THE COUNTING LEMMA FOR REGULAR k-UNIFORM HYPERGRAPHS

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Abstract. Szemerédi's Regularity Lemma proved to be a powerful tool in the area of extremal graph theory. Many of its applications are based on its accompanying Counting Lemma: *If G is an ℓ-partite graph with $V(G) = V_1 \cup \dots \cup V_\ell$ and $|V_i| = n$ for all $i \in [\ell]$, and all pairs (V_i, V_j) are ε-regular of density d for $1 \leq i < j \leq \ell$ and $\varepsilon \ll d$, then G contains $(1 \pm f_\ell(\varepsilon))d^{\binom{\ell}{2}} \times n^\ell$ cliques K_ℓ, where $f_\ell(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Recently, V. Rödl and J. Skokan generalized Szemerédi's Regularity Lemma from graphs to k-uniform hypergraphs for arbitrary $k \geq 2$. In this paper we prove a Counting Lemma accompanying the Rödl–Skokan hypergraph Regularity Lemma. Similar results were independently obtained by W. T. Gowers.

It is known that such results give combinatorial proofs to the density result of E. Szemerédi and some of the density theorems of H. Furstenberg and Y. Katznelson.

1. Introduction

Extremal problems are among the most central and extensively studied in combinatorics. Many of these problems concern thresholds for properties concerning deterministic structures and have proven to be difficult as well as interesting. An important recent trend in combinatorics has been to consider the analogous problems for random structures. Tools are then sometimes afforded for determining with what probability a random structure possesses certain properties.

The study of *quasi-random structures*, pioneered by the work of Szemerédi [51], merges features of deterministic and random settings. Roughly speaking, a quasi-random structure is one which, while deterministic, mimics the behavior of random structures from certain important points of view. The (quasi-random) combinatorial structures we consider in this paper are *hypergraphs*. We begin our discussion with graphs.

1.1. Szemerédi's regularity lemma for graphs.

In the course of proving his celebrated density theorem (see Theorem 4) concerning arithmetic progressions, Szemerédi established a lemma which decomposes the edge set of any graph into constantly many "blocks", almost all of which are quasi-random (cf. [25, 26, 52]). In what follows, we give a precise account of Szemerédi's lemma.

For a graph $G = (V, E)$ and two disjoint sets A, $B \subset V$, let $E(A, B)$ denote the set of edges $\{a, b\} \in E$ with $a \in A$ and $b \in B$ and set $e(A, B) = |E(A, B)|$. We also set $d(A, B) = d(G_{AB}) = e(A, B)/|A||B|$ be the *density* of the bipartite graph $G_{AB} = (A \cup B, E(A, B))$.

The concept central to Szemerédi's lemma is that of an *ε-regular pair*. Let $\varepsilon > 0$ be given. We say that the pair (A, B) is ε-regular if $|d(A, B) - d(A', B')| < \varepsilon$ holds whenever $A' \subset A$, $B' \subset B$, and $|A'||B'| > \varepsilon|A||B|$.

We call a partition $V = V_0 \cup V_1 \cup \dots \cup V_t$ an *equitable partition* if it satisfies $|V_1| = |V_2| = \dots = |V_t|$ and $|V_0| < t$; we call an equitable partition *ε-regular* if all but $\varepsilon\binom{t}{2}$ pairs V_i, V_j are ε-regular. Szemerédi's lemma may then be stated as follows.

Theorem 1 (Szemerédi's Regularity Lemma)**.** *Let $\varepsilon > 0$ be given and let t_0 be a positive integer. There exist positive integers $T_0 = T_0(\varepsilon, t_0)$ and $n_0 = n_0(\varepsilon, t_0)$ such that any graph $G = (V, E)$ with $|V| = n \geq n_0$ vertices admits an ε-regular equitable partition $V = V_0 \cup V_1 \cup \dots \cup V_t$ with t satisfying $t_0 \leq t \leq T_0$.*

Szemerédi's Regularity Lemma is a powerful tool in the area of extremal graph theory. One of its most important features is that, in appropriate circumstances, it can be used to show a given graph contains a fixed subgraph. Suppose that a (large) graph is given along with an ε-regular partition $V = V_0 \cup V_1 \cup \dots \cup V_t$

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and let H be a fixed graph. If an appropriate collection of pairs $I_H \subseteq \binom{[t]}{2}$ have each (V_i, V_j), $\{i, j\} \in I_H$, ε-regular and sufficiently dense (with respect to ε), one is guaranteed a copy of H within this collection of bipartite graphs $E(V_i, V_j)$, $\{i, j\} \in I_H$. This observation is due to the following well-known fact, which may be appropriately called the Counting Lemma. We denote by $G[V_i, V_j]$ the bipartite graph of G induced on V_i and V_j, and we say that $G[V_i, V_j]$ is ε-regular if (V_i, V_j) is ε-regular.

Fact 2 (Counting Lemma)**.** *For every integer ℓ and positive reals d and γ there exists $\varepsilon > 0$ so that the following holds. Let $G = \bigcup_{1 \le i < j \le \ell} G^{ij}$ be an ℓ-partite graph with ℓ-partition $V_1 \cup \dots \cup V_\ell$ where $G^{ij} = G[V_i, V_j]$, $1 \le i < j \le \ell$, and $|V_1| = \ldots = |V_\ell| = n$. Suppose further that all graphs G^{ij} are ε-regular with density d. Then the number of copies of the ℓ-clique K_ℓ in G is within the interval $(1 \pm \gamma) d^{\binom{\ell}{2}} n^\ell$.*

1.2. **Extensions of Szemerédi's lemma to hypergraphs.** Several hypergraph regularity lemmas were considered by various authors [3, 4, 8, 10, 32]. None of these regularity lemmas seemed to admit a companion counting result (i.e., a corresponding generalization of Fact 2). The first attempt of developing a hypergraph regularity lemma together with a corresponding Counting Lemma was undertaken in [9]. In that paper, Frankl and Rödl established an extension of Szemerédi's Regularity Lemma to 3-graphs, hereafter called the FR-Lemma.

Analogously to the feature that Szemerédi's Regularity Lemma decomposes a given graph into an ε-regular partition, the FR-Lemma decomposes the edge set of a given 3-graph into constantly many "blocks", almost all of which are, in a specific sense, "quasi-random". The concept of 3-graph regularity which plays the analogous rôle of the ε-regular pair is, unfortunately, considerably more technical than its graph counterpart. It is not necessary at this time to know this precise definition in order to understand the current introduction. We therefore postpone precise discussion until later (see Section 2.2).

Just as Fact 2, the Counting Lemma, is an important companion statement to Szemerédi's Regularity Lemma, most applications of the FR-Lemma require a similar companion lemma - the "3-graph Counting Lemma". Analogously to Fact 2, the 3-graph Counting Lemma estimates the number of copies of the clique $K_\ell^{(3)}$ (i.e., the complete 3-graph on ℓ vertices) contained in an appropriate collection of "dense and regular blocks" within a regular partition provided by the FR-Lemma. This 3-graph Counting Lemma was established in [9] for the special case $K_4^{(3)}$ and subsequently established by Nagle and Rödl in [28] (see [30, 31] for alternative proofs of the 3-graph Counting Lemma as well as [5, 20] for an algorithmic version of both the FR-Lemma and the 3-graph Counting Lemma).

Recently, Rödl and Skokan [40] established a generalization of the FR-Lemma to k-graphs for $k \ge 3$ (see Section 3.2). We will refer to this lemma as the RS-Lemma. In [41], they also succeeded to prove a companion Counting Lemma in the special case of $K_5^{(4)}$. In this paper,

we prove the general k-graph Counting Lemma corresponding to the RS-Lemma.

Our Counting Lemma, the main theorem of this paper, requires some notation. Therefore, we defer its precise statement to Section 2.3 (see Theorem 12).

We mention that a different Regularity Lemma as well as a corresponding Counting Lemma for k-graphs was recently proved by Gowers [15, 16]. However, the approach in [15, 16] is different from the one taken in [40] and this paper.

Finally, as we briefly discuss in Section 9, combining the methods of this paper together with those used in [40] allows one to prove a Regularity Lemma for hypergraphs which is simpler to formulate and easier to use. This research is further developed in [36].

1.3. **Applications of the regularity method.** Szemerédi's Regularity Lemma, together with its corresponding Counting Lemma, Fact 2, has numerous applications (see [25, 26] for comprehensive surveys). The FR-Lemma and the companion 3-graph Counting Lemma [28] were exploited in a variety of extremal hypergraph problems (cf. [9, 21, 22, 23, 27, 34, 35, 45, 50]).

The methods developed in this paper together with the RS-Lemma for general k-uniform hypergraphs were already applied to several combinatorial problems [29, 37, 39, 47, 48]. In particular, the following theorem, which was conjectured by Erdős, Frankl, and Rödl [7], is an immediate consequence of the regularity method for hypergraphs.

Theorem 3 (Removal Lemma)**.** *For every $k \geq 2$ and $\varepsilon > 0$ there exist $\delta > 0$ such that the following holds. Suppose that a k-uniform hypergraph $\mathcal{H}^{(k)}$ on n vertices contains at most δn^{k+1} copies of $K_{k+1}^{(k)}$, the complete k-uniform hypergraph on $k+1$ vertices. Then one can delete εn^k edges of $\mathcal{H}^{(k)}$ to make it $K_{k+1}^{(k)}$-free.*

Ruzsa and Szemerédi [44] proved this result for graphs ($k = 2$) and Frankl and Rödl [9] proved Theorem 3 for $k = 3$. We give a proof of Theorem 3 for general k in Section 8 following the ideas of [9]. Theorem 3 also holds if $K_{k+1}^{(k)}$ is replaced by an arbitrary k-uniform hypergraph $\mathcal{F}^{(k)}$ and δn^{k+1} replaced by $\delta n^{|V(\mathcal{F}^{(k)})|}$, which was proved in [7] for $k = 2$ and in [39] for general k. Theorem 3 implies several density versions of combinatorial partition theorems which we briefly discuss below.

Density theorems. In 1975 Szemerédi [51] solved a longstanding conjecture of Erdős and Turán [6] concerning the upper density of subsets of the integers which contain no arithmetic progression of fixed length.

Theorem 4 (Szemerédi's theorem)**.** *For every $\ell \geq 3$ and $\delta > 0$ there exist $n_0 = n_0(\ell, \delta)$ such that if $A \subseteq [n] = \{1, \dots, n\}$ with $n \geq n_0$ and $|A| \geq \delta n$, then A contains an arithmetic progression of length ℓ.*

Theorem 4 stimulated a lot of research in quite diverse areas of mathematics. Furstenberg [11] gave an alternative proof of Theorem 4 using methods of ergodic theory (see also [53] for a "finitized" version of Furstenberg's original argument due to Tao). Gowers also proved Szemerédi's theorem using, among others, methods of Fourier analysis [17]. In particular, the proof of Gowers gives the best known bounds on n_0 as a function of δ and ℓ. He showed that

$$n_0 \leq \exp(\exp(\delta^{-C_\ell})) \quad \text{where} \quad C_\ell = 2^{2^{\ell+9}}. \tag{1}$$

Theorem 3 also implies Szemerédi's theorem. This connection was first observed by Ruzsa and Szemerédi [44]. They showed that Theorem 3 for $k = 2$ implies Theorem 4 for $\ell = 3$, which was originally obtained by Roth [42, 43]. Later it was shown by Frankl and Rödl [9, 33] that the Removal Lemma for general k yields an alternative proof of Szemerédi's theorem. Subsequently, Solymosi [49, 50] proved that the Removal Lemma also yields alternative proofs of the following multidimenional version of Szemerédi's theorem, originally due to by Furstenberg and Katznelson [12].

Theorem 5. *For all $\ell \geq 2$, $d \geq 1$, and $\delta > 0$ there exist $n_0 = n_0(\ell, d, \delta)$ such that if $A \subseteq [n]^d = \{1, \dots, n\}^d$ with $n \geq n_0$ and $|A| \geq \delta n^d$, then A contains a homothetic copy of $[\ell]^d$, i.e., a set of the form $\boldsymbol{z} + j[\ell]^d$ for some $\boldsymbol{z} \in [n]^d$ and $j > 0$.*

The following theorem is also a corollary of Theorem 3, as shown in [38].

Theorem 6. *Let R be a finite ring with q elements. Then for every $\delta > 0$, there exists $n_0 = n_0(q, \delta)$ such that, for $n \geq n_0$, any subset $A \subset R^n$ with $|A| > \delta |R^n| = \delta q^n$ contains a coset of an isomorphic copy (as a left R-module) of R. In other words, there exist $\boldsymbol{r}$, $\boldsymbol{u} \in R^n$ such that $\boldsymbol{r} + \varphi(R) \subseteq A$, where $\varphi\colon R \hookrightarrow R^n$, $\varphi(\alpha) = \alpha \boldsymbol{u}$ for $\alpha \in R$, is an injection.*

We mention that Theorem 6 is similar to related work of Furstenberg and Katznelson [13] implying that dense subsets of high dimensional vector spaces over finite fields contain affine subspaces of fixed dimension.

It is worth mentioning that so far the only known proofs of the theorems of Furstenberg and Katznelson [12, 13] discussed above involve ergodic theory. The purely combinatorial proofs based on the RS-Lemma and the main result of this paper (or similarly, proofs based on the recent results of Gowers [15]) give the first quantitative proofs of those theorems. The bounds on n_0, however, are incomparably weaker to the one in (1). They belong to a level of the Ackermann hierarchy that depends on the input parameters.

The techniques introduced by Furstenberg and Katznelson [11, 12, 13], however, have been further extended to prove other generalizations of Theorems 4–6, among which are a density version of the Hales–Jewett theorem [18], again due to Furstenberg and Katznelson [14], and polynomial extensions of Szemerédi's theorem, due to Bergelson and Leibman [1] and Bergelson and McCutcheon [2]. At the time of this writing, it is not known whether Theorem 3 can be used to prove such stronger results.

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2. Statement of the main result

2.1. **Basic notation.** For reals x and y and a non-negative constant ξ we sometimes write $x = y \pm \xi$, if $y - \xi \le x \le y + \xi$. We denote by $[\ell]$ the set $\{1, \dots, \ell\}$. For a set V and an integer $k \ge 1$, let $\binom{V}{k}$ be the set of all k-element subsets of V. A subset $\mathcal{G}^{(k)} \subseteq \binom{V}{k}$ is a *k-uniform hypergraph* on the vertex set V. We identify hypergraphs with their edge sets. For a given k-uniform hypergraph $\mathcal{G}^{(k)}$, we denote by $V\left(\mathcal{G}^{(k)}\right)$ and $E\left(\mathcal{G}^{(k)}\right)$ its vertex and edge set, respectively. For $U \subseteq V\left(\mathcal{G}^{(k)}\right)$, we denote by $\mathcal{G}^{(k)}[U]$ the subhypergraph of $\mathcal{G}^{(k)}$ induced on U (i.e. $\mathcal{G}^{(k)}[U] = \mathcal{G}^{(k)} \cap \binom{U}{k}$). A k-uniform *clique* of *order* j, denoted by $K_j^{(k)}$, is a k-uniform hypergraph on $j \ge k$ vertices consisting of all $\binom{j}{k}$ many k-tuples (i.e., $K_j^{(k)}$ is isomorphic to $\binom{[j]}{k}$).

The central objects of this paper are ℓ-partite hypergraphs. Throughout this paper, the underlying vertex partition $V = V_1 \cup \dots \cup V_\ell$, $|V_1| = \dots = |V_\ell| = n$, is fixed. The vertex set itself can be seen as a 1-uniform hypergraph and, hence, we will frequently refer to the underlying fixed vertex set as $\mathcal{G}^{(1)}$. For integers $\ell \ge k \ge 1$ and vertex partition $V_1 \cup \dots \cup V_\ell$, we denote by $K_\ell^{(k)}(V_1, \dots, V_\ell)$ the *complete* ℓ-partite, k-uniform hypergraph (i.e. the family of all k-element subsets $K \subseteq \bigcup_{i \in [\ell]} V_i$ satisfying $|V_i \cap K| \le 1$ for every $i \in [\ell]$). Then, an *(n, ℓ, k)-cylinder* $\mathcal{G}^{(k)}$ is any subset of $K_\ell^{(k)}(V_1, \dots, V_\ell)$. Observe, that $\left|V\left(\mathcal{G}^{(k)}\right)\right| = \ell \times n$ for an (n, ℓ, k)-cylinder $\mathcal{G}^{(k)}$. Observe that the vertex partition $V_1 \cup \dots \cup V_\ell$ is an $(n, \ell, 1)$-cylinder $\mathcal{G}^{(1)}$. (This observation may seem artificial right now, but it will simplify later notation.) For $k \le j \le \ell$ and set $\Lambda_j \in \binom{[\ell]}{j}$, we denote by $\mathcal{G}^{(k)}[\Lambda_j] = \mathcal{G}^{(k)}\left[\bigcup_{\lambda \in \Lambda_j} V_\lambda\right]$ the subhypergraph of the (n, ℓ, k)-cylinder $\mathcal{G}^{(k)}$ induced on $\bigcup_{\lambda \in \Lambda_j} V_\lambda$.

For an (n, ℓ, j)-cylinder $\mathcal{G}^{(j)}$ and an integer $j \le i \le \ell$, we denote by $\mathcal{K}_i\left(\mathcal{G}^{(j)}\right)$ the family of all i-element subsets of $V\left(\mathcal{G}^{(j)}\right)$ which span complete subhypergraphs in $\mathcal{G}^{(j)}$ of order i. Note that $\left|\mathcal{K}_i\left(\mathcal{G}^{(j)}\right)\right|$ is the number of all copies of $K_i^{(j)}$ in $\mathcal{G}^{(j)}$.

Given an $(n, \ell, j-1)$-cylinder $\mathcal{G}^{(j-1)}$ and an (n, ℓ, j)-cylinder $\mathcal{G}^{(j)}$ on the same vertex partition, we say an edge J of $\mathcal{G}^{(j)}$ *belongs to* $\mathcal{G}^{(j-1)}$ if $J \in \mathcal{K}_j(\mathcal{G}^{(j-1)})$, i.e., J corresponds to a clique of order j in $\mathcal{G}^{(j-1)}$. Moreover, $\mathcal{G}^{(j-1)}$ *underlies* $\mathcal{G}^{(j)}$ if $\mathcal{G}^{(j)} \subseteq \mathcal{K}_j\left(\mathcal{G}^{(j-1)}\right)$, i.e., every edge of $\mathcal{G}^{(j)}$ belongs to $\mathcal{G}^{(j-1)}$. This brings us to one of the main concepts of this paper, the notion of a *complex*.

