2(u, \hat{U}_{q_\ell}^s \setminus X) \\ &\geq \frac{1}{|\hat{W}_\ell^s|} \left(\sum_{u \in \hat{W}_\ell^s} \deg_{i, \hat{U}}(u, \hat{U}_{q_\ell}^s \setminus X) \right)^2. \end{aligned}$$

Moreover,

$$\begin{aligned} &\sum_{u \in \hat{W}_\ell^s} \deg_{i, \hat{U}}(u, \hat{U}_{q_\ell}^s \setminus X) \\ &= \sum_{u \in \hat{U}} \deg_{i, \hat{U}}(u, \hat{U}_{q_\ell}^s \setminus X) - \sum_{u \in \hat{U} \setminus \hat{W}_\ell^s} \deg_{i, \hat{U}}(u, \hat{U}_{q_\ell}^s \setminus X) \\ &\stackrel{(45), (48)}{\geq} \frac{\xi'}{2} (1-\delta)^i q_\ell^i |E|. \end{aligned}$$

Combining the last two estimates and $\delta \leq 1/2$ we obtain

$$|\hat{W}_\ell^s| \geq \frac{(\xi')^2}{4^{2k+i+1} K} |V|.$$

In other words, in this case property (ii.a) holds with probability at least

$$1 - 2^{-\hat{b}q_\ell|V|} - 2^{-\bar{b}q_\ell|V|/2}$$

and in view of (41) and the choice of b_{\min} in (25) for sufficiently large n this yields the proof of Claim 1 in this case.

Case 2 ($H[\hat{U}]$ is not $(R(i, r+1), \zeta)$ -Ramsey). In this case we show that conclusion (ii.b) of Claim 1 holds. Due to (30), the assumption of this case combined with Fact 2.2, and the definition of the function $R(\cdot, \cdot)$ in Definition 2.3 we have

$$H[U \setminus \hat{U}] \text{ is } ((r+1) \cdot R(i+1, r), \zeta)\text{-Ramsey.} \quad (49)$$

Recall that

$$U \setminus \hat{U} = \bigcup_{j=1}^{\ell-1} W_j = \bigcup_{j=1}^{\ell-1} \bigcup_{s=1}^{r+1} W_j^s.$$

For $s \in [r+1]$ let

$$W^s = \{w \in U \setminus \hat{U} : w \in W_j^s \text{ for some } j \in [\ell-1]\} = \bigcup_{j=1}^{\ell-1} W_j^s.$$

Clearly,

$$W^1 \cup \dots \cup W^{r+1} = U \setminus \hat{U}.$$

We remark that this $W^1 \cup \dots \cup W^{r+1}$ is not necessarily a partition of $U \setminus \hat{U}$. However, it follows from Fact 2.2 and (49) that there exists a $t \in [r+1]$ such that

$$H[W^t] \text{ is } (R(i+1, r), \zeta)\text{-Ramsey.} \quad (50)$$

In particular, (50) implies that $e(W^t) \geq \zeta|E|$. Moreover, invoking the (K, \mathbf{p}) -boundedness of \mathbf{H} (see Definition 2.4) for $i = k-1$ and $q' = 1$ shows that the assumptions of Fact 2.12 are satisfied by H and W^t . Consequently, we have

$$|W^t| \geq \frac{\zeta^2}{4K} |V|. \quad (51)$$

Moreover, due to (50) we can apply the induction assumption for $i+1$ and r to $H[W^t]$ (cf. (35)). This yields

$$\begin{aligned} \mathbb{P}\left(H[W_{(1-\delta)q_\ell}^t] \text{ is } (i+1, r, \xi^*, (1-\delta)q_\ell, W^t)\text{-Ramsey}\right) &\geq 1 - 2^{-b^*(1-\delta)q_\ell|W^t|} \\ &\geq 1 - 2^{-(b^*/2)q_\ell|W^t|}. \end{aligned}$$

Similarly as in the former case, we infer from Proposition 2.11 that with probability at least

$$1 - 2^{-\delta^2 q_\ell |W^t|/20} \stackrel{(22), (51)}{\geq} 1 - 2^{-\hat{b}q_\ell |V|} \quad (52)$$

the random set $W_{q_\ell}^t$ has the property that

$$H[W_{q_\ell}^t \setminus X] \text{ is } (i+1, r, \xi^*, (1-\delta)q_\ell, W^t)\text{-Ramsey} \quad (53)$$

for every $X \subseteq W_{q_\ell}^t$ with $|X| \leq \delta q_\ell |W^t|/2$.