Definition 7 (**(n, ℓ, k)-complex**). *Let $n \ge 1$ and $\ell \ge k \ge 1$ be integers. An (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}}$ is a collection of (n, ℓ, j)-cylinders $\{\mathcal{G}^{(j)}\}_{j=1}^k$ such that*

(*a*) *$\mathcal{G}^{(1)}$ is an $(n, \ell, 1)$-cylinder, i.e., $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$ with $|V_i| = n$ for $i \in [\ell]$,*
(*b*) *$\mathcal{G}^{(j-1)}$ underlies $\mathcal{G}^{(j)}$ for $2 \le j \le k$.*

2.2. **Regular complexes.** We begin with a notion of density for an (n, ℓ, j)-cylinder with respect to a family of $(n, \ell, j-1)$-cylinders.

Definition 8 (**density**). *Let $\mathcal{G}^{(j)}$ be an (n, ℓ, j)-cylinder and let $\boldsymbol{\mathcal{Q}}^{(j-1)} = \{\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}\}$ be a family of $(n, \ell, j-1)$-cylinders. We define the density of $\mathcal{G}^{(j)}$ w.r.t. the family $\boldsymbol{\mathcal{Q}}^{(j-1)}$ as*

$$d\left(\mathcal{G}^{(j)} \middle| \boldsymbol{\mathcal{Q}}^{(j-1)}\right) = \begin{cases} \frac{\left|\mathcal{G}^{(j)} \cap \bigcup_{s \in [r]} \mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right|}{\left|\bigcup_{s \in [r]} \mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right|} & \text{if} \quad \left|\bigcup_{s \in [r]} \mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right| > 0 \\ 0 & \text{otherwise}. \end{cases}$$

We now define a notion of regularity of an (n, j, j)-cylinder with respect to an $(n, j, j-1)$-cylinder.

Definition 9. *Let positive reals δ_j and d_j and a positive integer r be given along with an (n, j, j)-cylinder $\mathcal{G}^{(j)}$ and an underlying $(n, j, j-1)$-cylinder $\mathcal{G}^{(j-1)}$. We say $\mathcal{G}^{(j)}$ is (δ_j, d_j, r)-regular* w.r.t. *$\mathcal{G}^{(j-1)}$ if whenever $\boldsymbol{\mathcal{Q}}^{(j-1)} = \{\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}\}$, $\mathcal{Q}_s^{(j-1)} \subseteq \mathcal{G}^{(j-1)}$, $s \in [r]$, satisfies*

$$\left|\bigcup_{s \in [r]} \mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right| \ge \delta_j \left|\mathcal{K}_j\left(\mathcal{G}^{(j-1)}\right)\right|, \ \text{then} \quad d\left(\mathcal{G}^{(j)} \middle| \boldsymbol{\mathcal{Q}}^{(j-1)}\right) = d_j \pm \delta_j .$$

We extend the notion of (δ_j, d_j, r)-regularity from (n, j, j)-cylinders to (n, ℓ, j)-cylinders $\mathcal{G}^{(j)}$.

Definition 10 (**(δ_j, d_j, r)-regular**)**.** *We say an (n, ℓ, j)-cylinder $\mathcal{G}^{(j)}$ is (δ_j, d_j, r)-regular* w.r.t. *an $(n, \ell, j-1)$-cylinder $\mathcal{G}^{(j-1)}$ if for every $\Lambda_j \in \binom{[\ell]}{j}$ the restriction $\mathcal{G}^{(j)}[\Lambda_j] = \mathcal{G}^{(j)}\big[\bigcup_{\lambda\in\Lambda_j} V_\lambda\big]$ is (δ_j, d_j, r)-regular w.r.t. to the restriction $\mathcal{G}^{(j-1)}[\Lambda_j] = \mathcal{G}^{(j-1)}\big[\bigcup_{\lambda\in\Lambda_j} V_\lambda\big]$.*

*We sometimes write (δ_j, r)-*regular *to mean $\big(\delta_j, d\big(\mathcal{G}^{(j)}\big|\mathcal{G}^{(j-1)}\big), r\big)$-regular for cylinders $\mathcal{G}^{(j)}$ and $\mathcal{G}^{(j-1)}$.*

We close this section of basic definitions with the central notion of a regular complex.

Definition 11 (**$(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular complex**)**.** *Let vectors $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ and $\boldsymbol{d} = (d_2, \dots, d_k)$ of positive reals be given and let r be a positive integer. We say an (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$ is $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular if:*

- (a) *$\mathcal{G}^{(2)}$ is $(\delta_2, d_2, 1)$-regular w.r.t. $\mathcal{G}^{(1)}$ and*
- (b) *$\mathcal{G}^{(j)}$ is (δ_j, d_j, r)-regular w.r.t. $\mathcal{G}^{(j-1)}$ for $3 \le j \le k$.*

2.3. **Statement of the counting lemma.** The following assertion is the main theorem of this paper.

Theorem 12 (Counting Lemma)**.** *For all integers $2 \le k \le \ell$ the following is true: $\forall \gamma > 0\ \forall d_k > 0\ \exists \delta_k > 0\ \forall d_{k-1} > 0\ \exists \delta_{k-1} > 0\ \dots\ \forall d_2 > 0\ \exists \delta_2 > 0$ and there are integers r and n_0 so that, with $\boldsymbol{d} = (d_2, \dots, d_k)$ and $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ and $n \ge n_0$, whenever $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^k$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular (n, ℓ, k)-complex, then*

$$\Big|\mathcal{K}_\ell\big(\mathcal{G}^{(k)}\big)\Big| = (1 \pm \gamma) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell\,.$$

For given integers k and ℓ we shall refer to this theorem by $\mathbf{CL}_{k,\ell}$.

Observe from the quantification $\forall \gamma, d_k\ \exists \delta_k\ \forall d_{k-1}\ \exists \delta_{k-1}\ \dots\ \forall d_2\ \exists \delta_2$, the constants of Theorem 12 can satisfy $\delta_h \gg d_{h-1}$ for any $3 \le h \le k$. In particular, the hypothesis of Theorem 12 allows for the possibility that

$$\gamma, d_k \gg \delta_k \gg d_{k-1} \gg \delta_{k-1} \gg \dots \gg d_h \gg \delta_h \gg d_{h-1} \gg \dots \gg d_2 \gg \delta_2. \tag{2}$$

Consequently, the Counting Lemma includes the case when complexes $\{\mathcal{G}^{(h)}\}_{h=1}^k$ consist of fairly sparse hypergraphs. It seems that this is the main difficulty in proving Theorem 12.

2.4. **Generalization of the counting lemma.** The main result of this paper, Theorem 12, allows us to count complete hypergraphs of fixed order within a sufficiently regular complex. For some applications, it is more useful to consider slightly more general lemmas.

The following generalization enables us to estimate the number of copies of an arbitrary hypergraph $\mathcal{F}^{(k)}$ with vertices $\{1, \dots, \ell\}$ in an (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$ satisfying that $\mathcal{G}^{(j)}[\Lambda_j]$ is regular w.r.t. $\mathcal{G}^{(j-1)}[\Lambda_j]$ whenever $\Lambda_j \subseteq K$ for some edge K of $\mathcal{F}^{(k)}$. Rather than counting copies of $K_\ell^{(k)}$ in an "everywhere" regular complex, this lemma counts copies of $\mathcal{F}^{(k)}$ in the complex $\boldsymbol{\mathcal{G}}$ satisfying the less restrictive assumptions above. We introduce some more notation before we give the precise statement below (see Corollary 15).

For a fixed k-uniform hypergraph $\mathcal{F}^{(k)}$, we define the j-th shadow for $j \in [k]$ by

$$\Delta_j(\mathcal{F}^{(k)}) = \{J\colon\ |J| = j \text{ and } J \subseteq K \text{ for some } K \in \mathcal{F}^{(k)}\}\,.$$

We extend the notion of a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular complex to $(\boldsymbol{\delta}, \ge\!\boldsymbol{d}, r, \mathcal{F}^{(k)})$-regular complex.

Definition 13 (**$(\boldsymbol{\delta}, \ge\!\boldsymbol{d}, r, \mathcal{F}^{(k)})$-regular complex**)**.** *Let $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ and $\boldsymbol{d} = (d_2, \dots, d_k)$ be vectors of positive reals and let r be a positive integer. Let $\mathcal{F}^{(k)}$ be a k-uniform hypergraph on ℓ vertices $\{1, \dots, \ell\}$. We say an (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$ with $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$ is $\big(\boldsymbol{\delta}, \ge\!\boldsymbol{d}, r, \mathcal{F}^{(k)}\big)$-regular if:*

- (a) *for every $\Lambda_2 \in \Delta_2(\mathcal{F}^{(k)})$, the $(n, 2, 2)$-cylinder $\mathcal{G}^{(2)}[\Lambda_2]$ is $(\delta_2, d_2, 1)$-regular w.r.t. $\mathcal{G}^{(1)}[\Lambda_2]$,*
- (b) *for every $\Lambda_j \in \Delta_j(\mathcal{F}^{(k)})$, the (n, j, j)-cylinder $\mathcal{G}^{(j)}[\Lambda_j]$ is (δ_j, d_j, r)-regular w.r.t. $\mathcal{G}^{(j-1)}$ for $3 \le j < k$, and*
- (c) *for every $\Lambda_k \in \mathcal{F}^{(k)}$, the (n, k, k)-cylinder $\mathcal{G}^{(k)}[\Lambda_k]$ is $(\delta_k, d_{\Lambda_k}, r)$-regular w.r.t. $\mathcal{G}^{(k-1)}$ with $d_{\Lambda_k} \ge d_k$.*

The '$\ge$' in a $(\boldsymbol{\delta}, \ge\!\boldsymbol{d}, r, \mathcal{F}^{(k)})$-regular complex indicates that we only enforce a lower bound on the densities in the k-th layer of $\boldsymbol{\mathcal{G}}$ (cf. part (c) of the definition). This is the environment which usually appears in applications. We also observe that the Definition 13 imposes only a regular structure on those (m,k,k)-subcomplexes of $\boldsymbol{\mathcal{G}}$ which naturally correspond to edges of $\mathcal{F}^{(k)}$ (i.e., on a subcomplex induced on $V_{\lambda_1},\dots,V_{\lambda_k}$, where $\{\lambda_1,\dots,\lambda_k\}$ forms an edge in $\mathcal{F}^{(k)}$). We need one more definition before we can state the corollary.

Definition 14 (**partite isomorphic**)**.** *Suppose $\mathcal{F}^{(k)}$ is a k-uniform hypergraph with $V(\mathcal{F}^{(k)})=[\ell]$ and $\mathcal{G}^{(k)}$ is an (n,ℓ,k)-cylinder with vertex partition $V(\mathcal{G}^{(k)})=V_1\cup\dots\cup V_\ell$. We say a copy $\mathcal{F}_0^{(k)}$ of $\mathcal{F}^{(k)}$ in $\mathcal{G}^{(k)}$ is partite isomorphic to $\mathcal{F}^{(k)}$ if there is a labeling of $V(\mathcal{F}_0^{(k)})=\{v_1,\dots,v_\ell\}$ such that*

(*i*) *$v_\alpha\in V_\alpha$ for every $\alpha\in[\ell]$, and*

(*ii*) *$v_\alpha\mapsto\alpha$ is a hypergraph isomorphism (edge preserving bijection of the vertex sets) between $\mathcal{F}_0^{(k)}$ and $\mathcal{F}^{(k)}$.*

Corollary 15. *For all integers $2\le k\le\ell$ and $\forall\gamma>0$ $\forall d_k>0$ $\exists\delta_k>0$ $\forall d_{k-1}>0$ $\exists\delta_{k-1}>0$ $\dots$ $\forall d_2>0$ $\exists\delta_2>0$ and there are integers r and n_0 so that the following holds for $\boldsymbol{d}=(d_2,\dots,d_k)$, $\boldsymbol{\delta}=(\delta_2,\dots,\delta_k)$, and $n\ge n_0$. If $\mathcal{F}^{(k)}$ is a k-uniform hypergraph on vertices $\{1,\dots,\ell\}$ and $\boldsymbol{\mathcal{G}}=\{\mathcal{G}^{(h)}\}_{h=1}^k$ is a $(\boldsymbol{\delta},\ge\boldsymbol{d},r,\mathcal{F}^{(k)})$-regular (n,ℓ,k)-complex with $\mathcal{G}^{(1)}=V_1\cup\dots\cup V_\ell$, then the number of partite isomorphic copies of $\mathcal{F}^{(k)}$ in $\mathcal{G}^{(k)}$ is at least*

$$(1-\gamma)\prod_{h=2}^{k-1}d_h^{|\Delta_h(\mathcal{F}^{(k)})|}\times\prod_{\Lambda_k\in\mathcal{F}^{(k)}}d_{\Lambda_k}\times n^\ell\ge(1-\gamma)\prod_{h=2}^{k}d_h^{|\Delta_h(\mathcal{F}^{(k)})|}\times n^\ell\,.$$

Corollary 15 can be easily derived from Theorem 12. Below we briefly outline that proof. The full proof can be found in [46, Chapter 9].

The idea of the proof consists of two basic parts. For $2\le j\le k$, for each $\Lambda_j=\{\lambda_1,\dots,\lambda_j\}\notin\Delta_j(\mathcal{F}^{(k)})$, we replace the (n,j,j)-cylinder $\mathcal{G}^{(j)}[\Lambda_j]$ with the complete j-partite j-uniform system $K_j^{(j)}(V_{\lambda_1},\dots,V_{\lambda_j})$. Doing so over all $2\le j\le k$ and all $\Lambda_j\notin\Delta_j(\mathcal{F}^{(k)})$ clearly results in an "everywhere" regular complex, let us call it $\boldsymbol{\mathcal{H}}$, whose cliques $K_\ell^{(k)}$ correspond to copies of $\mathcal{F}^{(k)}$ in $\boldsymbol{\mathcal{G}}$.

One now wishes to apply the Counting Lemma, Theorem 12, to the complex $\boldsymbol{\mathcal{H}}$ to finish the job. The only minor technicality in doing so is that, unlike the hypothesis of Theorem 12, the complex $\boldsymbol{\mathcal{H}}$ potentially has, for each $2\le j\le k$, (n,j,j)-cylinders $\mathcal{H}^{(j)}[\Lambda_j]$, $\Lambda_j\in\binom{[\ell]}{j}$, of differing densities. This is handled, however, by "randomly slicing" the (n,j,j)-cylinders $\mathcal{H}^{(j)}[\Lambda_j]$, $\Lambda_j\in\binom{[\ell]}{j}$, into appropriately many pieces of the same density as formally required in Theorem 12. Consequently, we create a series of pairwise $K_\ell^{(k)}$-disjoint complexes $\boldsymbol{\mathcal{H}}_1,\boldsymbol{\mathcal{H}}_2,\dots$, each of which satisfies the hypothesis of the Counting Lemma. Theorem 12 applies to each of the newly created complexes $\boldsymbol{\mathcal{H}}_i$, $i\ge1$, and so we add the resulting number of cliques to finish the proof of Corollary 15.

3. Auxiliary results

In this section we review a few results that are essential for our proof of Theorem 12 in Section 4.

3.1. The dense counting lemma.

We recall that Theorem 12 is formulated under the involved quantification $\forall d_k$ $\exists\delta_k$ $\forall d_{k-1}$ $\exists\delta_{k-1}\dots\forall d_2$ $\exists\delta_2$ and that the difficulty we have encountered in the proof the Counting Lemma is due to the sparseness arising from this quantification. If the quantification can be simplified so that

$$\min_{2\le j\le k}d_j\gg\max_{2\le j\le k}\delta_j \tag{3}$$

is ensured, then the so-called Dense Counting Lemma (see Theorem 16 below) is known to be true. This was proved by Kohayakawa, Rödl, and Skokan (see Theorem 6.5 in [24]). Observe that (3) represents the 'dense case' in contrast to the 'sparse case' (2), since all densities are bigger then the measure of regularity $\max\delta_j$.

Theorem 16 (Dense Counting Lemma)**.** *For all integers $2\le k\le\ell$ and any positive constants $d_2,\dots,d_k$, there exist $\varepsilon>0$ and integer m_0 so that, with $\boldsymbol{d}=(d_2,\dots,d_k)$ and $\boldsymbol{\varepsilon}=(\varepsilon,\dots,\varepsilon)\in\mathbb{R}^{k-1}$ and $m\ge m_0$, whenever $\boldsymbol{\mathcal{H}}=\{\mathcal{H}^{(j)}\}_{j=1}^k$ is a $(\boldsymbol{\varepsilon},\boldsymbol{d},1)$-regular (m,ℓ,k)-complex, then*

$$\left|\mathcal{K}_\ell(\mathcal{H}^{(k)})\right|=(1\pm g_{k,\ell}(\varepsilon))\prod_{h=2}^{k}d_h^{\binom{\ell}{h}}\times m^\ell$$

where $g_{k,\ell}(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

While the quantification of the main theorem, Theorem 12, does not allow us to assume (3), Peng, Rödl, and Skokan in [31] used Theorem 16 to prove Theorem 12 for $k = 3$ by reducing the harder 'sparse case' to the easier 'dense case'. This is not unlike the idea of our current proof, although our reduction scheme is entirely different and allows an extension for arbitrary k.

3.2. **Partitions.** One of the major tools we use in our proof of Theorem 12 is the recently developed regularity lemma of Rödl and Skokan [40] for k-uniform hypergraphs. In this section, we shall describe the partition structure this lemma provides (in fact, we shall employ an 'ℓ-partite' version of this lemma so that below, the partition structure is adapted to an ℓ-partite scenario). We formulate the regularity lemma in Section 3.3.

The regularity lemma provides partitions of all the complete (n, ℓ, j)-cylinders $K_\ell^{(j)}(V_1, \dots, V_\ell)$ for $j \in [k-1]$. We shall describe the families $\mathscr{R}$ of these partitions in two distinct but equivalent ways. First, we describe $\mathscr{R} = \{\mathscr{R}^{(1)}, \dots, \mathscr{R}^{(k-1)}\}$ inductively. This first description is simpler and perhaps more intuitive. The second description, however, is somewhat better suited for our proof and was also used in [40] to state the regularity lemma.

Partitions (inductive description). We begin with an inductive description of the families $\mathscr{R}$. Let k be a fixed integer and $V_1 \cup \dots \cup V_\ell$ be a partition of V with $|V_\lambda| = n$ for every $\lambda \in [\ell]$. Let $\mathscr{R}^{(1)}$ be a partition of V which refines the given partition $V_1 \cup \dots \cup V_\ell$. Suppose that partitions $\mathscr{R}^{(h)}$ of $K_\ell^{(h)}(V_1, \dots, V_\ell)$ for $1 \le h \le j-1$ have been given. Our goal is to describe the structure of partition $\mathscr{R}^{(j)}$ of $K_\ell^{(j)}(V_1, \dots, V_\ell)$.