Note that in the statement above only partitions into r classes are considered, while we have to deal with $(r+1)$ -partitions here. Let $\chi_\ell: U_{q_\ell} \rightarrow [r+1]$ be an arbitrary partition. Depending on the cardinality of $\chi_\ell^{-1}(t) \cap W_{q_\ell}^t$ we will argue in two different ways. In fact, if

$$|\chi_\ell^{-1}(t) \cap W_{q_\ell}^t| \geq \frac{\delta}{4} q_\ell |W^t|, \quad (54)$$

then we infer from the fact that $W^t = \bigcup_{j=1}^{\ell-1} W_j^t$ and (36)

$$\begin{aligned} \left| E_U^{i+1} \left(\bigcup_{j=1}^\ell U_{q_j}^t \right) \right| &= \left| \left\{ e \in E(U) : \left| e \cap \bigcup_{j=1}^\ell U_{q_j}^t \right| \geq i+1 \right\} \right| \\ &\geq \frac{1}{k} \sum_{u \in \chi_\ell^{-1}(t) \cap W_{q_\ell}^t} \deg_{i,U} \left(u, \bigcup_{j=1}^\ell U_{q_j}^t \right) \\ &\geq \frac{1}{k} \sum_{u \in \chi_\ell^{-1}(t) \cap W_{q_\ell}^t} \max_{j \in [\ell-1]} \deg_{i,U} \left(u, U_{q_j}^t \right) \\ &\geq \frac{1}{k} \cdot \frac{\delta}{4} q_\ell |W^t| \cdot \frac{\xi^t(1-\delta)^i}{2} q_1^i \frac{|E|}{|V|}. \end{aligned}$$

Hence, if χ_ℓ satisfies (54), then since $q_1 \leq q_\ell$

$$\begin{aligned} \left| E_U^{i+1} \left(\bigcup_{j=1}^\ell U_{q_j}^t \right) \right| &\stackrel{(51)}{\geq} \frac{\delta \zeta^2 \xi' (1-\delta)^i}{32kK} q_1^{i+1} |E| \\ &\stackrel{(31)}{\geq} \frac{\delta \zeta^2 \xi' (1-\delta)^i}{32kK} \left(\frac{B-1}{B^{\ell-1}-1} \right)^{i+1} q^{i+1} |E| \stackrel{(26)}{\geq} \xi q^{i+1} |E|. \end{aligned}$$

In other words, if χ_ℓ satisfies (54), then the resulting partition satisfies conclusion (ii.b) of Claim 1.

If, on the other hand, (54) does not hold, then setting

$$X = (\chi_\ell^{-1}(t) \cap W_{q_\ell}^t) \cup \bigcup_{j=1}^{\ell-1} U_{q_j}$$

we have

$$\begin{aligned} |X| &\stackrel{(39)}{\leq} \frac{\delta}{4} q_\ell |W^t| + 2|U| \sum_{j=1}^{\ell-1} q_j \leq \frac{\delta}{4} q_\ell |W^t| + 2|V| \sum_{j=1}^{\ell-1} q_j \\ &\leq \frac{\delta}{4} q_\ell |W^t| + \frac{8K}{\zeta^2} |W^t| \sum_{j=1}^{\ell-1} q_j \stackrel{(33)}{\leq} \frac{\delta}{2} q_\ell |W^t|. \end{aligned}$$