To this end, we introduce special j-partite, $(j-1)$-uniform, hypergraphs which we call *polyads*. Note that it follows by induction that for every $(j-1)$-tuple I in $K_\ell^{(j-1)}(V_1, \dots, V_\ell)$, there exists a unique $\mathcal{R}^{(j-1)} = \mathcal{R}^{(j-1)}(I) \in \mathscr{R}^{(j-1)}$ so that $I \in \mathcal{R}^{(j-1)}$. For every j-tuple J in $K_\ell^{(j)}(V_1, \dots, V_\ell)$, we define $\hat{\mathcal{R}}^{(j-1)}(J)$, the *polyad of* J, by

$$\hat{\mathcal{R}}^{(j-1)}(J) = \bigcup \left\{ \mathcal{R}^{(j-1)}(I) \colon I \in \tbinom{J}{j-1} \right\}. \tag{4}$$

In other words, $\hat{\mathcal{R}}^{(j-1)}(J)$ is the unique collection of j partition classes of $\mathscr{R}^{(j-1)}$ each containing a $(j-1)$-element subset $I \in \binom{J}{j-1}$. Observe that $\hat{\mathcal{R}}^{(j-1)}(J)$ can be viewed as a j-partite, $(j-1)$-uniform, hypergraph. To emphasize the special rôle these objects play in this paper, we use the additional symbol '$\hat{\ }$' in $\hat{\mathcal{R}}^{(j-1)}$.

As a final concept needed to describe partition $\mathscr{R}^{(j)}$ of $K_\ell^{(j)}(V_1, \dots, V_\ell)$, we define $\hat{\mathscr{R}}^{(j-1)}$, the family of all polyads, as

$$\hat{\mathscr{R}}^{(j-1)} = \left\{ \hat{\mathcal{R}}^{(j-1)}(J) \colon J \in K_\ell^{(j)}(V_1, \dots, V_\ell) \right\}.$$

Note that $\hat{\mathcal{R}}^{(j-1)}(J_1)$ and $\hat{\mathcal{R}}^{(j-1)}(J_2)$ are not necessarily distinct for different j-tuples J_1 and J_2. As such, $\hat{\mathscr{R}}^{(j-1)}$ can be viewed as a set of equivalence classes, or more simply, as a set. We then have that $\{\mathcal{K}_j(\hat{\mathcal{R}}^{(j-1)}) \colon \hat{\mathcal{R}}^{(j-1)} \in \hat{\mathscr{R}}^{(j-1)}\}$ is a partition of $K_\ell^{(j)}(V_1, \dots, V_\ell)$.

The structural requirement on the partition $\mathscr{R}^{(j)}$ of $K_\ell^{(j)}(V_1, \dots, V_\ell)$ is

$$\mathscr{R}^{(j)} \prec \{\mathcal{K}_j(\hat{\mathcal{R}}^{(j-1)}) \colon \hat{\mathcal{R}}^{(j-1)} \in \hat{\mathscr{R}}^{(j-1)}\}, \tag{5}$$

where '$\prec$' denotes the refinement relation of set partitions. In other words, we require that the set of cliques spanned by a polyad in $\hat{\mathscr{R}}^{(j-1)}$ is sub-partitioned in $\mathscr{R}^{(j)}$, and that every partition class in $\mathscr{R}^{(j)}$ belongs to precisely one polyad in $\hat{\mathscr{R}}^{(j-1)}$.

Definition 17 (cohesive). *For* $j \in [k-1]$ *let* $\mathscr{R}^{(j)}$ *be a partition of* $K_\ell^{(j)}(V_1, \dots, V_\ell)$. *We say the family of partitions* $\mathscr{R} = \{\mathscr{R}^{(1)}, \dots, \mathscr{R}^{(k-1)}\}$ *is* cohesive *if* (5) *holds for every* $j = 2, \dots, k-1$.

In this paper, we also consider j-partite, h-uniform hypergraphs formed by partition classes from $\mathscr{R}^{(h)}$ for $h < j-1$. For that we extend the definition in (4) and for $1 \le h < j$, we set for $J \in K_\ell^{(j)}(V_1, \dots, V_\ell)$

$$\mathcal{R}^{(h)}(J) = \bigcup \left\{ \mathcal{R}^{(h)}(H) \colon H \in \tbinom{J}{h} \right\} \qquad \text{and} \qquad \boldsymbol{\mathcal{R}}(J) = \left\{ \mathcal{R}^{(h)}(J) \right\}_{h=1}^{j-1}. \tag{6}$$

Note, that

$$\hat{\mathcal{R}}^{(j-1)}(J) \text{ (defined in (4)) and } \mathcal{R}^{(j-1)}(J) \text{ (defined in (6)) are identical.} \tag{7}$$

Finally we observe that if $\mathscr{R} = \{\mathscr{R}^{(1)}, \dots, \mathscr{R}^{(k-1)}\}$ is a cohesive family of partitions, then $\boldsymbol{\mathcal{R}}(J)$ defined in (6) is a complex.

This concludes our inductive description of the families $\mathscr{R}$ that will be provided by the regularity lemma. At this moment, we could, in fact, state the regularity lemma in the language above. We choose to postpone the statement of the regularity lemma, however, until we have fully developed the second description (in terms of 'addresses') of these same partitions. Once both languages are then developed, we will state the regularity lemma from [40] in Section 3.3.

Addresses. We consider partitions described by labeling every element of $K_\ell^{(j)}(V_1, \dots, V_\ell)$ with an 'address'. For every $j \in [k-1]$, let $a_j \in \mathbb{N}$ and let φ_j be a function such that

$$\varphi_j \colon K_\ell^{(j)}(V_1, \dots, V_\ell) \to [a_j]\,.$$

Note, for every $\lambda \in [\ell]$, mapping φ_1 defines a partition $V_\lambda = V_{\lambda,1} \cup \dots \cup V_{\lambda,a_1}$, where $V_{\lambda,\alpha} = \varphi_1^{-1}(\alpha) \cap V_\lambda$ for all $\alpha \in [a_1]$. Here, we only consider functions φ_1 such that

$$\big||\varphi_1^{-1}(\alpha) \cap V_\lambda| - |\varphi_1^{-1}(\alpha') \cap V_\lambda|\big| = \big||V_{\lambda,\alpha}| - |V_{\lambda,\alpha'}|\big| \le 1 \tag{8}$$

for every $\lambda \in [\ell]$ and $\alpha,\, \alpha' \in [a_1]$. Consequently, we have $\lfloor n/a_1 \rfloor \le |V_{\lambda,\alpha}| \le \lceil n/a_1 \rceil$.

Remark 18. *For convenience, we delete all floors and ceilings and simply write $|V_{\lambda,\alpha}| = n/a_1$ for every $\lambda \in [\ell]$ and $\alpha \in [a_1]$.*

Let $\binom{[\ell]}{j}_< = \{(\lambda_1, \dots, \lambda_j) \in [\ell]^j \colon \lambda_1 < \dots < \lambda_j\}$ be the set of vectors that naturally correspond to the totally ordered j-element subsets of $[\ell]$. More generally, for a totally ordered set Π of cardinality at least j, let $\binom{\Pi}{j}_<$ be the family of totally ordered j-element subsets of Π. For $j \in [k-1]$, we consider the projection π_j of $K_\ell^{(j)}(V_1, \dots, V_\ell)$ onto $[\ell]$;

$$\pi_j \colon K_\ell^{(j)}(V_1, \dots, V_\ell) \to \binom{[\ell]}{j}_<\,,$$

mapping $J \in K_\ell^{(j)}(V_1, \dots, V_\ell)$ to the totally ordered set $\pi_j(J) = (\lambda_1, \dots, \lambda_j) \in \binom{[\ell]}{j}_<$ satisfying $|J \cap V_{\lambda_h}| = 1$ for every $h \in [j]$. Moreover, for every $1 \le h \le |J|$, let

$$\Phi_h(J) = (x_{\pi_h(H)})_{H \in \binom{J}{h}} \text{ where } x_{\pi_h(H)} = \varphi_h(H) \text{ for every } H \in \binom{J}{h}\,. \tag{9}$$

In other words, $\Phi_h(J)$ is a vector of length $\binom{|J|}{h}$ and its entries, which are φ_h values of h-subsets of J, are indexed by elements from $\binom{\pi_j(J)}{h}_<$. For our purposes, it will be convenient to assume that the entries of $\Phi_h(J)$ are ordered lexicographically w.r.t. their indices. Observe that for $0 < h \le |J|$

$$\Phi_h(J) \in [a_h] \times \dots \times [a_h] = [a_h]^{\binom{j}{h}}\,.$$

We define

$$\boldsymbol{\Phi}^{(j)}(J) = (\pi_j(J), \Phi_1(J), \dots, \Phi_j(J))\,. \tag{10}$$

Note that $\boldsymbol{\Phi}^{(j)}(J)$ is a vector with $j + 2^j - 1$ entries. Observe that if we set $\boldsymbol{a} = (a_1, a_2, \dots, a_{k-1})$ and

$$A(j, \boldsymbol{a}) = \binom{[\ell]}{j}_< \times \prod_{h=1}^{j} [a_h]^{\binom{j}{h}},$$

then $\boldsymbol{\Phi}^{(j)}(J) \in A(j, \boldsymbol{a})$ for every set $J \in K_\ell^{(j)}(V_1, \dots, V_\ell)$. In other words, to each edge J of cardinality j, we assign $\pi_j(J)$ and a vector $(x_{\pi_h(H)})_{\varnothing \ne H \subseteq J}$ with each entry $x_{\pi_h(H)}$ corresponding to a non-empty subset H of J such that $x_{\pi_h(H)} = \varphi_h(H)$, where $h = |H|$.

For two edges $J_1, J_2 \in K_\ell^{(j)}(V_1, \dots, V_\ell)$, the equality $\boldsymbol{\Phi}^{(j)}(J_1) = \boldsymbol{\Phi}^{(j)}(J_2)$ defines an equivalence relation on $K_\ell^{(j)}(V_1, \dots, V_\ell)$ into at most

$$|A(j, \boldsymbol{a})| = \binom{\ell}{j} \times \prod_{h=1}^{j} a_h^{\binom{j}{h}}$$

parts. Recalling (9) and (10), we have

$$\mathbf{\Phi}^{(j)}(J_1) = \mathbf{\Phi}^{(j)}(J_2) \quad \iff \quad \begin{array}{l}\pi_j(J_1) = \pi_j(J_2) \text{ and for every } h \in [j]\,,\ H_1 \in \binom{J_1}{h} \text{ and } H_2 \in \binom{J_2}{h} \\ \text{which satisfy } \pi_h(H_1) = \pi_h(H_2) \text{ we have } x_{\pi_h(H_1)} = x_{\pi_h(H_2)}\,. \end{array} \tag{11}$$

For each $j < k$, we define a partition $\mathscr{R}^{(j)}$ of $K_\ell^{(j)}(V_1, \dots, V_\ell)$ with partition classes corresponding to the equivalence relation defined above. In this way, each partition class in $\mathscr{R}^{(j)}$ has a unique address $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$. While $\boldsymbol{x}^{(j)}$ is a $(j+2^j-1)$-dimensional vector, we will frequently view it as a $(j+1)$-dimensional vector $(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_j)$, where $\boldsymbol{x}_0 = (\lambda_1, \dots, \lambda_j) \in \binom{[\ell]}{j}_<$ is a totally ordered set and $\boldsymbol{x}_h = (x_\Xi\colon\ \Xi \in \binom{\boldsymbol{x}_0}{h}_<) \in [a_h]^{\binom{j}{h}}$ for $1 \le h \le j$. For each address $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$, we denote its corresponding partition class from $\mathscr{R}^{(j)}$ by

$$\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)}) = \left\{J \in K_\ell^{(j)}(V_1, \dots, V_\ell)\colon\ \mathbf{\Phi}^{(j)}(J) = \boldsymbol{x}^{(j)}\right\}. \tag{12}$$

To give a precise description of the family of partitions of $K_\ell^{(j)}(V_1, \dots, V_\ell)$, we summarize the notation above in the following Setup in which we work.

Setup 19. *Let $2 \le k \le \ell$ and n be fixed positive integers, let $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$ be a $(n, \ell, 1)$-cylinder, and $\boldsymbol{a} = \boldsymbol{a}_{\mathscr{R}} = (a_1, a_2, \dots, a_{k-1})$ be a vector of positive integers. For every $j \in [k-1]$ let*

$$A(j, \boldsymbol{a}) = \binom{[\ell]}{j}_< \times \prod_{h=1}^{j} [a_h]^{\binom{j}{h}}\,,$$

and let $\varphi_j\colon\ K_\ell^{(j)}(V_1, \dots, V_\ell) \to [a_j]$ be a mapping. Moreover, suppose that φ_1 satisfies (8) for every $\lambda \in [\ell]$ and α, $\alpha' \in [a_1]$. Set $\boldsymbol{\varphi} = \{\varphi_j\colon\ j \in [k-1]\}$.

We now define the family of partitions of $K_\ell^{(j)}(V_1, \dots, V_\ell)$.

Definition 20 (family of partitions). *Given Setup 19, for every $j \in [k-1]$, define a partition $\mathscr{R}^{(j)}$ of $K_\ell^{(j)}(V_1, \dots, V_\ell)$ by*

$$\mathscr{R}^{(j)} = \left\{\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)})\colon\ \boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})\right\}.$$

We also define the family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(j)}\}_{j=1}^{k-1}$ and the rank of $\mathscr{R}$ by

$$\operatorname{rank} \mathscr{R} = |A(k-1, \boldsymbol{a})|\,.$$

We wish to claim that a family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ as defined in Definition 20 is, in fact, cohesive in the sense of Definition 17 (see Claim 21 below). To this end, we shall need a notion of polyad (as in (4) (cf. (6), (7))) corresponding to Definition 20. Fix $1 \le h \le j \le k-1$ and $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$ and choose an *arbitrary* $J \in \mathcal{R}^{(j)}(\boldsymbol{x}^{(j)})$. We define

$$\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)}) = \bigcup \left\{\mathcal{R}^{(h)}(H)\colon\ H \in \tbinom{J}{h}\right\}. \tag{13}$$

Note that $\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})$ is well-defined, i.e., independent of the choice of $J \in \mathcal{R}^{(j)}(\boldsymbol{x}^{(j)})$. Indeed, for every J_1, $J_2 \in \mathcal{R}^{(j)}(\boldsymbol{x}^{(j)})$, we have

$$\mathcal{R}^{(h)}(J_1) \overset{(6)}{=} \bigcup \left\{\mathcal{R}^{(h)}(H_1)\colon\ H_1 \in \tbinom{J_1}{h}\right\} \overset{(11),(12)}{=} \bigcup \left\{\mathcal{R}^{(h)}(H_2)\colon\ H_2 \in \tbinom{J_2}{h}\right\} \overset{(6)}{=} \mathcal{R}^{(h)}(J_2)\,.$$

The object $\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})$ given in (13) is an $(n/a_1, j, h)$-cylinder and corresponds to the objects described in of (6). When $h = j-1$, the object $\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)})$ is, in fact, a polyad of $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ (see (6) and (7)). While we discuss polyads in more detail momentarily, we first return to our claim that $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ is cohesive.

Claim 21. *Let $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ be a family of partitions as in Definition 20. Then, for every $1 \le j \le k-1$ and every $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$, $\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)}) \subseteq \mathcal{K}_j(\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)}))$. In particular, $\boldsymbol{\mathcal{R}}(\boldsymbol{x}^{(j)}) = \left\{\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})\right\}_{h=1}^{j}$ is an $(n/a_1, j, j)$-complex.*

Proof. The first assertion (cohesion) is immediate from (13). Indeed, let $J \in \mathcal{R}^{(j)}(\boldsymbol{x}^{(j)})$. Then (13) gives $\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)}) = \bigcup\{\mathcal{R}^{(j-1)}(H) : H \in \binom{J}{j-1}\}$ so that, in particular, $H \in \mathcal{R}^{(j-1)}(H)$ for all $H \in \binom{J}{j-1}$. Consequently, $J \in \mathcal{K}_j(\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)}))$. The second assertion then follows from the first. □

We now focus further attention on polyads $\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)})$ of a family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$.

Polyads. Let $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ be a family of partitions as defined in Definition 20 and fix $1 \le j \le k-1$ and $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$. Note that $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$ is of the form $\boldsymbol{x}^{(j)} = (\hat{\boldsymbol{x}}^{(j-1)}, \alpha)$, where $\alpha \in [a_j]$ and $\hat{\boldsymbol{x}}^{(j-1)}$ is a $(j+2^j-2)$-dimensional vector. Importantly, note that the polyad $\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)})$ depends only on $\hat{\boldsymbol{x}}^{(j-1)}$ (and not α). As such, we wish to allocate $\hat{\boldsymbol{x}}^{(j-1)}$ as the 'address' of the polyad $\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)})$, and now proceed to formalize this effort.

We define the set $\hat{A}(j-1, \boldsymbol{a})$ of $(j+2^j-2)$-dimensional vectors for $j \in [k-1]$ by

$$\hat{A}(j-1, \boldsymbol{a}) = \binom{[\ell]}{j}_{<} \times \prod_{h=1}^{j-1} [a_h]^{\binom{j}{h}} \quad \text{so that} \quad \big|\hat{A}(j-1, \boldsymbol{a})\big| = \binom{\ell}{j}_{<} \times \prod_{h=1}^{j-1} a_h^{\binom{j}{h}} . \tag{14}$$

Consider a vector $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_{j-1}) \in \hat{A}(j-1, \boldsymbol{a})$ with $\hat{\boldsymbol{x}}_0 = (\lambda_1, \dots, \lambda_j) \in \binom{[\ell]}{j}_{<}$. Then, for every $h \in [j-1]$, vector $\hat{\boldsymbol{x}}_h$ can be written as $\hat{\boldsymbol{x}}_h = (x_\Xi \colon \Xi \in \binom{\hat{\boldsymbol{x}}_0}{h}_{<})$, i.e., its entries are labeled by the ordered h-element subsets of the ordered set $\hat{\boldsymbol{x}}_0 \in \binom{[\ell]}{j}_{<}$ in lexicographic order w.r.t. the indices. For every $u \in [j]$, we set

$$\partial_u \hat{\boldsymbol{x}}_h = \left(x_\Xi \colon \Xi \in \binom{\hat{\boldsymbol{x}}_0 \setminus \{\lambda_u\}}{h}_{<} \right) . \tag{15}$$

In other words, vector $\partial_u \hat{\boldsymbol{x}}_h$ contains precisely those entries of $\hat{\boldsymbol{x}}_h$ which are labeled by the h-element subsets of $\hat{\boldsymbol{x}}_0$ not containing λ_u. Clearly, $\partial_u \hat{\boldsymbol{x}}_h$ has $\binom{j-1}{h}$ entries from $[a_h]$. Furthermore, we set

$$\partial_u \hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0 \setminus \{\lambda_u\}, \partial_u \hat{\boldsymbol{x}}_1, \partial_u \hat{\boldsymbol{x}}_2, \dots, \partial_u \hat{\boldsymbol{x}}_{j-1})$$

and observe that $\partial_u \hat{\boldsymbol{x}}^{(j-1)}$ is a $(j-1+2^{j-1}-1)$-dimensional vector belonging to $A(j-1, \boldsymbol{a})$. Finally, for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$, we set the corresponding polyad equal to

$$\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}) = \bigcup \Big\{ \mathcal{R}^{(j-1)}\big(\partial_u \hat{\boldsymbol{x}}^{(j-1)}\big) \colon u \in [j] \Big\} .$$

Note that $\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ and $\mathcal{R}^{(j-1)}((\hat{\boldsymbol{x}}^{(j-1)}, \alpha))$ are identical for any $\alpha \in [a_j]$.

We make a final definition. Note that since $\mathscr{R}$ is cohesive, we have that the $(n/a_1, j, j-1)$-complexes $\boldsymbol{\mathcal{R}}(J_1)$ and $\boldsymbol{\mathcal{R}}(J_2)$, defined in (6), are identical for all J_1, $J_2 \in \mathcal{K}_j\big(\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)$. Consequently, if $\mathcal{K}_j\big(\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) \neq \varnothing$, then we define

$$\boldsymbol{\mathcal{R}}(\hat{\boldsymbol{x}}^{(j-1)}) = \boldsymbol{\mathcal{R}}(J) \text{ for some } J \in \mathcal{K}_j\big(\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) . \tag{16}$$

In the rather uninteresting case $\mathcal{K}_j\big(\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = \varnothing$, we set $\boldsymbol{\mathcal{R}}(\hat{\boldsymbol{x}}^{(j-1)}) = \varnothing$. The following observation is a direct consequence from the definitions.

Claim 22. *Let $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ be a family of partitions as in Definition 20. For every vector $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_{j-1}) \in \hat{A}(j-1, \boldsymbol{a})$, the following statements are true.*

(a) $\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ *is an $(n/a_1, j, j-1)$-cylinder;*
(b) $\boldsymbol{\mathcal{R}}(\hat{\boldsymbol{x}}^{(j-1)})$ *is an $(n/a_1, j, j-1)$-complex.*

For every polyad $\hat{\mathcal{R}}^{(j-1)}$, there exists a unique vector $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$ so that $\hat{\mathcal{R}}^{(j-1)} = \hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$. Hence, each polyad $\hat{\mathcal{R}}^{(j-1)}$ uniquely defines an $(n/a_1, j, j-1)$-complex $\boldsymbol{\mathcal{R}}(\hat{\boldsymbol{x}}^{(j-1)}) = \big\{\mathcal{R}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})\big\}_{h=1}^{j-1}$ such that $\hat{\mathcal{R}}^{(j-1)} = \hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$.

3.3. **Regular partitions and the regularity lemma.** In addition to the structural properties discussed in the preceding section, the family of the partitions provided by the regularity lemma features additional "regularity" properties. Loosely speaking, for "most" $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{a})$ the $(n/a_1, k, k-1)$-complex $\boldsymbol{\mathcal{R}}(\hat{\boldsymbol{x}}^{(k-1)})$ is regular (cf. (16) and Definition 11).

In the two definitions, below we introduce two concepts central to regular partitions. We use the notation δ', d' and r' to be consistent with the context in which we apply the Regularity Lemma (Theorem 25 below).

Definition 23 (**($\mu, \boldsymbol{\delta'}, \boldsymbol{d'}, r'$)-equitable**)**.** *Let μ be a number in the interval $(0,1]$, let $\boldsymbol{\delta'} = (\delta'_2, \ldots, \delta'_k)$ and $\boldsymbol{d'} = (d'_2, \ldots, d'_k)$ be two arbitrary but fixed vectors of real numbers between 0 and 1 and let r' be a positive integer. We say that a family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ (as defined in Definition 20) is $(\mu, \boldsymbol{\delta'}, \boldsymbol{d'}, r')$-equitable if all but μn^k edges of $K_\ell^{(k)}(V_1, \ldots, V_\ell)$ belong to $(\boldsymbol{\delta'}, \boldsymbol{d'}, r')$-regular complexes $\boldsymbol{\mathcal{R}}(\hat{\boldsymbol{x}}^{(k-1)})$ with $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{a})$.*

Before finally stating the regularity lemma, we define regular partitions.

Definition 24 (**regular partition**)**.** *Let $\mathcal{G}^{(k)}$ be a (n, ℓ, k)-cylinder and let $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ be a $(\mu, \boldsymbol{\delta'}, \boldsymbol{d'}, r')$-equitable family of partitions.*

We say $\mathscr{R}$ is a (δ'_k, r')-regular w.r.t. $\mathcal{G}^{(k)}$ if all but at most $\delta'_k n^k$ edges of $K_\ell^{(k)}(V_1, \ldots, V_\ell)$ belong to polyads $\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ such that $\mathcal{G}^{(k)}$ is (δ'_k, r')-regular with respect to $\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$.

We now state the Regularity Lemma of Rödl and Skokan. In what follows, $\boldsymbol{D} = (D_2, \ldots, D_{k-1})$ is a vector of positive real variables and A_1 is an integer variable.

Theorem 25 (Regularity Lemma (ℓ-partite version))**.** *For all integers $\ell \geq k \geq 2$ and all positive reals δ'_k and μ and all positive valued functions*

$$\boldsymbol{\delta'}(\boldsymbol{D}) = \left(\delta'_{k-1}(D_{k-1}), \ldots, \delta'_2(D_2, \ldots, D_{k-1})\right), \qquad r'(A_1, \boldsymbol{D}) = r'(A_1, D_2, \ldots, D_{k-1}),$$

there exist integers n_k and L_k such that the following holds.

For every (n, ℓ, k)-cylinder $\mathcal{G}^{(k)}$ with $n \geq n_k$ there exist an integer vector $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$, a density vector $\boldsymbol{d'} = \left(d'_2, \ldots, d'_{k-1}\right)$, and a $\left(\mu, \boldsymbol{\delta'}(\boldsymbol{d'}), \boldsymbol{d'}, r'(a_1, \boldsymbol{d'})\right)$-equitable $\left(\delta'_k, r'(a_1, \boldsymbol{d'})\right)$-regular family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ with $\operatorname{rank} \mathscr{R} = |A(k-1, \boldsymbol{a})| \leq L_k$.

In the upcoming Corollary 27, we state a modification of Theorem 25 whose formulation is convenient for us in our proof of Theorem 12. Before stating Corollary 27, we outline the main differences between Theorem 25 and its corollary below. For that we need the following definition.

Definition 26 (**refinement**)**.** *Let $\{\mathcal{G}^{(h)}\}_{h=1}^j$ be an (n, ℓ, j)-complex and let $\mathscr{R} = \mathscr{R}(j, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(h)}\}_{h=1}^j$ be a family of partitions of $K_\ell^{(h)}(V_1, \ldots, V_\ell)$ for $h \in [j]$. We say $\mathscr{R}$ refines $\{\mathcal{G}^{(h)}\}_{h=1}^j$ if for every $h \in [j]$ and every $\boldsymbol{x}^{(h)} \in A(h, \boldsymbol{a})$ either $\mathcal{R}^{(h)}(\boldsymbol{x}^{(h)}) \subseteq \mathcal{G}^{(h)}$ or $\mathcal{R}^{(h)}(\boldsymbol{x}^{(h)}) \cap \mathcal{G}^{(h)} = \varnothing$.*

Moreover, adding an additional layer $\mathcal{G}^{(j+1)} \subseteq \mathcal{K}_{j+1}(\mathcal{G}^{(j)})$ to $\{\mathcal{G}^{(h)}\}_{h=1}^j$, we will also say that $\mathscr{R} = \{\mathscr{R}^{(h)}\}_{h=1}^j$ refines the $(n, \ell, j+1)$-complex $\{\mathcal{G}^{(h)}\}_{h=1}^j \cup \{\mathcal{G}^{(j+1)}\}$ if $\mathscr{R}$ refines $\{\mathcal{G}^{(h)}\}_{h=1}^j$.

It is a well known fact that the proof of Szemerédi's regularity lemma not only yields the existence of a regular partition for any graph $\mathcal{G}^{(2)}$, but also shows that *any* given equitable, initial partition $\mathcal{G}^{(1)} = V_1 \cup \cdots \cup V_\ell$ of the vertex set $V = V\left(\mathcal{G}^{(2)}\right)$ has a regular refinement. Similarily, the proof of Theorem 25 (which is proved by induction on k) yields immediately the existence of a regular and equitable partition $\mathscr{R}$ which refines a given equitable partition of the underlying structure. In particular, regularizing the k-th layer $\mathcal{G}^{(k)}$ of a given (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$, one can obtain a partition $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ satisfying the following property: *for any $1 \leq j \leq k-1$ and every $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$, either $\mathcal{R}^{(j)}\left(\boldsymbol{x}^{(j)}\right) \subseteq \mathcal{G}^{(j)}$ or $\mathcal{R}^{(j)}\left(\boldsymbol{x}^{(j)}\right) \cap \mathcal{G}^{(j)} = \varnothing$.* In other words, $\mathscr{R}$ refines the partitions given by $\mathcal{G}^{(j)} \cup \overline{\mathcal{G}^{(j)}} = K^{(j)}(V_1, \ldots, V_\ell)$ for every $j \in [k-1]$.

One can maintain yet another property of the $(\mu, \boldsymbol{\delta'}(\boldsymbol{d'}), \boldsymbol{d'}, r')$-equitable family of partitions $\mathscr{R}$ with density vector $\boldsymbol{d'}$. In the proof of Theorem 25 (cf. [40]), the $\boldsymbol{d'}$ are chosen explicitly and there is some freedom to choose them (more precisely there is no necessary lower bound on each d'_j, $2 \leq j \leq k-1$). Hence, we shall assume, without loss of generality, that for any given *fixed* $\sigma_2, \ldots, \sigma_{k-1}$, we may arrange the constants d'_j, $2 \leq j \leq k-1$, so that the quotients σ_j / d'_j, $2 \leq j \leq k-1$, are integers.

Summarizing the discussion above we arrive at the following Corollary 27 (stated below) of Theorem 25. The full proof of Corollary 27 is identical to the proof of Theorem 25 with the two minor adjustments indicated above (see [40, Lemma 12.1]).

Corollary 27. *For all integers $\ell \geq k \geq 2$ and all positive reals $\sigma_2, \ldots, \sigma_{k-1}$, δ'_k, and μ and all positive functions*

$$\boldsymbol{\delta'}(\boldsymbol{D}) = \left(\delta'_{k-1}(D_{k-1}), \ldots, \delta'_2(D_2, \ldots, D_{k-1})\right), \qquad r'(A_1, \boldsymbol{D}) = r'(A_1, D_2, \ldots, D_{k-1}),$$

there exist integers n_k and L_k such that the following holds.

For every (n,ℓ,k)-complex $\boldsymbol{\mathcal{G}}=\{\mathcal{G}^{(j)}\}_{j=1}^{k}$ with $n\geq n_k$ there exists an integer vector $\boldsymbol{a}=(a_1,\dots,a_{k-1})$, a density vector $\boldsymbol{d'}=\big(d'_2,\dots,d'_{k-1}\big)$, and a $(\mu,\boldsymbol{\delta'}(\boldsymbol{d'}),\boldsymbol{d'},r'(a_1,\boldsymbol{d'}))$-equitable $(\delta'_k,r'(a_1,\boldsymbol{d'}))$-regular (w.r.t. $\mathcal{G}^{(k)}$) family of partitions $\mathscr{R}=\mathscr{R}\left(k-1,\boldsymbol{a},\boldsymbol{\varphi}\right)$ such that

- (i) *$\mathscr{R}$ refines $\boldsymbol{\mathcal{G}}$,*
- (ii) *σ_j/d'_j is an integer for $j=2,\dots,k-1$, and*
- (iii) $\operatorname{rank}\mathscr{R}=|A\left(k-1,\boldsymbol{a}\right)|\leq L_k$.

3.3.1. *Statement of cleaning phase I.* The proof of the main theorem, Theorem 12, presented in Section 4 uses the following lemma, Lemma 29, which follows from Corollary 27 and the induction assumption on Theorem 12.

We use Lemma 29 in the proof of Theorem 12 instead of Corollary 27 since it allows a simpler presentation of the later arguments. For $k=2$, Lemma 29 is a straightforward reformulation of Szemerédi's lemma and reduces to the statement that for any graph $\mathcal{G}^{(2)}=(V,E)$, there is a graph $\tilde{\mathcal{G}}^{(2)}$ for which $\left|\mathcal{G}^{(2)}\triangle\,\tilde{\mathcal{G}}^{(2)}\right|$ is small and where $\tilde{\mathcal{G}}^{(2)}=(V,\tilde{E})$ admits a "perfectly equitable" partition, i.e., $V=V_1\cup\dots\cup V_t$ with $|V_1|=\dots=|V_t|$ and all pairs (V_i,V_j) are ε-regular for $1\leq i<j\leq t$. Lemma 29 will generalize this concept for $\mathcal{G}^{(k)}$ with $k>2$.

The following definition reflects the ideal situation when, for each $2\leq j<k$, every partition class is regular. Similarly to Definition 23 and Definition 24, we use tilde-notation in the next definition to be consistent with the context in which it is used later.

Definition 28 (**perfect** $(\tilde{\boldsymbol{\delta}},\tilde{\boldsymbol{d}},\tilde{r},\boldsymbol{b})$**-family**). *Let $\tilde{\boldsymbol{\delta}}=\big(\tilde{\delta}_2,\dots,\tilde{\delta}_{k-1}\big)$ and $\tilde{\boldsymbol{d}}=\big(\tilde{d}_2,\dots,\tilde{d}_{k-1}\big)$ be vectors of reals, $\tilde{r}>0$ be an integer and $\boldsymbol{b}=(b_1,\dots,b_{k-1})$ be a vector of positive integers.*

We say that a family of partitions $\mathscr{P}=\mathscr{P}(k-1,\boldsymbol{b},\boldsymbol{\psi})$ (as defined in Definition 20) is a perfect $\big(\tilde{\boldsymbol{\delta}},\tilde{\boldsymbol{d}},\tilde{r},\boldsymbol{b}\big)$-family of partitions if the following holds:

- (i) *$\tilde{d}_j=1/b_j$ for every $2\leq j\leq k-1$, and*
- (ii) *for every $2\leq j\leq k-1$, for every $\hat{\boldsymbol{x}}^{(j-1)}\in\hat{A}\left(j-1,\boldsymbol{b}\right)$ and for every $\beta\in[b_j]$, the $(n/b_1,j,j)$-cylinder $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)},\beta)\big)$ is $\big(\tilde{\delta}_j,\tilde{d}_j,\tilde{r}\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}\big(\hat{\boldsymbol{x}}^{(j-1)}\big)$.*

A perfect family of partitions $\mathscr{P}$ has the property that for *all* $\boldsymbol{x}^{(k-1)}\in A(k-1,\boldsymbol{b})$ the $(n/b_1,k-1,k-1)$-complex $\boldsymbol{\mathcal{P}}^{(k-1)}(\boldsymbol{x}^{(k-1)})=\{\mathcal{P}^{(j)}(\boldsymbol{x}^{(k-1)})\}_{j=1}^{k-1}$ (cf. Claim 21) is $\big((\tilde{\delta}_2,\dots,\tilde{\delta}_{k-1}),(\tilde{d}_2,\dots,\tilde{d}_{k-1}),\tilde{r}\big)$-regular.

The statement of the next lemma is very much tailored for its application in Section 4.4. While the constants $\delta_3,\dots,\delta_k$ are inessential for the proof of the following lemma, they allow a presentation which is consistent with the context in which the lemma will be applied. As well, the constants $d_2,\dots,d_k$ are formulated as rational numbers, for reasons to which we return in the next section.

Lemma 29 (Cleaning Phase I)**.** *For every vector $\boldsymbol{d}=(d_2,\dots,d_k)$ of positive rationals, for every choice of $\delta_3,\dots,\delta_k$, for any positive real $\tilde{\delta}_k$ and all positive functions*

$$\tilde{\boldsymbol{\delta}}\big(\boldsymbol{D}\big)=\big(\tilde{\delta}_{k-1}\big(D_{k-1}\big),\dots,\tilde{\delta}_2\big(D_2,\dots,D_{k-1}\big)\big)\,,\qquad\qquad \tilde{r}\big(B_1,\boldsymbol{D}\big)=\tilde{r}\big(B_1,D_2,\dots,D_{k-1}\big)\,,$$

there exist integers $\tilde{n}_k$, $\tilde{L}_k$, a vector of positive reals $\tilde{\boldsymbol{c}}=(\tilde{c}_2,\dots,\tilde{c}_{k-1})$ and a positive constant δ_2 so that the following holds:

For every $(\boldsymbol{\delta}=(\delta_2,\dots,\delta_k),\boldsymbol{d},1)$-regular (n,ℓ,k)-complex $\boldsymbol{\mathcal{G}}=\{\mathcal{G}^{(j)}\}_{j=1}^{k}$ with $n\geq\tilde{n}_k$ there exist an (n,ℓ,k)-complex $\tilde{\boldsymbol{\mathcal{G}}}=\{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$, an integer vector $\boldsymbol{b}=(b_1,\dots,b_{k-1})$, a density vector $\tilde{\boldsymbol{d}}=\big(\tilde{d}_2,\dots,\tilde{d}_{k-1}\big)$ componentwise bigger than $\tilde{\boldsymbol{c}}$, and a perfect $\big(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}),\tilde{\boldsymbol{d}},\tilde{r}(b_1,\tilde{\boldsymbol{d}}),\boldsymbol{b}\big)$-family of partitions $\mathscr{P}=\mathscr{P}\left(k-1,\boldsymbol{b},\boldsymbol{\psi}\right)=\{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ refining $\tilde{\boldsymbol{\mathcal{G}}}$ so that:

- (i) *$\tilde{\mathcal{G}}^{(k)}$ is $\big(\tilde{\delta}_k,\tilde{r}(b_1,\tilde{\boldsymbol{d}})\big)$-regular with respect to $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ for every $\hat{\boldsymbol{x}}^{(k-1)}\in\hat{A}\left(k-1,\boldsymbol{b}\right)$,*
- (ii) *$\tilde{\mathcal{G}}^{(1)}=\mathcal{G}^{(1)}$, $\tilde{\mathcal{G}}^{(2)}=\mathcal{G}^{(2)}$, and $\tilde{\mathcal{G}}^{(j)}\subseteq\mathcal{G}^{(j)}$ for every $3\leq j\leq k$,*
- (iii) *for every $3\leq j\leq k$ and every $j\leq i\leq\ell$, the following holds:*

$$\big|\mathcal{K}_i(\mathcal{G}^{(j)})\triangle\,\mathcal{K}_i(\tilde{\mathcal{G}}^{(j)})\big|=\big|\mathcal{K}_i(\mathcal{G}^{(j)})\setminus\mathcal{K}_i(\tilde{\mathcal{G}}^{(j)})\big|\leq\tilde{\delta}_k\prod_{h=2}^{j}d_h^{\binom{i}{h}}\times n^i\,,$$

(iv) *for every* $\hat{\boldsymbol{x}}^{(1)} = ((\lambda_1, \lambda_2), (\beta_1, \beta_2)) \in \hat{A}(1, \boldsymbol{b})$*, the graph* $\tilde{\mathcal{G}}^{(2)}[V_{\lambda_1,\beta_1}, V_{\lambda_2,\beta_2}]$ *is* $(\tilde{L}_k^2\delta_2, d_2, 1)$*-regular w.r.t.* $\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}}^{(1)}) = V_{\lambda_1,\beta_1} \cup V_{\lambda_2,\beta_2}$*,*

(v) $\operatorname{rank} \mathscr{P} \leq \tilde{L}_k$ *and consequently* $|\hat{A}(k-1, \boldsymbol{b})| \leq (\tilde{L}_k)^k$*, and*

(vi) $d_j/\tilde{d}_j$ *is an integer.*

In the proof of Lemma 29 (see Section 5) we construct an (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}}$ admitting a perfect family of partitions $\mathscr{P}$. Moreover, $\tilde{\boldsymbol{\mathcal{G}}}$ is almost identical to a given (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}}$ (see (ii) and (iii) of Lemma 29). In particular, $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$, while we allow a small difference between $\tilde{\mathcal{G}}^{(j)}$ and $\mathcal{G}^{(j)}$ for $j \geq 3$.