Since $W^t \subseteq U$ and $\chi_\ell^{-1}(s) \cap (W_{q_\ell}^t \setminus X) \subseteq \bigcup_{j=1}^{\ell-1} U_{q_j}^s$, it follows from (53) that there exists some $s \in [r+1] \setminus \{t\}$ such that

$$\begin{aligned} \left| E_U^{i+1} \left(\bigcup_{j=1}^\ell U_{q_j}^s \right) \right| &\geq \left| E_{W^t}^{i+1} (\chi_\ell^{-1}(s) \cap (W_{q_\ell}^t \setminus X)) \right| \\ &\geq \xi^* (1-\delta)^{i+1} q_\ell^{i+1} |E| \stackrel{(31),(26)}{\geq} \xi q^{i+1} |E|, \end{aligned}$$

which again implies conclusion (ii.b) of Claim 1.

Summarizing, it follows from (52) that in this case conclusion (ii.b) of Claim 1 holds for any $\chi_\ell: U_{q_\ell} \rightarrow [r+1]$ with probability at least $1 - 2^{-\hat{b}q_\ell|V|}$. This combined with (41) concludes the proof of Claim 1, since

$$1 - 2^{-\hat{b}q_\ell|V|} - 2^{-2b_{\min}q_\ell|V|} \stackrel{(25)}{\geq} 1 - 2^{-b_{\min}q_\ell|V|}$$

for sufficiently large n . □

3. PROOF OF THE NEW RESULTS

In this section we deduce Theorem 1.1 and Theorem 1.2 from Theorem 2.5.

Proof of Theorem 1.1. Note that due to (2) it suffices to verify the 1-statement of Theorem 1.1 and we will show that this follows from Theorem 2.5. Let A be an irredundant, partition regular $(\ell \times k)$ -integer matrix. It follows from elementary linear algebra that $m_A = m_{A'}$, whenever $\mathcal{L}(A)$ and $\mathcal{L}(A')$ have the same set of solutions. Hence, without loss of generality we may assume that A has full rank (i.e., $\text{rank}(A) = \ell$), since we are only interested in the set of solutions $\mathcal{L}(A)$. As mentioned in the introduction, it follows from Rado's characterization of partition regular matrices that $k \geq \ell + 2$.

For every $n \in \mathbb{N}$ we consider the k -uniform hypergraph $H_n = ([n], E_n)$ where the edges of H_n are the k -sets $\{x_1, \dots, x_k\} \subseteq [n]$ such that (for some ordering) the

vector (x_1, \dots, x_k) is a solution of $\mathcal{L}(A)$. (Note that we disregard solutions of $\mathcal{L}(A)$ which consist of less than k distinct integers). Let $p_n = n^{-1/m_A}$ (cf. (1)).

The conclusion of Theorem 2.5 yields Theorem 1.1, since (by definition) $H[V_{n,q_n}]$ is r -Ramsey if and only if $[n]_{q_n} \rightarrow (A)_r$ and we have to show that \mathbf{H} and \mathbf{p} satisfy the assumptions of Theorem 2.5. This means, we have to verify the following

- (a) $p_n^k |E_n| \rightarrow \infty$ as $n \rightarrow \infty$,
- (b) for every $R \in \mathbb{N}$ exists some $\zeta > 0$ such that \mathbf{H} is (R, ζ) -Ramsey, and
- (c) \mathbf{H} is (K, \mathbf{p}) -bounded for some $K \geq 1$.

It was shown in [30, Proposition 2.2 (ii)] that $m_A \geq k - 1$ and due to Rado's characterization we have $k - \ell \geq 2$, which yields $|E_n| = \Omega(n^2)$. Therefore, we have

$$p^k |E_n| = \Omega(n^{-k/(k-1)} \cdot n^2) = \Omega(n^{\frac{k-2}{k-1}}).$$

Moreover, it follows from [5, Theorem 1] that for every $R \in \mathbb{N}$ there exists some $\zeta > 0$ for which \mathbf{H} is (R, ζ) -Ramsey. Consequently, it suffices to verify that \mathbf{H} is (K, \mathbf{p}) -bounded for some $K \geq 1$. For $i \in [k - 1]$ and $q \geq n^{-1/m_A}$ we have to show that

$$\mu_i(H_n, q) = O\left(q^{2i} \frac{|E_n|^2}{n}\right).$$

Recalling the definition of $\mu_i(H_n, q)$ in (7) and $H_n = ([n], E_n)$ we have

$$\mu_i(H_n, q) = \mathbb{E} \left[\sum_{x \in [n]} \deg_i^2(x, V_{n,q}) \right] = \sum_{x \in [n]} \mathbb{E} [\deg_i^2(x, V_{n,q})]. \quad (55)$$