On the other hand, note that the partition $\mathscr{P}$ given by Lemma 29 is perfect in the sense that for $2 \leq j \leq k$ every j-tuple of $\tilde{\mathcal{G}}^{(j)}$ belongs to a regular polyad of $\mathscr{P}$. This feature will later give us a significant notational advantage. Moreover, important in view of Theorem 12, Lemma 29 (iii) ensures that the two complexes $\boldsymbol{\mathcal{G}}$ and $\tilde{\boldsymbol{\mathcal{G}}}$ differ by few cliques only.

3.3.2. *The slicing lemma.* The following lemma whose proof is based on the fact that *randomly chosen subcylinders of a regular cylinder are regular*, which follows from the concentration of the binomial distribution, was proved in [40]. We will find it useful in this paper as well.

Lemma 30 (Slicing Lemma)**.** *Suppose* ϱ*,* δ *are two real numbers such that* $0 < \delta/2 < \varrho \leq 1$*. There is an* $m_0 = m_0(\varrho, \delta)$ *such that the following holds for all* $m \geq m_0$*. If* $\hat{\mathcal{P}}^{(j-1)}$ *is an* $(m, j, j-1)$*-cylinder so that* $\left|\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\right| \geq m^j/\ln m$ *and if* $\mathcal{F}^{(j)} \subseteq \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})$ *is an* (m, j, j)*-cylinder which is* $(\delta, \varrho, r_{\mathrm{SL}})$*-regular w.r.t.* $\hat{\mathcal{P}}^{(j-1)}$*, then for every* $0 < p < 1$*, where* $3\delta < p\varrho$ *and* $u = \lfloor 1/p \rfloor$*, there exists a decomposition of* $\mathcal{F}^{(j)} = \mathcal{F}_0^{(j)} \cup \mathcal{F}_1^{(j)} \cup \cdots \cup \mathcal{F}_u^{(j)}$ *such that* $\mathcal{F}_i^{(j)}$ *is* $(3\delta, p\varrho, r_{\mathrm{SL}})$*-regular w.r.t.* $\hat{\mathcal{P}}^{(j-1)}$ *for* $1 \leq i \leq u$*.*

Moreover if $1/p$ *is an integer then* $\mathcal{F}_0^{(j)} = \varnothing$*.*

Remark 31. *The proof of Lemma 30 given in [40] is probabilistic. In fact, there the following stronger statement can be obtained by the same proof. Under the assumption of Lemma 30, let* $\mathcal{F}_p^{(j)}$ *be the random subhypergraph of* $\mathcal{F}^{(j)}$ *where edges are chosen independently with probability* p*. Let* A *be the event that* $\mathcal{F}_p^{(j)}$ *is* $(3\delta, p\varrho, r_{\mathrm{SL}})$*-regular w.r.t.* $\hat{\mathcal{P}}^{(j-1)}$*, then* $\mathbb{P}(A) = 1 - o(1)$*, with* $o(1) \to 0$ *as* $m \to \infty$*.*

4. Proof of the Counting Lemma

In this section, we provide the outline of our proof of the Counting Lemma, Theorem 12. Our proof of the Counting Lemma follows an induction on $k \geq 2$, but our argument is interwoven in a substantial way with upcoming Theorem 34 (stated below). We require some preparation to present this inductive outline, and begin by discussing its base case and inductive assumption.

4.1. **Induction assumption on the counting lemma.** We prove the Counting Lemma by induction on $k \geq 2$. For $k = 2$, the Counting Lemma, i.e., Fact 2, is a well known fact (see, e.g., [25, 26]) and for $k = 3$ it was proved by Nagle and Rödl in [28]. Therefore, from now on let $k \geq 4$ be a fixed integer. (For the inductive proof presented here it suffices to assume Fact 2 as the base case.)

Induction hypothesis. *We assume that*

$$\mathbf{CL}_{j,i} \text{ holds for } 2 \leq j \leq k-1 \text{ and } i \geq j\,. \tag{17}$$

We prove $\mathbf{CL}_{k,\ell}$ holds for all integers $\ell \geq k$. For that, throughout the proof, let $\ell \geq k$ be fixed.

Rather than quoting various forms of our induction hypothesis $\mathbf{CL}_{j,i}$ (for varying $2 \leq j \leq k-1$ and $j \leq i \leq \ell$) involving different δ's and γ's, we summarize all such statements in one. The following statement, which we denote by $\mathbf{IHC}_{k-1,\ell}$, is a reformulation of our induction hypothesis.

Statement 32 (Induction hypothesis on counting)**.** *The following is true:* $\forall \eta > 0\ \forall d_{k-1} > 0\ \exists \delta_{k-1} > 0\ \forall d_{k-2} > 0\ \exists \delta_{k-2} > 0\ \ldots\ \forall d_2 > 0\ \exists \delta_2 > 0$ *and there exist integers* r *and* $m_{k-1,\ell}$ *so that for all integers* j *and* i *with* $2 \leq j \leq k-1$ *and* $j \leq i \leq \ell$ *the following holds.*

If $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^{j}$ *is a* $\big((\delta_2, \ldots, \delta_j), (d_2, \ldots, d_j), r\big)$*-regular* (m, i, j)*-complex with* $m \geq m_{k-1,\ell}$*, then*

$$\left|\mathcal{K}_i(\mathcal{G}^{(j)})\right| = (1 \pm \eta) \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times m^i\,.$$

The following fact confirms that Statement 32 is an easy consequence of our Induction Hypothesis in (17).

Fact 33. *If* $\mathbf{CL}_{j,i}$ *holds for all integers* $2 \le j \le k-1$ *and* $j \le i \le \ell$, *then* $\mathbf{IHC}_{k-1,\ell}$ *holds.*

Note that Fact 33 is trivial to prove and only requires confirming the constants may be chosen appropriately; when given $\eta, d_{k-1}, \delta_{k-1}, \dots, \delta_{j+1}$ and d_j, choose δ_j to be the minimum of all δ_j's from the statements $\mathbf{CL}_{h,i}$ with $j \le h \le k-1$ and $h \le i \le \ell$, where δ_j appears. Similarly, we set r and $m_{k-1,\ell}$ to the maximum of the corresponding constants in $\mathbf{CL}_{h,i}$.

As a consequence of the induction assumption stated in (17) and Fact 33, we may assume for the remainder of this paper that

$$\mathbf{IHC}_{k-1,\ell} \text{ holds}. \tag{18}$$

It now remains to prove the inductive step $\mathbf{IHC}_{k-1,\ell} \implies \mathbf{CL}_{k,\ell}$.

4.2. **Proof of the inductive step.** In what follows, we aim to reduce the inductive step $\mathbf{IHC}_{k-1,\ell} \implies \mathbf{CL}_{k,\ell}$ to Theorem 34 stated momentarily (the message of Theorem 34 won't directly discuss 'counting', but from it, counting will be easily inferred). The purpose of our current Section 4.2 is to verify the implication

$$\text{Theorem 34} \implies \mathbf{CL}_{k,\ell}, \tag{19}$$

a proof which isn't very difficult, but which brings a formal conclusion to the inductive step. Upon verifying the implication (19), the remainder of our paper is devoted to the significant task of proving Theorem 34 from the induction hypothesis $\mathbf{IHC}_{k-1,\ell}$.

Before proceeding to the statement of Theorem 34, we continue with a brief and useful[1] observation concerning $\mathbf{CL}_{k,\ell}$. Consider the special case of $\mathbf{CL}_{k,\ell}$ in which all positive constants d_i, $2 \le i \le k$, are chosen to be rational numbers, and let us refer to this special case as the 'rational-$\mathbf{CL}_{k,\ell}$'. It is not too difficult to show, as we do momentarily in upcoming Fact 35, that in order to verify $\mathbf{CL}_{k,\ell}$, it suffices to prove rational-$\mathbf{CL}_{k,\ell}$. As such, the flow of the argument our current Section 4.2 presents is given by

$$\mathbf{IHC}_{k-1,\ell} \implies \text{Theorem 34} \implies \text{rational-}\mathbf{CL}_{k,\ell} \overset{\text{Fact 35}}{\implies} \mathbf{CL}_{k,\ell}\,. \tag{20}$$

We now proceed to the precise statements of Theorem 34 and Fact 35.

Theorem 34. *The following is true:* $\forall \gamma > 0\ \forall d_k \in \mathbb{Q}_+\ \exists \delta_k > 0\ \forall d_{k-1} \in \mathbb{Q}_+\ \exists \delta_{k-1} > 0\ \dots\ \forall d_2 \in \mathbb{Q}_+, \varepsilon > 0\ \exists \delta_2 > 0$ *and there exist integers* r *and* n_0 *so that, with* $\boldsymbol{d} = (d_2, \dots, d_k)$ *and* $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ *and* $n \ge n_0$, *whenever* $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^{k}$ *is a* $(\boldsymbol{\delta}, \boldsymbol{d}, r)$*-regular* (n, ℓ, k)*-complex, then there exists an* (n, ℓ, k)*-complex* $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(h)}\}_{h=1}^{k}$ *such that*

(i) $\boldsymbol{\mathcal{F}}$ *is* $(\boldsymbol{\varepsilon}, \boldsymbol{d}, 1)$*-regular, with* $\boldsymbol{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in \mathbb{R}^{(k-1)}$,
(ii) $\mathcal{F}^{(1)} = \mathcal{G}^{(1)}$ *and* $\mathcal{F}^{(2)} = \mathcal{G}^{(2)}$, *and*
(iii) $\big|\mathcal{K}_\ell\big(\mathcal{G}^{(k)}\big) \triangle \mathcal{K}_\ell\big(\mathcal{F}^{(k)}\big)\big| \le (\gamma/2) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell$.

We mention that Theorem 34 has some interesting implications of its own which we discuss in Section 9.

We now proceed to Fact 35. In our formulation of Fact 35 below, we assert something slightly stronger than advertised. Let us refer to the special case of $\mathbf{CL}_{k,\ell}$ in which all d_i, $2 \le i \le k$, are reciprocals of integers by 'reciprocal-$\mathbf{CL}_{k,\ell}$'. We prove the following.

Fact 35. *Statements* $\mathbf{IHC}_{k-1,\ell}$ *and reciprocal-*$\mathbf{CL}_{k,\ell}$ *imply* $\mathbf{CL}_{k,\ell}$.

The remainder of this section is organized as follows. In Sections 4.2.1 and 4.2.2, we verify implication (19). More prcisely, we deduce rational-$\mathbf{CL}_{k,\ell}$ from Theorem 34 in Section 4.2.1 and prove Fact 35 in Section 4.2.2 The proof of Theorem 34 (based on the induction assumption $\mathbf{IHC}_{k-1,\ell}$) is outlined in Section 4.3 and given in Section 4.4.

[1] Some arguments in our proof of Theorem 34 will simplify formally somewhat, e.g., avoiding calculations with integer parts, if we assume that all constants d_i, $2 \le i \le k$, are rational numbers. Intuitively, this may not seem to represent any restriction since the numbers d_i, $2 \le i \le k$, essentially represent relative densities of hypergraphs (which are clearly rational). However, due to the complex quantification of the Counting Lemma, Theorem 12, the argument that such an assumption can be made does not appear to be completely straightforward.

4.2.1. *Rational-*$\mathbf{CL}_{k,\ell}$ *follows from Theorem 34.* The proof of this reduction is based on the Dense Counting Lemma, Theorem 16.

Proof: Theorem 34 $\Longrightarrow$ *rational-*$\mathbf{CL}_{k,\ell}$. We begin by describing the constants involved. Observe that with the exception of ε, rational-$\mathbf{CL}_{k,\ell}$ and Theorem 34 involve the same constants under the same quantification. Hence, given γ and $d_k \in \mathbb{Q}_+$ from rational-$\mathbf{CL}_{k,\ell}$, we let δ_k be the $\delta_k\big(\text{Thm.}34(\gamma, d_k)\big)$ from Theorem 34. In general, given $d_j \in \mathbb{Q}_+$, $3 \le j \le k$, we set

$$\delta_j = \delta_j\big(\text{Thm.}34(\gamma, d_k, \delta_k, d_{k-1}, \dots, \delta_{j+1}, d_j)\big)\,.$$

Having fixed $\gamma, d_k, \delta_k, d_{k-1}, \dots, \delta_4, d_3, \delta_3$, now let $d_2 \in \mathbb{Q}_+$ be given by rational-$\mathbf{CL}_{k,\ell}$. Next, we fix ε for Theorem 34 so that

$$\varepsilon \le \varepsilon\big(\text{Thm.}16(d_2, \dots, d_k)\big) \quad \text{and} \quad g_{k,\ell}(\varepsilon) \le \frac{\gamma}{2}\,, \tag{21}$$

where $g_{k,\ell}$ is given by the Dense Counting Lemma, Theorem 16. Moreover, let $m_0\big(\text{Thm.}16(d_2, \dots, d_k)\big)$ be the lower bound on the number of vertices given by Theorem 16 applied to $d_2, \dots, d_k$.

Then, Theorem 34 yields

$$\begin{aligned} \delta_2\big(\text{Thm.}34(\gamma, d_k, \delta_k, \dots, \delta_3, d_2, \varepsilon)\big)\,, \quad r\big(\text{Thm.}34(\gamma, d_k, \delta_k, \dots, \delta_3, d_2, \varepsilon)\big)\,, \\ \text{and} \quad n_0\big(\text{Thm.}34(\gamma, d_k, \delta_k, \dots, \delta_3, d_2, \varepsilon)\big)\,. \end{aligned} \tag{22}$$

Finally, we set δ_2 and r for rational-$\mathbf{CL}_{k,\ell}$ to its corresponding constants given in (22). Also, we set n_0 for rational-$\mathbf{CL}_{k,\ell}$ to

$$n_0 = \max\big\{n_0\big(\text{Thm.}34(\gamma, d_k, \delta_k, \dots, \delta_3, d_2, \varepsilon)\big), m_0\big(\text{Thm.}16(d_2, \dots, d_k)\big)\big\}\,.$$

Now, let $\boldsymbol{\mathcal{G}}$ be a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular (n, ℓ, k)-complex satisfying $n \ge n_0$. Then, Theorem 34 yields an $(\boldsymbol{\varepsilon}, \boldsymbol{d}, 1)$-regular (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}}$ satisfying (i)–(iii) of Theorem 34. Consequently, by (21) and (i), we may apply the Dense Counting Lemma to $\boldsymbol{\mathcal{F}}$. Therefore,

$$\Big|\mathcal{K}_\ell\big(\mathcal{F}^{(k)}\big)\Big| = \Big(1 \pm \frac{\gamma}{2}\Big) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell$$

and rational-$\mathbf{CL}_{k,\ell}$ follows from (iii) of Theorem 34. □

We note that the proof of rational $\mathbf{CL}_{k,\ell}$ did not use the full strength of Theorem 34. In particular, we made no use of (ii) here. However, (ii) is important with respect to further consequences of Theorem 34 discussed in Section 9.

4.2.2. $\mathbf{CL}_{k,\ell}$ *follows from reciprocal-*$\mathbf{CL}_{k,\ell}$. The proof of this reduction is based on a simple probabilistic argument and the induction assumption on the Counting Lemma, $\mathbf{IHC}_{k-1,\ell}$.

Proof of Fact 35. We begin by discussing the constants involved, and for simplicity, we set $C = \{1/s : s \in \mathbb{N}\}$ to be the set of integer-reciprocals.

Let γ and d_k be given as in $\mathbf{CL}_{k,\ell}$. Choose any $c_k \in C$ satisfying $d_k/2 \le c_k \le d_k$. Let $\check{\delta}_k = \delta_k(\text{reciprocal}-\mathbf{CL}_{k,\ell}(\gamma/2, c_k))$. We set $\delta_k = \check{\delta}_k/(9 \times 2^{2^k - 1})$. For $2 \le j < k$, we proceed in a similar way. Indeed, after d_j is revealed as in $\mathbf{CL}_{k,\ell}$, choose any $c_j \in C$ such that $d_j/2 \le c_j \le d_j$. Set

$$\begin{aligned} \check{\delta}_j = \min\Big\{&\delta_j\big(\text{reciprocal-}\,\mathbf{CL}_{k,\ell}(\tfrac{\gamma}{2}, c_k, \dots, c_j)\big)\,, \\ &\delta_j\big(\,\mathbf{IHC}_{k-1,\ell}(\tfrac12, c_{k-1}, \dots, c_j)\big)\,, \delta_j\big(\,\mathbf{IHC}_{k-1,\ell}(\tfrac12, d_{k-1}, \dots, d_j)\big)\Big\}^2\,. \end{aligned}$$

and

$$\delta_j = \frac{\check{\delta}_j}{9 \times 2^{2^j - 1}}\,.$$

Finally, we set

$$\begin{aligned} r = \min\Big\{&r\big(\text{reciprocal-}\,\mathbf{CL}_{k,\ell}(\tfrac{\gamma}{2}, c_k, \dots, c_2)\big)\,, \\ &r\big(\,\mathbf{IHC}_{k-1,\ell}(\tfrac12, c_{k-1}, \dots, c_2)\big)\,, r\big(\,\mathbf{IHC}_{k-1,\ell}(\tfrac12, d_{k-1}, \dots, d_2)\big)\Big\}\,. \end{aligned}$$

Let a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^k$ be given, where $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ and $\boldsymbol{d} = (d_2, \dots, d_k)$ and where n is sufficiently large. For $j = 2, \dots, k$, let $\mathcal{G}_{p_j}^{(j)}$ be a random subhypergraph of $\mathcal{G}^{(j)}$ with edges chosen independently, each with probability $p_j = c_j/d_j$. We consider the "random" (n, ℓ, k)-subcomplex $\check{\boldsymbol{\mathcal{G}}} = \{\check{\mathcal{G}}^{(h)}\}_{h=1}^k$ of $\boldsymbol{\mathcal{G}}$ recursively defined by

$$\check{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)} \qquad \text{and} \qquad \check{\mathcal{G}}^{(j)} = \mathcal{G}_{p_j}^{(j)} \cap \mathcal{K}_j\big(\check{\mathcal{G}}^{(j-1)}\big)\,.$$

Since we want to apply reciprocal-$\mathbf{CL}_{k,\ell}$ to $\check{\boldsymbol{\mathcal{G}}}$, our next goal is to verify that $\check{\boldsymbol{\mathcal{G}}}$ is a $(\check{\boldsymbol{\delta}}, \boldsymbol{c}, r)$-regular (n, ℓ, k)-complex where $\check{\boldsymbol{\delta}} = (\check{\delta}_2, \dots, \check{\delta}_k)$ and $\boldsymbol{c} = (c_2, \dots, c_k)$.

Indeed, let A_j be the event that $\check{\boldsymbol{\mathcal{G}}}^{(j)} = \{\check{\mathcal{G}}^{(h)}\}_{h=1}^j$ is $\big((\check{\delta}_2, \dots, \check{\delta}_j), (c_2, \dots, c_j), r\big)$-regular. Invoking Remark 31, we infer that, with probability $1 - o(1)$, all the bipartite graphs $\check{\mathcal{G}}^{(2)}[\Lambda_2] = \mathcal{G}_{p_2}^{(2)}[\Lambda_2]$, $\Lambda_2 \in \binom{[\ell]}{2}$ are $(3\delta_2, c_2)$-regular. In other words, since $3\delta_2 \le \check{\delta}_2$, we have $\mathbb{P}(A_2) = 1 - o(1)$. Next, we assume that A_{j-1} holds for some $3 \le j \le k$ and fix an arbitrary j-tuple $\Lambda_j \in \binom{[\ell]}{j}$. Due to the assumption $\check{\boldsymbol{\mathcal{G}}}^{(j-1)}[\Lambda_j] = \{\check{\mathcal{G}}^{(h)}[\Lambda_j]\}_{h=1}^{j-1}$ is a $\big((\check{\delta}_2, \dots, \check{\delta}_{j-1}), (c_2, \dots, c_{j-1}), r\big)$-regular $(n, j, j-1)$-complex, and consequently, owing to $\mathbf{IHC}_{k-1,\ell}$ and $c_h \ge d_h/2$ for all $h = 2, \dots, j-1$, we have

$$\big|\mathcal{K}_j\big(\check{\mathcal{G}}^{(j-1)}[\Lambda_j]\big)\big| \ge \frac{1}{2} \prod_{h=2}^{j-1} c_h^{\binom{j}{h}} \times n^j \ge \frac{1}{2^{2^j}} \prod_{h=2}^{j-1} d_h^{\binom{j}{h}} \times n^j\,.$$

On the other hand, applying $\mathbf{IHC}_{k-1,\ell}$ to $\boldsymbol{\mathcal{G}}^{(j-1)}[\Lambda_j] = \{\mathcal{G}^{(h)}[\Lambda_j]\}_{h=1}^{j-1}$ gives

$$\big|\mathcal{K}_j\big(\mathcal{G}^{(j-1)}[\Lambda_j]\big)\big| \le \frac{3}{2} \prod_{h=2}^{j-1} d_h^{\binom{j}{h}} \times n^j\,.$$

Hence, it follows directly from Definition 9 that $\check{\mathcal{G}}^{(j)} = \mathcal{G}^{(j)}[\Lambda_j] \cap \mathcal{K}_j(\check{\mathcal{G}}^{(j-1)}[\Lambda_j])$ is still $(3 \times 2^{2^j-1}\delta_j = \check{\delta}_j/3, d_j, r)$ regular w.r.t. $\check{\mathcal{G}}^{(j-1)}[\Lambda_j]$. Consequently, and again from Remark 31, we infer that, with probability $1 - o(1)$, the random subhypergraph $\check{\mathcal{G}}^{(j)}[\Lambda_j]$ is $(\check{\delta}_j, c_j, r)$-regular w.r.t. $\check{\mathcal{G}}^{(j-1)}[\Lambda_j]$. Since the number of possible choices for Λ_j is $\binom{\ell}{j} = O(1)$, we infer that $\mathbb{P}(A_j|A_{j-1}) = 1 - o(1)$ for every $j \ge 2$. Consequently, $\mathbb{P}(A_k) = \mathbb{P}(A_2)\prod_{j=3}^k \mathbb{P}(A_j|A_{j-1}) = 1 - o(1)$, or in other words, $\check{\boldsymbol{\mathcal{G}}}$ is a $(\check{\boldsymbol{\delta}}, \boldsymbol{c}, r)$-regular (n, ℓ, k)-complex with probability $1 - o(1)$.

Next, consider the random variable $X = |\mathcal{K}_\ell(\check{\mathcal{G}}^{(k)})|$ counting the cliques in $\check{\mathcal{G}}^{(k)}$. From the discussion above and the choices of $\check{\delta}_j$, $j = 2, \dots, k$, we infer that with probability $1 - o(1)$

$$X = \Big(1 \pm \frac{\gamma}{2}\Big) \prod_{h=2}^k c_h^{\binom{\ell}{h}} \times n^\ell\,.$$

On the other hand, the expectation of X is $\mathbb{E}[X] = |\mathcal{K}_\ell(\mathcal{G}^{(k)})| \times \prod_{j=2}^k p_j^{\binom{\ell}{j}}$. Also X is sharply concentrated as $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = O(n^{2\ell-1})$, and hence by Chebyshev's inequality,

$$\mathbb{P}\bigg(|X - E[X]| > \frac{\gamma}{2} \prod_{h=2}^k c_h^{\binom{\ell}{h}} \times n^\ell\bigg) = o(1)\,.$$

Consequently, there exists a complex $\check{\mathcal{G}}$ such that $|\mathcal{K}_\ell(\check{\mathcal{G}}^{(k)})| = \mathbb{E}[X] \pm \frac{\gamma}{2}\prod_{h=2}^k c_h^{\binom{\ell}{h}} \times n^\ell$, implying

$$\big|\mathcal{K}_\ell(\mathcal{G}^{(k)})\big| \times \prod_{j=2}^k p_j^{\binom{\ell}{j}} = \mathbb{E}[X] = (1 \pm \gamma) \prod_{h=2}^k c_h^{\binom{\ell}{h}} \times n^\ell\,.$$

We then conclude the proof of $\mathbf{CL}_{k,\ell}$ by the choice of $p_j = c_j/d_j$. □

Having verified the implication (19) it is left to deduce Theorem 34 from $\mathbf{IHC}_{k-1,\ell}$.

4.3. **Outline of the proof of Theorem 34.** Given an (n,ℓ,k)-complex $\boldsymbol{\mathcal{G}}=\{\mathcal{G}^{(j)}\}_{j=1}^{k}$, Theorem 34 ensures the existence of an appropriate (n,ℓ,k)-complex $\boldsymbol{\mathcal{F}}=\{\mathcal{F}^{(j)}\}_{j=1}^{k}$. This complex is constructed successively in three phases outlined below.

The first phase, which we call Cleaning Phase I, is a variant of the RS-Lemma (see Theorem 25). The lemma corresponding to Cleaning Phase I, Lemma 29, was already stated in Section 3.3.1. Given a $(\boldsymbol{\delta},\boldsymbol{d},r)$-regular input complex $\boldsymbol{\mathcal{G}}$ with $\boldsymbol{\delta}=(\delta_2,\dots,\delta_k)$ and $\boldsymbol{d}=(d_2,\dots,d_k)$, we fix

$$\tilde{\delta}_k \ll \varepsilon' \ll \min\{\varepsilon, d_2,\dots,d_k\}. \tag{23}$$

Lemma 29 alters $\boldsymbol{\mathcal{G}}$ slightly (this is measured by $\tilde{\delta}_k$) to obtain an (n,ℓ,k)-complex $\tilde{\boldsymbol{\mathcal{G}}}=\{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$ together with a perfect $(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}),\tilde{\boldsymbol{d}},\tilde{r}(\tilde{\boldsymbol{d}}),\boldsymbol{b})$-family of partitions which is $(\tilde{\delta}_k,\tilde{r}(\tilde{\boldsymbol{d}}))$-regular w.r.t. $\tilde{\mathcal{G}}^{(k)}$ (cf. Lemma 29 and Figure 2 in Section 4.4.2). Importantly, Lemma 29 (iii) will ensure that

$$\left|\mathcal{K}_\ell\big(\mathcal{G}^{(k)}\big)\triangle\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)\right| \quad \text{is “small”}. \tag{24}$$

Cleaning Phase I enables us to work with a complex $\tilde{\boldsymbol{\mathcal{G}}}$ admitting a partition with no irregular polyads. These details are done largely for convenience to help ease subsequent parts of the proof.

We next proceed to Cleaning Phase II with the complex $\tilde{\boldsymbol{\mathcal{G}}}$ and a perfect $(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}),\tilde{\boldsymbol{d}},\tilde{r}(\tilde{\boldsymbol{d}}),\boldsymbol{b})$-family of partitions $\mathscr{P}=\mathscr{P}(k-1,\boldsymbol{b},\boldsymbol{\psi})$, rank $\mathscr{P}\le\tilde{L}_k$ (cf. Lemma 29). Since $\tilde{\boldsymbol{\mathcal{G}}}$ differs from $\boldsymbol{\mathcal{G}}$ only slightly (this is measured by $\tilde{\delta}_k$), it follows from the choice of constants (argued in Fact 45) that $\tilde{\boldsymbol{\mathcal{G}}}$ inherits $(2\boldsymbol{\delta},\boldsymbol{d},r)$-regularity from $\boldsymbol{\mathcal{G}}$. Moreover, the choice of r ensuring $r\ge\tilde{L}_k$ (cf. (40)) implies that the density $d\big(\tilde{\mathcal{G}}^{(j)}|\hat{\mathcal{P}}^{(j-1)}\big)$ is close to what it “should be”, namely, d_j, for “most” polyads $\hat{\mathcal{P}}^{(j-1)}$ from $\mathscr{P}$ with $\hat{\mathcal{P}}^{(j-1)}\subseteq\tilde{\mathcal{G}}^{(j-1)}$.

The goal in Cleaning Phase II is to perfect the small number of polyads having aberrant density. More specifically, Cleaning Phase II constructs ‘uni-dense’ (n,ℓ,k)-complex $\boldsymbol{\mathcal{F}}=\{\mathcal{F}^{(j)}\}_{j=1}^{k}$ where $d\big(\mathcal{F}^{(j)}|\hat{\mathcal{P}}^{(j-1)}\big)=d_j\pm\varepsilon'$ for every polyad $\hat{\mathcal{P}}^{(j-1)}$ from $\mathscr{P}$ with $\hat{\mathcal{P}}^{(j-1)}\subseteq\mathcal{F}^{(j-1)}$. The importance of “uni-density” is that it allows us to apply the Union Lemma, Lemma 41. Then the “final product” of the Union Lemma, the (n,ℓ,k)-complex $\boldsymbol{\mathcal{F}}$, will satisfy (i) of Theorem 34. We now further examine the details of Cleaning Phase II.

Cleaning Phase II splits into two parts. In Part 1 of Cleaning Phase II (cf. Lemma 37), we correct the first $k-1$ layers of possible imperfections of $\tilde{\boldsymbol{\mathcal{G}}}=\{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$, $j<k$, by constructing a “uni-dense” $(n,\ell,k-1)$-complex $\boldsymbol{\mathcal{H}}^{(k-1)}=\{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$. For the construction of $\boldsymbol{\mathcal{H}}^{(k-1)}$, we need to count cliques in such a complex and use $\mathbf{IHC}_{k-1,\ell}$ which we have available by the induction assumption ($\mathbf{CL}_{j,\ell}$ for $2\le j\le k-1$).

We next remedy imperfections on the k-th layer $\tilde{\mathcal{G}}^{(k)}$. However, in the absence of our induction assumption herein, we have to proceed more carefully.

We first construct a still “somewhat imperfect” (n,ℓ,k)-cylinder $\mathcal{H}^{(k)}$ so that $\boldsymbol{\mathcal{H}}=\{\mathcal{H}^{(j)}\}_{j=1}^{k}$ is an (n,ℓ,k)-complex and $d\big(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}\big)=d_k\pm\sqrt{\delta_k}$ for every polyad $\hat{\mathcal{P}}^{(k-1)}$ from $\mathscr{P}$ with $\hat{\mathcal{P}}^{(k-1)}\subseteq\mathcal{H}^{(k-1)}$. While $\tilde{\mathcal{G}}^{(k)}$ satisfies that for “most” $\hat{\mathcal{P}}^{(k-1)}\subseteq\tilde{\mathcal{G}}^{(k-1)}$, its density is close to d_k, the new $\mathcal{H}^{(k)}$ has density close to d_k for “all” $\hat{\mathcal{P}}^{(k-1)}\subseteq\mathcal{H}^{(k-1)}$. Moreover, we can construct $\mathcal{H}^{(k)}$ in such a way that

$$\left|\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)\triangle\mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big)\right| \quad \text{is “small”}. \tag{25}$$

Part 2 of Cleaning Phase II deals with $\mathcal{H}^{(k)}$, the k-th layer of the complex $\boldsymbol{\mathcal{H}}$. In this part, we construct uni-dense (w.r.t. $\mathcal{H}^{(k-1)}$ and $\mathscr{P}^{(k-1)}$) (n,ℓ,k)-cylinders $\mathcal{H}_-^{(k)}$ and $\mathcal{H}_+^{(k)}$ (with densities $d_k-\sqrt{\delta_k}$ and $d_k+\sqrt{\delta_k}$, respectively) and $\mathcal{F}^{(k)}$ where each of these cylinders, together with $\boldsymbol{\mathcal{H}}^{(k-1)}=\{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$, forms an (n,ℓ,k)-complex. In the construction, we will also ensure that

$$\mathcal{H}_-^{(k)}\subseteq\mathcal{H}^{(k)}\subseteq\mathcal{H}_+^{(k)} \qquad\text{and}\qquad \mathcal{H}_-^{(k)}\subseteq\mathcal{F}^{(k)}\subseteq\mathcal{H}_+^{(k)}. \tag{26}$$

We then set $\mathcal{F}^{(j)}=\mathcal{H}^{(j)}$ for $j<k$ and $\boldsymbol{\mathcal{F}}=\{\mathcal{F}^{(j)}\}_{j=1}^{k}$.

We now discuss how we infer (i) and (iii) of Theorem 34 for $\boldsymbol{\mathcal{F}}$ (property (ii) is somewhat technical and we omit it from the outline given here). For part (i) of Theorem 34, we need to show that $\mathcal{F}^{(j)}$ is $(\varepsilon,d_j,1)$-regular w.r.t. $\mathcal{F}^{(j-1)}$, $2\le j\le k$. To this end, we take the union of all “partition blocks” from $\mathscr{P}^{(j)}$ (which are subhypergraphs of $\mathcal{F}^{(j)}$). Note that all these blocks are very regular (w.r.t. their underlying polyads (which are subhypergraphs of $\mathcal{F}^{(j-1)}$)) and have the same relative density (due to the uni-density). In fact, these blocks will be so regular that their union is $(\varepsilon',d_j,1)$-regular and therefore also $(\varepsilon,d_j,1)$-regular (cf. (23)).

Consequently, as proved in the Union Lemma, Lemma 41, we obtain that $\boldsymbol{\mathcal{F}}$ is $\big((\varepsilon,\dots,\varepsilon),\boldsymbol{d},1\big)$-regular, which proves (i) of Theorem 34.

We now outline the proof of part (iii). Observe that

$$\begin{aligned}\Big|\mathcal{K}_\ell\big(\mathcal{G}^{(k)}\big)\triangle\mathcal{K}_\ell\big(\mathcal{F}^{(k)}\big)\Big| \le \Big|\mathcal{K}_\ell\big(\mathcal{G}^{(k)}\big)\triangle\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)\Big|+&\\ +\Big|\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)\triangle\mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big)\Big|+&\\ +\Big|\mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big)\triangle\mathcal{K}_\ell\big(\mathcal{F}^{(k)}\big)\Big|.&\end{aligned} \tag{27}$$

The first two terms of the right-hand side are small as mentioned earlier (see (24) and (25)). Let us say a few words on how to bound the third quantity.

Because of (26), we have

$$\Big|\mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big)\triangle\mathcal{K}_\ell\big(\mathcal{F}^{(k)}\big)\Big| \le \Big|\mathcal{K}_\ell\big(\mathcal{H}_+^{(k)}\big)\triangle\mathcal{K}_\ell\big(\mathcal{H}_-^{(k)}\big)\Big| = \Big|\mathcal{K}_\ell\big(\mathcal{H}_+^{(k)}\big)\Big| - \Big|\mathcal{K}_\ell\big(\mathcal{H}_-^{(k)}\big)\Big|. \tag{28}$$

Since both complexes $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ are uni-dense, we can show that $\boldsymbol{\mathcal{H}}_*$ is $\big((\varepsilon',\dots,\varepsilon'),\boldsymbol{d}_*,1\big)$-regular for $*\in\{+,-\}$ where $\boldsymbol{d}_* = (d_2,\dots,d_{k-1},d_k^*)$ and $d_k^* = d_k + \sqrt{\delta_k}$ for $*=+$ and $d_k^* = d_k - \sqrt{\delta_k}$ for $*=-$. Similarly to the proof of (i), where the Union Lemma was applied to $\boldsymbol{\mathcal{F}}$, we can use it here for $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$. Consequently, owing to (23), we can apply the Dense Counting Lemma, Theorem 16, to bound the right-hand side of (28) and thus the right-hand side of (27). This yields part (iii) of Theorem 34.

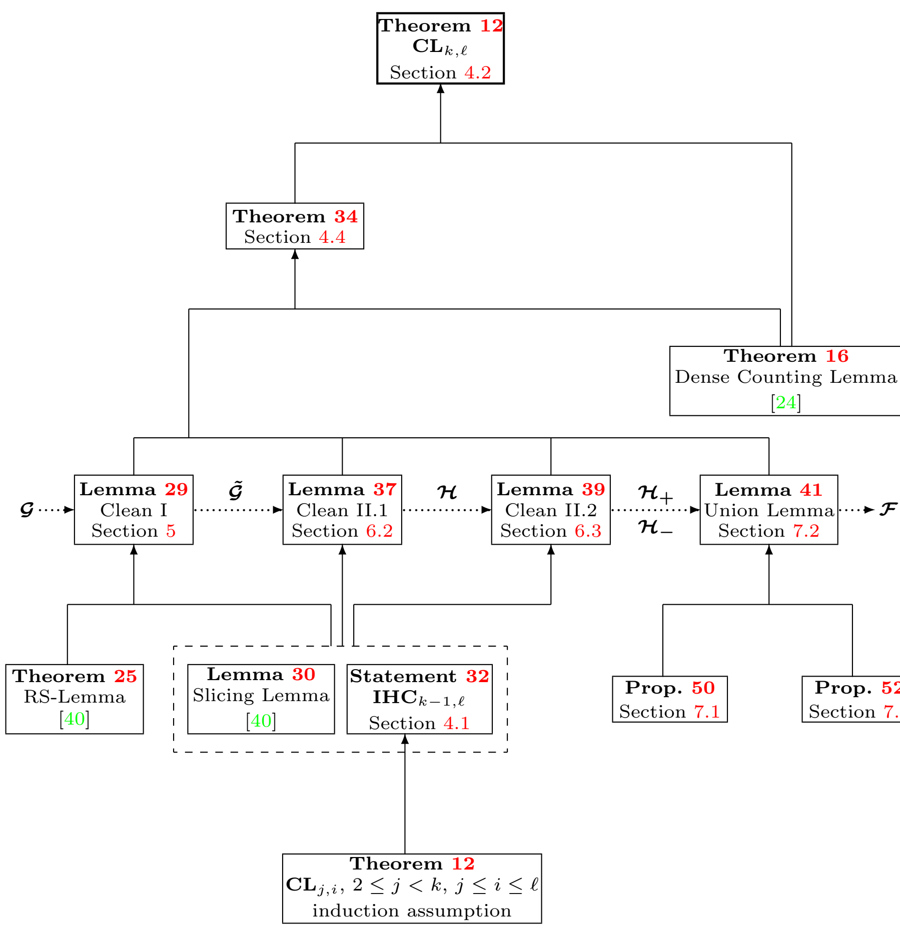

FIGURE 1. Structure of the proof of Theorem 12

The flowchart in Figure 1 gives a sketch of the connection of theorems and lemmas involved in the proof of $\mathbf{CL}_{k,\ell}$, Theorem 12. Each box represents a theorem or lemma containing a reference for its proof. Vertical arcs indicate which statements are needed to prove the statement to which the arc points. The horizontal arcs indicate the alteration of the involved complexes outlined above.