Note that $\mathbb{E} [\deg_i^2(x, V_{n,q})]$ is the expected number of pairs $(X, Y) \in [n]^k \times [n]^k$ such that

- (i) $x \in X \cap Y$,
- (ii) $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$ are solutions of $\mathcal{L}(A)$, where

$$A\mathbf{x} = A\mathbf{y} = \mathbf{0}$$

- for $\mathbf{x} = (x_1, \dots, x_k)^t$ and $\mathbf{y} = (y_1, \dots, y_k)^t$, and
- (iii) $|X \cap ([n]_q \setminus \{x\})| \geq i$ and $|Y \cap ([n]_q \setminus \{x\})| \geq i$.

For fixed x and (X, Y) let $w \geq 1$ be the largest integer such that there exist indices i_1, \dots, i_w and j_1, \dots, j_w for which

$$x_{i_1} = y_{j_1}, \dots, x_{i_w} = y_{j_w}. \quad (56)$$

Consequently,

$$x \in \{x_{i_1}, \dots, x_{i_w}\} = \{y_{j_1}, \dots, y_{j_w}\} \quad (57)$$

Set $W_1 = \{i_1, \dots, i_w\}$ and $W_2 = \{j_1, \dots, j_w\}$.

For fixed sets $W_1, W_2 \subseteq [k]$ we are going to describe all $(2k - w)$ -tuples $X \cup Y$ satisfying (ii) and (56). To this end consider the $2\ell \times (2k - w)$ matrix B , which arises from two copies A_1 and A_2 of A with permuted columns. Recall that for a set of columns W of a matrix A we denote by A_W the matrix obtained from A by restricting to the columns indexed by W . We set $A_1 = (A_{\overline{W}_1} \mid A_{W_1})$ and $A_2 = (A_{W_2} \mid A_{\overline{W}_2})$ where for every $\alpha = 1, \dots, w$ the column of A_{W_1} which is indexed by i_α aligns with that column of A_{W_2} which is indexed by j_α . Then let

$$B = \left(\begin{array}{c|c|c} A_{\overline{W}_1} & A_{W_1} & \mathbf{0} \\ \hline \mathbf{0} & A_{W_2} & A_{\overline{W}_2} \end{array} \right).$$

Without loss of generality we may assume that $\text{rank}(A_{\overline{W}_1}) \geq \text{rank}(A_{\overline{W}_2})$ and, therefore,

$$\text{rank}(B) \geq \text{rank}(A) + \text{rank}(A_{\overline{W}_1}).$$

Clearly, the number of $(2k-w)$ -tuples $X \cup Y$ satisfying (ii) and (56) equals the number of solutions of the homogeneous system given by B , which is $\Theta(n^{2k-w-\text{rank}(B)})$. Since A is an irredundant, partition regular matrix, it follows from [30, Proposition 2.2 (i)] that $\text{rank}(A') = \text{rank}(A)$ for every matrix A' obtained from A by removing one column. Consequently, any matrix B' obtained from B by removing one of the middle columns (i.e., one of the w columns of B which consist of a column of A_{W_1} and a column of A_{W_2}) satisfies

$$\text{rank}(B') \geq \text{rank}(A) + \text{rank}(A_{\overline{W}_1}) = \ell + \text{rank}(A_{\overline{W}_1}).$$

Therefore, it follows from (57) that the number of such $(2k-w)$ -tuples that also satisfy condition (i) for some fixed $x \in [n]$ is at most

$$O(n^{2k-w-1-\ell-\text{rank}(A_{\overline{W}_1})}). \quad (58)$$

Finally, we estimate the probability that a $(2k-w)$ -tuple $X \cup Y$ satisfying (i), (ii), and (56) also satisfies (iii). Let $j = |X \cap Y \cap ([n]_q \setminus \{x\})|$. Since $j \leq w-1$ and $q \leq 1$ this probability is bounded by

$$\sum_{j=0}^{w-1} q^{2i-j} = O(q^{2i-w+1}).$$

In view of (58) we obtain

$$\begin{aligned} & \sum_{x \in [n]} \mathbb{E}[\text{deg}_i^2(x, V_{n,q})] \\ &= \sum_{x \in [n]} \sum_{w=1}^k \sum_{\substack{W_1, W_2 \subseteq [k] \\ |W_1|=|W_2|=w}} O(n^{2k-w-1-\ell-\text{rank}(A_{\overline{W}_1})} q^{2i-w+1}). \end{aligned} \quad (59)$$

Note that if $w=1$, then again due to [30, Proposition 2.2 (i)] we have $\text{rank}(A_{\overline{W}_1}) = \ell$ and, therefore, the contribution of those terms satisfies

$$\sum_{x \in [n]} \sum_{\substack{W_1, W_2 \subseteq [k] \\ |W_1|=|W_2|=1}} O(n^{2k-2\ell-2} q^{2i}) = O(n^{2k-2\ell-1} q^{2i}) = O\left(q^{2i} \frac{|E_n|^2}{n}\right). \quad (60)$$

For $w \geq 2$ and $W_1 \subseteq [k]$ with $|W_1| = w$ we obtain from the definition of m_A in (1) and $q \geq n^{-1/m_A}$ that

$$q^{w-1} \geq n^{-w+1-\text{rank}(A_{\overline{W}_1})+\ell}.$$

Consequently,

$$\begin{aligned}
& \sum_{x \in [n]} \sum_{w=2}^k \sum_{\substack{W_1, W_2 \subseteq [k] \\ |W_1|=|W_2|=w}} O(n^{2k-w-1-\ell-\text{rank}(A_{\overline{W_1}})} q^{2i-w+1}) \\
&= \sum_{x \in [n]} \sum_{w=2}^k \sum_{\substack{W_1, W_2 \subseteq [k] \\ |W_1|=|W_2|=w}} O(n^{2k-2-2\ell} q^{2i}) \\
&= O(n^{2k-2\ell-1} q^{2i}) = O\left(q^{2i} \frac{|E_n|^2}{n}\right). \quad (61)
\end{aligned}$$

Finally, combining (55), (59), (60), and (61) we obtain

$$\mu_i(H_n, q) = O\left(q^{2i} \frac{|E_n|^2}{n}\right),$$

which concludes the proof of Theorem 1.1. \square

Proof of Theorem 1.2. Let F be an ℓ -uniform hypergraph with $k = e(F) \geq \Delta(F) \geq 2$ edges. For every $n \in \mathbb{N}$ we consider the k -uniform hypergraph $H_n = (V_n, E_n)$, where $V_n = E(K_n^{(\ell)})$ and where the edges of H_n correspond to (unlabeled) copies of F in $K_n^{(\ell)}$. Furthermore, let $p_n = n^{-1/m_F}$ (cf. (5)). It is easy to see that the conclusion of Theorem 2.5 yields Theorem 1.2, i.e., $H[V_n, q_n]$ is r -Ramsey if and only if $G^{(\ell)}(n, q_n) \rightarrow (F)_r$. Therefore, it is left to show that \mathbf{H} and \mathbf{p} satisfy the assumptions of Theorem 2.5 (see properties (a)-(c) in the proof of Theorem 1.1).

Since $\Delta(F) \geq 2$ it follows from the definition of m_F in (5) that $p_n \geq n^{-\ell+1}$. Consequently, the expected number of hyperedges in $G^{(\ell)}(n, p_n)$ is at least n . The definition of m_F ensures that the expected number of copies of F in $G^{(\ell)}(n, p_n)$ is at least of the same order as the expected number of edges in $G^{(\ell)}(n, p_n)$. Consequently, $p_n^k |E_n| = \Omega(n)$.