4.4. **Proof of Theorem 34.** In this section, we give all the details of the proof of Theorem 34 outlined in the last section. The proof of Theorem 34 splits into four parts. We separate these parts across Sections 4.4.1–4.4.4.

4.4.1. *Constants.* The hierarchy of the involved constants plays an important rôle in our proof. The choice of the constants breaks into two steps.

Step 1. Let an integer ℓ be given. We first recall the quantification of Theorem 34:

$$\forall \gamma, d_k \ \exists \delta_k \ \forall d_{k-1} \dots \exists \delta_3 \ \forall d_2, \varepsilon \ \exists \delta_2, r, n_0 \,,$$

where $d_2, \dots, d_k$ are rationals in $(0, 1]$. Given γ and d_k we choose δ_k such that

$$\delta_k \ll \min\{\gamma, d_k\} \tag{29}$$

holds. Now, let d_{k-1} be given. We set

$$\eta = 1/4 \,. \tag{30}$$

(Our proof is not too sensitive to the choice of η, representing the multiplicative error for $\mathbf{IHC}_{k-1,\ell}$.) We then choose δ_{k-1} in such a way that $\delta_{k-1} \ll \min\{\delta_k, d_{k-1}\}$ and $\delta_{k-1} \le \delta_{k-1}\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1})\big)$ where $\delta_{k-1}\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1})\big)$ is the value of δ_{k-1} given by Statement 32 for η and d_{k-1}. We then proceed and define δ_j for $j = k-2, \dots, 3$, in the similar way. Summarizing the above, for $j = k-1, \dots, 3$, we choose δ_j such that

$$\delta_j \ll \min\{\delta_{j+1}, d_j\} \qquad \text{and} \qquad \delta_j \le \delta_j\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1}, \delta_{k-1}, d_{k-2}, \dots, \delta_{j+1}, d_j)\big) \,. \tag{31}$$

We mention that after d_2 is revealed, we pause before defining δ_2.

Indeed, next we choose an auxiliary constant ε' so that

$$\begin{aligned} &\varepsilon' \le \min\Big\{\varepsilon\big(\text{Thm.16}(d_2, \dots, d_{k-1}, d_k - \sqrt{\delta_k})\big), \varepsilon\big(\text{Thm.16}(d_2, \dots, d_{k-1}, d_k + \sqrt{\delta_k})\big)\Big\} \,, \\ &\varepsilon' \ll \min\big\{\delta_3, d_2, \varepsilon\big\} \,, \qquad \text{and} \qquad g_{k,\ell}(\varepsilon') \ll \delta_k \,, \end{aligned} \tag{32}$$

where $g_{k,\ell}$ is given by the Dense Counting Lemma, Theorem 16. We then fix $\tilde{\eta}$ and $\tilde{\delta}_k$ to satisfy

$$\varepsilon' \gg \tilde{\eta} \gg \tilde{\delta}_k \qquad \text{and} \qquad \tilde{\delta}_k \le 1/8 \,. \tag{33}$$

This completes Step 1 of the choice of the constants. We summarize the choices above in the following flowchart:

$$\begin{array}{ccccccccccccc} & & d_k & & \dots & & d_3 & & d_2, \varepsilon & & & & \\ & & \curlyvee\!\!\curlyvee & & \dots & & \curlyvee\!\!\curlyvee & & \curlyvee\!\!\curlyvee & & & & \\ \gamma & \gg & \delta_k & & \gg \dots \gg & & \delta_3 & \gg & \varepsilon' & \gg & \tilde{\eta} & \gg & \tilde{\delta}_k \end{array} \tag{34}$$

Step 2. The definition of the constants here is more subtle. Our goal is to extend (34) with the additional constants $\tilde{d}_j$, $\tilde{\delta}_j$ (for $j = k-1, \dots, 2$), $\tilde{r}$, $\tilde{L}_k$, δ_2, and r so that

$$\begin{array}{ccccccccc} & & \tilde{d}_{k-1} & & \dots & \tilde{d}_3 & & \tilde{d}_2 & & \\ & & \curlyvee\!\!\curlyvee & & \dots & \curlyvee\!\!\curlyvee & & \curlyvee\!\!\curlyvee & & \\ \tilde{\delta}_k & \gg & \tilde{\delta}_{k-1} & & \gg \dots \gg & \tilde{\delta}_3 & \gg & \tilde{\delta}_2, 1/\tilde{r} & \gg & \tilde{L}_k^2 \delta_2, 1/r \end{array}$$

In our proof, we apply Lemma 29 to the (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$. Lemma 29 has functions $\tilde{\delta}_j(D_j, \dots, D_{k-1})$ for $j = 2, \dots, k-1$ and $\tilde{r}(B_1, D_2, \dots, D_{k-1})$ in variables $B_1, D_2, \dots, D_{k-1}$ as part of its input. The application of Lemma 29 results in a perfect $\big(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, \tilde{r}(\tilde{\boldsymbol{d}}), \boldsymbol{b}\big)$-family of partitions $\mathscr{P}$ with some $\tilde{\boldsymbol{d}} = (\tilde{d}_2, \dots, \tilde{d}_{k-1})$. We want to be able to count cliques within the polyads of the family of regular partitions

$\mathscr{P}$ by applying $\mathbf{IHC}_{k-1,\ell}$. Therefore, we choose the functions $\tilde{\delta}_j(D_j,\dots,D_{k-1})$ for Lemma 29 in such a way that they comply with the quantification of $\mathbf{IHC}_{k-1,\ell}$, Statement 32.

To this end let $\tilde{\delta}_{k-1}(D_{k-1})$ be a function satisfying

$$\tilde{\delta}_{k-1}(D_{k-1}) \ll \min\{\tilde{\delta}_k, D_{k-1}\} \qquad \text{and} \qquad \tilde{\delta}_{k-1}(D_{k-1}) \le \delta_{k-1}\big(\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, D_{k-1})\big)\,.$$

We next choose the function $\tilde{\delta}_{k-2}(D_{k-2},D_{k-1})$ in a similar way, making sure that

$$\tilde{\delta}_{k-2}(D_{k-2},D_{k-1}) \ll \min\left\{\tilde{\delta}_{k-1}(D_{k-1}), D_{k-2}\right\},$$

and

$$\tilde{\delta}_{k-2}(D_{k-2},D_{k-1}) \le \delta_{k-2}\big(\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, D_{k-1}, \tilde{\delta}_{k-1}(D_{k-1}), D_{k-2})\big)\,.$$

(Since $\tilde{\delta}_{k-1}(D_{k-1})$ is a function of D_{k-1} and $\tilde{\eta}$ was fixed in (33) already, we indeed also have that the right-hand sides of the first two inequalities above depend on the variables D_{k-2} and D_{k-1} only.) In general for $j = k-1,\dots,2$, we choose $\tilde{\delta}_j(D_j,\dots,D_{k-1})$ so that

$$\begin{aligned}&\tilde{\delta}_j(D_j,\dots,D_{k-1}) \ll \min\left\{D_j, \tilde{\delta}_{j+1}(D_{j+1},\dots,D_{k-1})\right\},\\ &\text{and}\\ &\tilde{\delta}_j(D_j,\dots,D_{k-1}) \le \delta_j\big(\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, D_{k-1}, \tilde{\delta}_{k-1}(D_{k-1}), D_{k-2},\dots,\tilde{\delta}_{j+1}(D_{j+1},\dots,D_{k-1}), D_j)\big)\,.\end{aligned} \tag{35}$$

We may assume, without loss of generality, that the functions defined in (35) are componentwise monotone decreasing. Since for every $h \ge j+1$ the $\tilde{\delta}_h$ was constructed as a function of $D_h,\dots,D_{k-1}$ only, as before, we may view the right-hand sides of the first two inequalities of (35) as a function of $D_j,\dots,D_{k-1}$ only. Consequently, $\tilde{\delta}_j$ is a function of $D_j,\dots,D_{k-1}$, as promised. Furthermore, we set $\tilde{r}(D_2,\dots,D_{k-1})$ to be a componentwise monotone increasing function such that

$$\begin{aligned}&\tilde{r}(D_2,\dots,D_{k-1}) \gg \max\left\{1/D_2, 1/\tilde{\delta}_3(D_3,\dots,D_{k-1})\right\}\\ &\text{and}\\ &\tilde{r}(D_2,\dots,D_{k-1}) \ge r\big(\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, D_{k-1}, \tilde{\delta}_{k-1}(D_{k-1}),\dots D_2)\big)\,.\end{aligned} \tag{36}$$

As a result of Lemma 29 applied to the constants $\boldsymbol{d} = (d_2,\dots,d_k)$, $\delta_3,\dots,\delta_k$, $\tilde{\delta}_k$, and the functions $\tilde{\delta}_{k-1}(D_{k-1}),\dots,\tilde{\delta}_2(D_2,\dots,D_{k-1})$ and $\tilde{r}(D_2,\dots,D_{k-1})$ we obtain integers $\tilde{n}_k$, $\tilde{L}_k$, a vector of positive reals $\tilde{\boldsymbol{c}} = (\tilde{c}_2,\dots,\tilde{c}_{k-1})$ and a constant $\delta_2^{\text{Lem.29}}$. (Here we did not use the variable B_1 for the function $\tilde{r}(D_2,\dots,D_{k-1})$.) Next, we disclose δ_2 and r promised by Theorem 12. For that we apply the functions $\tilde{\delta}_2(D_2,\dots,D_{k-1})$ and $\tilde{r}(D_2,\dots,D_{k-1})$, defined in (35) and (36), to $\tilde{\boldsymbol{c}}$. We set δ_2 and r so that

$$\begin{aligned}&(\tilde{L}_k^2\delta_2) \ll \min\left\{\tilde{\delta}_2(\tilde{\boldsymbol{c}}), \tilde{r}(\tilde{\boldsymbol{c}})\right\}, \qquad \delta_2 \le \delta_2^{\text{Lem.29}}\\ &\text{and} \quad \delta_2 \le \delta_2\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1}, \delta_{k-1}, d_{k-2},\dots,\delta_3, d_2)\big)\,,\end{aligned} \tag{37}$$

and

$$r \gg \max\left\{1/\tilde{\delta}_2(\tilde{\boldsymbol{c}}), \tilde{r}(\tilde{\boldsymbol{c}}), 2^\ell \tilde{L}_k^k\right\} \quad \text{and} \quad r \ge r\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1}, \delta_{k-1}, d_{k-2},\dots,\delta_3, d_2)\big)\,. \tag{38}$$

Finally, we set n_0 so that

$$n_0 \gg \max\left\{\tilde{n}_k,\ 1/\delta_2,\ r,\ m_0\,\tilde{L}_k m_{k-1,\ell},\ \tilde{L}_k \tilde{m}_{k-1,\ell}\right\}, \tag{39}$$

where

$$m_0 = \max\left\{m_0\big(\text{Thm.16}(d_2,\dots,d_k - \sqrt{\delta_k})\big), m_0\big(\text{Thm.16}(d_2,\dots,d_k + \sqrt{\delta_k})\big)\right\}$$

is given by Theorem 16 applied to $d_2,\dots,d_{k-1}$ and $d_k - \sqrt{\delta_k}$ and $d_k + \sqrt{\delta_k}$, respectively, and similarly

$$m_{k-1,\ell} = m_{k-1,\ell}\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1}, \delta_{k-1},\dots,d_2,\delta_2)\big)$$

and

$$\tilde{m}_{k-1,\ell} = m_{k-1,\ell}\big(\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, \tilde{c}_{k-1}, \tilde{\delta}_{k-1}(\tilde{c}_{k-1}),\dots,\tilde{c}_2, \tilde{\delta}_2(\tilde{\boldsymbol{c}}))\big)$$

come from Statement 32.

This way we defined all constants involved in the statement Theorem 34. Moreover, we defined the functions and constants needed for Lemma 29. This brings us to the next part of the proof, Cleaning Phase I.

4.4.2. *Cleaning phase I.* Let $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$ be a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular (n, ℓ, k)-complex where $n \geq n_0$ and $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$, $\boldsymbol{d} = (d_2, \ldots, d_k)$ and r are chosen as described in Section 4.4.1. We apply Lemma 29 to $\boldsymbol{\mathcal{G}}$ with the constant $\tilde{\delta}_k$, the functions $\tilde{\boldsymbol{\delta}}(\boldsymbol{D}) = (\tilde{\delta}_{k-1}(D_{k-1}), \ldots, \tilde{\delta}_2(D_2, \ldots, D_{k-1}))$ and the function $\tilde{r}(\boldsymbol{D})$ as given in (33), (35) and (36). Lemma 29 renders an (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^k$, a real vector of positive coordinates $\tilde{\boldsymbol{d}} = (\tilde{d}_2, \ldots, \tilde{d}_{k-1})$ componentwise bigger than $\tilde{\boldsymbol{c}}$ and a perfect $(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{b}, \boldsymbol{\psi})$ refining $\tilde{\boldsymbol{\mathcal{G}}}$ (cf. Definition 26 and Definition 28). Note that the choice of r in (38) and (v) of Lemma 29 ensures that for $2 \leq j \leq k$,

$$r \geq 2^\ell \tilde{L}_k^k \geq 2^\ell \left|\hat{A}(k-1, \boldsymbol{b})\right| \geq \left|\hat{A}(j-1, \boldsymbol{b})\right|. \tag{40}$$

For $2 \leq j \leq k-1$, we finally fix the constants

$$\tilde{\delta}_j = \tilde{\delta}_j(\tilde{d}_j, \ldots, \tilde{d}_{k-1}) \qquad \text{and} \qquad \tilde{r} = \tilde{r}(\tilde{\boldsymbol{d}}). \tag{41}$$

¿From the monotonicity of the functions $\tilde{\delta}_2$ and $\tilde{r}$ and (37) and (38), we infer

$$\tilde{\delta}_2 = \tilde{\delta}_2(\tilde{\boldsymbol{d}}) \geq \tilde{\delta}_2(\tilde{\boldsymbol{c}}) \gg \delta_2 \quad \text{and} \quad \tilde{r} = \tilde{r}(\tilde{\boldsymbol{d}}) \leq \tilde{r}(\tilde{\boldsymbol{c}}) \ll r. \tag{42}$$

For future reference, we summarize, in Figure 2, (29)–(42) and for the remainder of this paper, all constants are fixed as summarized in Figure 2.

$$\begin{array}{ccccccccccc}
 & & d_k & & \cdots & & d_3 & & d_2, \varepsilon & & \\
 & & \Vvdash & & \cdots & & \Vvdash & & \Vvdash & & \\
\gamma & \gg & \delta_k & \gg & \cdots & \gg & \delta_3 & \gg & \varepsilon' & & \\
 & & & & & & & & \Vvdash & & \\
 & & & & & & & & \tilde{\eta} & & \\
 & & & & & & & & \Vvdash & & \tilde{d}_{k-1} \quad \cdots \quad \tilde{d}_3 \quad \tilde{d}_2 \\
 & & & & & & & & \Vvdash & & \Vvdash \quad \cdots \quad \Vvdash \quad \Vvdash \\
 & & & & & & & & \tilde{\delta}_k & \gg & \tilde{\delta}_{k-1} \gg \cdots \gg \tilde{\delta}_3 \gg \tilde{\delta}_2, 1/\tilde{r} \\
 & & & & & & & & & & \Vvdash \\
 & & & & & & & & & & \tilde{L}_k^2 \delta_2, 1/r
\end{array}$$

FIGURE 2. Flowchart of the constants

Observe that due to part (i) of Definition 28 and due to part (vi) of Lemma 29, we have for every $2 \leq j < k$

$$\tilde{d}_j b_j = 1 \qquad \text{and} \qquad d_j b_j = d_j / \tilde{d}_j \text{ is an integer.} \tag{43}$$

For future reference, we summarize the results of Cleaning Phase I.

Setup 36 (After Cleaning Phase I)**.** *Let all constants be chosen as summarized in Figure 2 and* (43)*. Let $\boldsymbol{\mathcal{G}}$ be the $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular (n, ℓ, k)-complex from the input Theorem 34. Let $\tilde{\boldsymbol{\mathcal{G}}}$ be the (n, ℓ, k)-complex and $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{b}, \boldsymbol{\psi})$ be the perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family of partitions refining $\tilde{\boldsymbol{\mathcal{G}}}$ given after Cleaning Phase I, i.e., after an application of Lemma 29.*

We now mention a few comments to motivate our next step in the proof. The family of partitions $\mathscr{P}$ given by Lemma 29 (cf. Setup 36) is a perfect family (cf. Definition 28). Moreover, by Lemma 29 part (i), $\tilde{\mathcal{G}}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{b})$. However, while every component of the partition is regular, it is possible that the densities $d\big(\tilde{\mathcal{G}}^{(j)} \big| \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)$ may vary across different $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{b})$ for which $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq \tilde{\mathcal{G}}^{(j-1)}$.

The goal of the next cleaning phase is to alter $\tilde{\boldsymbol{\mathcal{G}}}$ to form a complex $\boldsymbol{\mathcal{F}}$ where all densities are appropriately uniform. Importantly, we show that the two complexes $\tilde{\boldsymbol{\mathcal{G}}}$ and $\boldsymbol{\mathcal{F}}$ share mostly all their respective cliques. (For technical reasons, we will also need to construct two auxiliary complexes $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$.)