Moreover, the original proof of Ramsey's theorem (see also [3]) implies that for every $R \in \mathbb{N}$ there exists some $\zeta > 0$ such that the sequence $\mathbf{H} = (H_n)_{n \in \mathbb{N}}$ is (R, ζ) -Ramsey. Consequently, it is left to verify that \mathbf{H} is (K, \mathbf{p}) -bounded for some constant $K \geq 1$.

To this end we observe that H_n is a regular hypergraph with $\binom{n}{\ell}$ vertices. Moreover, every vertex is contained in $\Theta(n^{v(F)-\ell})$ edges and $|E_n| = \Theta(n^{v(F)})$. We will show that for $q \geq n^{-1/m_F}$ and $i \in [k-1]$ we have

$$\mu_i(H_n, q) = \mathbb{E} \left[\sum_{v \in V_n} \deg_i^2(v, V_{n,q}) \right] = \sum_{v \in V_n} \mathbb{E} [\deg_i^2(v, V_{n,q})] = O\left(q^{2i} \frac{|E_n|^2}{|V_n|}\right).$$

Due to the definition of \mathbf{H} every $v \in V_n$ corresponds to an edge $e(v)$ in $K_n^{(\ell)}$. Therefore, the number $\mathbb{E} [\deg_i^2(v, V_{n,q})]$ is the expected number of pairs (F_1, F_2) of copies F_1 and F_2 of F in $K_n^{(\ell)}$ satisfying $e(v) \in E(F_1) \cap E(F_2)$ and both copies F_1 and F_2 have at least i edges in $E(G^{(\ell)}(n, q)) \setminus \{e(v)\}$. Summing over all such pairs

F_1 and F_2 we obtain

$$\begin{aligned} \mathbb{E} [\deg_i^2(v, V_{n,q})] &\leq \sum_{F_1, F_2: e(v) \in E(F_1) \cap E(F_2)} \sum_{j=0}^{|E(F_1) \cap E(F_2)|-1} q^{2i-j} \\ &= O \left(\sum_{F_1, F_2: e(v) \in E(F_1) \cap E(F_2)} q^{2i-(|E(F_1) \cap E(F_2)|-1)} \right) \end{aligned} \quad (62)$$

since $q \leq 1$. Furthermore,

$$\begin{aligned} \sum_{F_1, F_2: e(v) \in E(F_1) \cap E(F_2)} q^{2i-(|E(F_1) \cap E(F_2)|-1)} \\ = O \left(\sum_{J: e(v) \in E(J)} n^{2v(F)-v(J)-\ell} q^{2i-(e(J)-1)} \right), \end{aligned} \quad (63)$$

where the sum on the right-hand side is indexed by all hypergraphs $J \subseteq K_n^{(\ell)}$ which contain $e(v)$ and which are isomorphic to a subhypergraph of F . It follows from the definition of m_F and $q \geq n^{-1/m_F}$ that $n^{v(J)} q^{e(J)} = \Omega(qn^\ell)$, i.e., $n^{-v(J)} q^{-e(J)} = O(q^{-1}n^{-\ell})$. Combining this with (62) and (63) we obtain

$$\begin{aligned} \mathbb{E} [\deg_i^2(v, V_{n,q})] &= O \left(\sum_{J: e(v) \in E(J)} n^{2v(F)-v(J)-\ell} q^{2i-e(J)+1} \right) \\ &= O \left(\sum_{J: e(v) \in E(J)} n^{2v(F)-2\ell} q^{2i} \right). \end{aligned}$$

Consequently,

$$\mu_i(H_n, q) = \sum_{v \in V_n} O(n^{2v(F)-2\ell} q^{2i}) = O(n^{2v(F)-\ell} q^{2i}) = O \left(q^{2i} \frac{|E_n|^2}{|V_n|} \right),$$

which concludes the proof of Theorem 1.2. \square

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