4.4.3. *Cleaning phase II.* The aim of this section is to construct the complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^k$ which is promised by Theorem 34. For the proof of part (*iii*) of Theorem 34, we construct two auxiliary complexes $\boldsymbol{\mathcal{H}}_+ = \{\mathcal{H}_+^{(j)}\}_{j=1}^k$ and $\boldsymbol{\mathcal{H}}_- = \{\mathcal{H}_-^{(j)}\}_{j=1}^k$. Later, in the final phase (see Section 4.4.4), our goal is to apply the Dense Counting Lemma to these auxiliary complexes.

The construction of $\boldsymbol{\mathcal{H}}_+$, $\boldsymbol{\mathcal{H}}_-$ and $\boldsymbol{\mathcal{F}}$ splits into two parts. First (cf. upcoming Lemma 37), we construct an auxiliary (n, ℓ, k)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ which will have the required properties for $1 \le j < k$ (we have $\mathcal{H}_+^{(j)} = \mathcal{H}_-^{(j)} = \mathcal{F}^{(j)} = \mathcal{H}^{(j)}$ for $1 \le j < k$).

In the second part, we use the upcoming Lemma 39 to overcome a 'slight imperfection' of $\mathcal{H}^{(k)}$ and construct $\mathcal{H}_+^{(k)}$, $\mathcal{H}_-^{(k)}$, and $\mathcal{F}^{(k)}$ so that $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ (as we will later show in Lemma 41) satisfy the assumptions of the Dense Counting Lemma. Moreover, $\mathcal{H}_+^{(k)}$ and $\mathcal{H}_-^{(k)}$ will "sandwich" $\mathcal{F}^{(k)}$ and $\mathcal{H}^{(k)}$ (i.e. $\mathcal{H}_+^{(k)} \supseteq \mathcal{F}^{(k)} \supseteq \mathcal{H}_-^{(k)}$ and $\mathcal{H}_+^{(k)} \supseteq \mathcal{H}^{(k)} \supseteq \mathcal{H}_-^{(k)}$).

We need the following definition in order to state Lemma 37. For a $(j-1)$-uniform hypergraph $\mathcal{H}^{(j-1)}$, we denote by $\hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right) \subseteq \hat{A}(j-1, \boldsymbol{b})$ the set of polyad addresses $\hat{\boldsymbol{x}}^{(j-1)}$ such that

$$\hat{\mathcal{P}}^{(j-1)}\left(\hat{\boldsymbol{x}}^{(j-1)}\right) \subseteq \mathcal{H}^{(j-1)}\,. \tag{44}$$

Lemma 37 (Cleaning Phase II, Part 1)**.** *Given Setup 36, there exists an (n, ℓ, k)-complex $\boldsymbol{\mathcal{H}} = \left\{\mathcal{H}^{(j)}\right\}_{j=1}^k$ such that:*

(*a*) *$\mathcal{H}^{(1)} = \tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$ (and consequently $\hat{A}\left(\mathcal{H}^{(1)}, 1, \boldsymbol{b}\right) = \hat{A}\left(1, \boldsymbol{b}\right)$) and $\mathcal{H}^{(2)} = \tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$.*

(*b*) *For every $2 \le j < k$, the following holds:*

- (*b1*) *For every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right)$, there exists an index set $I\big(\hat{\boldsymbol{x}}^{(j-1)}\big) \subseteq [b_j]$ of size $\left|I\big(\hat{\boldsymbol{x}}^{(j-1)}\big)\right| = d_j b_j \in \mathbb{N}$ (see (43)) such that*

$$\mathcal{H}^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = \bigcup_{\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\left(\big(\hat{\boldsymbol{x}}^{(j-1)}, \alpha\big)\right).$$

  (In this way, $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{b}, \boldsymbol{\psi})$ refines the $(n, \ell, k-1)$-complex $\{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$ (cf. Definition 26).)

- (*b2*) *For every $2 \le j \le i \le \ell$ (where $j < k$),*

$$\left|\mathcal{K}_i\big(\mathcal{H}^{(j)}\big) \triangle \mathcal{K}_i\big(\tilde{\mathcal{G}}^{(j)}\big)\right| \le \delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right) n^i\,.$$

(*c*) *Finally, the (n, ℓ, k)-cylinder $\mathcal{H}^{(k)}$ satisfies the following two properties:*

- (*c1*) *For every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}\big(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}\big)$, $\mathcal{H}^{(k)}$ is $\big(\tilde{\delta}_k, \bar{d}_k(\hat{\boldsymbol{x}}^{(k-1)}), \tilde{r}\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$ where $\bar{d}_k\big(\hat{\boldsymbol{x}}^{(k-1)}\big) = d_k \pm \sqrt{\delta_k}$.*
- (*c2*)

$$\left|\mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big) \triangle \mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)\right| \le \delta_k^{1/3} \left(\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\right) n^\ell\,.$$

We prove Lemma 37 in Section 6.2.

Consider the subcomplex $\boldsymbol{\mathcal{H}}^{(k-1)} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$. The complex $\boldsymbol{\mathcal{H}}^{(k-1)}$ is 'absolutely perfect' by having the following two properties for every $2 \le j < k$:

perfectly equitable (*PE*): For every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ and every $\beta \in I(\hat{\boldsymbol{x}}^{(j-1)})$, the $(n/b_1, j, j)$-cylinder $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \beta)\big)$ is $(\tilde{\delta}_j, \tilde{d}_j, \tilde{r})$-regular w.r.t. its underlying polyad $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$.

uniformly dense (*UD*): For every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$,

$$d\big(\mathcal{H}^{(j)} \big| \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = (d_j b_j)(\tilde{d}_j \pm \tilde{\delta}_j)\,. \tag{45}$$

Property (PE) is an immediate consequence of the fact that $\mathscr{P}$ is a perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family of partitions. Property (UD) easily follows from $(b1)$ combined with (PE).

We now rewrite the right-hand side of (45) in a more convenient form. It follows from (43) and choice of the constants $\tilde{\delta}_j \ll \tilde{d}_j$ (cf. Figure 2) that

$$d\big(\mathcal{H}^{(j)} \big| \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = (d_j b_j)(\tilde{d}_j \pm \tilde{\delta}_j) = d_j \pm d_j \tilde{\delta}_j / \tilde{d}_j = d_j \pm \sqrt{\tilde{\delta}_j}\,. \tag{46}$$

Remark 38. *Observe that the last two "equality signs" in (46) are used in a non-symmetric way. For example, the last equality sign abbreviates the validity of the two inequalities*

$$d_j - d_j\tilde{\delta}_j/\tilde{d}_j \geq d_j - \sqrt{\tilde{\delta}_j} \qquad \text{and} \qquad d_j + d_j\tilde{\delta}_j/\tilde{d}_j \leq d_j + \sqrt{\tilde{\delta}_j}\,.$$

We will use this notation occasionally in the calculations throughout this paper.

For each $2 \leq j < k$ consider $\mathcal{H}^{(j)}$ as the union

$$\mathcal{H}^{(j)} = \bigcup \big\{\mathcal{H}^{(j)} \cap \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})(\hat{\boldsymbol{x}}^{(j-1)}) \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})\big\}\,.$$

¿From property (PE) and (46) we will infer that $\mathcal{H}^{(j)}$ is $(\tilde{\delta}_j^{1/3}, d_j, 1)$-regular (and, therefore, also $(\varepsilon', d_j, 1)$-regular) w.r.t. $\mathcal{H}^{(j-1)}$ (this will be verified in the proof of Lemma 41 in Section 7.2). This means, however, that the complex $\boldsymbol{\mathcal{H}}^{(k-1)}$ is 'ready' for an application of the Dense Counting Lemma, Theorem 16.

The proof of $(b2)$ is based on the induction assumption (cf. $\mathbf{IHC}_{k-1,\ell}$). The treatment of $\mathcal{H}^{(k)}$ will necessarily have to be different. We shall construct two (n, ℓ, k)-cylinders $\mathcal{H}_+^{(k)}$ and $\mathcal{H}_-^{(k)}$ so that $\mathcal{H}_+^{(k)} \supseteq \mathcal{H}^{(k)} \supseteq \mathcal{H}_-^{(k)}$. Moreover, we construct $\mathcal{F}^{(k)}$, incomparable with respect to $\mathcal{H}^{(k)}$, but with $\mathcal{H}_+^{(k)} \supseteq \mathcal{F}^{(k)} \supseteq \mathcal{H}_-^{(k)}$. To this end, we use the following lemma whose proof we defer to Section 6.3.

Lemma 39 (Cleaning Phase II, Part 2)**.** *Given Setup 36 and the (n, ℓ, k)-complex $\boldsymbol{\mathcal{H}}$ from Part 1 of Cleaning Phase II, Lemma 37, there are (n, ℓ, k)-cylinders $\mathcal{H}_-^{(k)} \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{H}_+^{(k)}$ such that:*

- (α) *$\boldsymbol{\mathcal{H}}_- = \{\mathcal{H}^{(i)}\}_{i=1}^{k-1} \cup \{\mathcal{H}_-^{(k)}\}$, $\boldsymbol{\mathcal{H}}_+ = \{\mathcal{H}^{(i)}\}_{i=1}^{k-1} \cup \{\mathcal{H}_+^{(k)}\}$, and $\boldsymbol{\mathcal{F}} = \{\mathcal{H}^{(i)}\}_{i=1}^{k-1} \cup \{\mathcal{F}^{(k)}\}$ are (n, ℓ, k)-complexes and $\mathcal{H}_-^{(k)} \subseteq \mathcal{H}^{(k)} \subseteq \mathcal{H}_+^{(k)}$.*
- (β) *For every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}\left(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}\right)$, the following holds:*
  - $(\beta 1)$ *$\mathcal{H}_-^{(k)}$ is $\left(3\tilde{\delta}_k, d_k - \sqrt{\delta_k}, \tilde{r}\right)$-regular w.r.t. to $\hat{\mathcal{P}}^{(k-1)}\left(\hat{\boldsymbol{x}}^{(k-1)}\right)$ and*
  - $(\beta 2)$ *$\mathcal{H}_+^{(k)}$ is $\left(3\tilde{\delta}_k, d_k + \sqrt{\delta_k}, \tilde{r}\right)$-regular w.r.t. to $\hat{\mathcal{P}}^{(k-1)}\left(\hat{\boldsymbol{x}}^{(k-1)}\right)$,*
  - $(\beta 3)$ *$\mathcal{F}^{(k)}$ is $\left(21\tilde{\delta}_k, d_k, \tilde{r}\right)$-regular w.r.t. to $\hat{\mathcal{P}}^{(k-1)}\left(\hat{\boldsymbol{x}}^{(k-1)}\right)$.*

Cleaning Phase II is now concluded. For future reference, we summarize the effects of Cleaning Phase II.

Setup 40 (After Cleaning Phase II)**.** *Let all constants be chosen as summarized in Figure 2 and (43).*

- *Let $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{b}, \boldsymbol{\psi})$ be the perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family of partitions given after Cleaning Phase I, i.e., after an application of Lemma 29.*
- *Let $\boldsymbol{\mathcal{H}}$ be the (n, ℓ, k)-complex given from Part 1 of Cleaning Phase II, Lemma 37. For every $2 \leq j < k$ and $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$, let $I(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq [b_j]$ be the index set satisfying (b1) of Lemma 37.*
- *Moreover, let $\boldsymbol{\mathcal{H}}_+$, $\boldsymbol{\mathcal{H}}_-$, and $\boldsymbol{\mathcal{F}}$ be the (n, ℓ, k)-complexes given by Part 2 of Cleaning Phase II, Lemma 39.*

4.4.4. *The final phase.* We now finish the proof Theorem 34. The first goal is to show that $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ satisfy the assumptions of the Dense Counting Lemma. To this end, we use the upcoming Union Lemma, Lemma 41, stated below. After stating the Union Lemma, we finish the proof of Theorem 34.

Lemma 41 (Union lemma)**.** *Given Setup 40 and $* \in \{+, -\}$, the complex $\boldsymbol{\mathcal{H}}_*$ is $(\boldsymbol{\varepsilon}', \boldsymbol{d}_*, 1)$-regular where $\boldsymbol{\varepsilon}' = (\varepsilon', \dots, \varepsilon') \in \mathbb{R}^{k-1}$ and $\boldsymbol{d}_* = (d_2^*, \dots, d_k^*)$ with*

$$d_j^* = \begin{cases} d_j & \text{if} \quad 2 \leq j \leq k-1 \\ d_k + \sqrt{\delta_k} & \text{if} \quad j = k \text{ and } * = + \\ d_k - \sqrt{\delta_k} & \text{if} \quad j = k \text{ and } * = -\,. \end{cases} \tag{47}$$

Similarly, the (n,ℓ,k)-complex $\boldsymbol{\mathcal{F}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1} \cup \{\mathcal{F}^{(k)}\}$ is $(\boldsymbol{\varepsilon'}, \boldsymbol{d}, 1)$-regular where $\boldsymbol{\varepsilon'} = (\varepsilon', \dots, \varepsilon') \in \mathbb{R}^{k-1}$ and $\boldsymbol{d} = (d_2, \dots, d_k)$.

We give the proof of Lemma 41 in Section 7. We now finish this section with the proof of Theorem 34.

Proof of Theorem 34. Set $\mathcal{F}^{(j)} = \mathcal{H}^{(j)}$ for $1 \le j < k$ and let $\mathcal{F}^{(k)}$ be given by Lemma 39. Consequently, $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^{k}$ is an (n,ℓ,k)-complex and Lemma 41 and the choice $\varepsilon' \ll \varepsilon$ gives (i) of Theorem 34. Moreover, due to part (a) of Lemma 37, we have $\mathcal{G}^{(1)} = \mathcal{H}^{(1)} = \mathcal{F}^{(1)}$ and $\mathcal{G}^{(2)} = \mathcal{H}^{(2)} = \mathcal{F}^{(2)}$ which yields (ii) of Theorem 34. It is left to verify part (iii) of the theorem.

As an intermediate step, we first consider $\mathcal{K}_\ell(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell(\mathcal{F}^{(k)})$. Since $\mathcal{H}_+^{(k)} \supseteq \mathcal{H}^{(k)} \cup \mathcal{F}^{(k)}$ and $\mathcal{H}^{(k)} \cap \mathcal{F}^{(k)} \supseteq \mathcal{H}_-^{(k)}$ (cf. Lemma 39), we have

$$\left|\mathcal{K}_\ell(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell(\mathcal{F}^{(k)})\right| \le \left|\mathcal{K}_\ell(\mathcal{H}_+^{(k)}) \setminus \mathcal{K}_\ell(\mathcal{H}_-^{(k)})\right|. \tag{48}$$

We infer from Lemma 41 and the choice of ε' in (32) and n_0 in (39) that $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ satisfy the assumptions of the Dense Counting Lemma, Theorem 16. Consequently,

$$\begin{aligned}\left|\mathcal{K}_\ell(\mathcal{H}_+^{(k)})\right| &\le \left(1+\sqrt{\delta_k}\right)\left(d_k+\sqrt{\delta_k}\right)^{\binom{\ell}{k}} \prod_{h=2}^{k-1} d_h^{\binom{\ell}{h}} \times n^\ell \le \left(1+\sqrt{\delta_k}\right)\left(1+2\binom{\ell}{k}\frac{\sqrt{\delta_k}}{d_k}\right) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell \\ &\le \left(1+\delta_k^{1/3}\right) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell.\end{aligned} \tag{49}$$

Similarly,

$$\left|\mathcal{K}_\ell(\mathcal{H}_-^{(k)})\right| \ge \left(1-\delta_k^{1/3}\right) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell\,. \tag{50}$$

Therefore, from (48), (49) and (50), we infer

$$\left|\mathcal{K}_\ell(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell(\mathcal{F}^{(k)})\right| \le 2\delta_k^{1/3} \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell\,. \tag{51}$$

We now prove (iii) of Theorem 34. Using the triangle-inequality, we infer

$$\begin{aligned}\left|\mathcal{K}_\ell(\mathcal{G}^{(k)}) \triangle \mathcal{K}_\ell(\mathcal{F}^{(k)})\right| \le &\left|\mathcal{K}_\ell(\mathcal{G}^{(k)}) \triangle \mathcal{K}_\ell(\tilde{\mathcal{G}}^{(k)})\right| + \\ &+ \left|\mathcal{K}_\ell(\tilde{\mathcal{G}}^{(k)}) \triangle \mathcal{K}_\ell(\mathcal{H}^{(k)})\right| + \\ &+ \left|\mathcal{K}_\ell(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell(\mathcal{F}^{(k)})\right|.\end{aligned} \tag{52}$$

Then part (iii) of Lemma 29, $(c2)$ of Lemma 37, and (51) bound the right-hand side of (52) and, hence,

$$\left|\mathcal{K}_\ell(\mathcal{G}^{(k)}) \triangle \mathcal{K}_\ell(\mathcal{F}^{(k)})\right| \le \left(\tilde{\delta}_k + 3\delta_k^{1/3}\right) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell\,. \tag{53}$$

part (iii) of Theorem 34 now follows from $\gamma \gg \delta_k \gg \tilde{\delta}_k$ (cf. Figure 2). This concludes the proof of Theorem 34. □

5. Proof of cleaning phase I

The proof of Lemma 29 is organized as follows. We first fix all constants involved in the proof. We then inductively construct the perfect family of partitions $\mathscr{P}$ and the complex $\tilde{\boldsymbol{\mathcal{G}}}$ promised by Lemma 29. Finally, we verify that $\mathscr{P}$ and $\tilde{\boldsymbol{\mathcal{G}}}$ have the desired properties.

5.1. **Constants.** Let $\boldsymbol{d} = (d_2, \dots, d_k)$ be a vector of positive rational numbers. Suppose

$$d_j = \frac{p_j}{q_j} \text{ for some integers } p_j \text{ and } q_j.$$

Moreover, let $\delta_3, \dots, \delta_k$ satisfy $0 \le \delta_j \le d_j/2$ for $j = 3, \dots, k$. Furthermore, let $\tilde{\delta}_k$ be a positive real and let $\tilde{\boldsymbol{\delta}}(\boldsymbol{D})$ and $\tilde{r}(B_1, \boldsymbol{D})$ be the arbitrary positive functions in variables $\boldsymbol{D} = (D_2, \dots, D_{k-1})$ and B_1 given by the lemma. The proof of Lemma 29 relies on the Regularity Lemma, and more specifically, on Corollary 27. The proof also relies on the Induction Hypothesis on the Counting Lemma ($\mathbf{IHC}_{k-1,\ell}$), Statement 32, with $\ell = k$. Therefore, for the proof of Lemma 29 presented here, we assume that $\mathbf{IHC}_{k-1,\ell}$ holds (cf. (18)).

We set

$$\eta = \frac{1}{4}, \qquad \sigma_j = \frac{1}{q_j}, \qquad \text{and} \qquad \delta'_k = \mu = \frac{\tilde{\delta}_k}{2^{\ell+k}} \prod_{h=2}^{k} d_h^{\binom{\ell}{h}}. \tag{54}$$

We also fix functions δ'_j in the variables $D_j, \dots, D_{k-1}$ for $j = k-1, \dots, 2$ so that

$$\begin{aligned} &\delta'_j\left(D_j, \dots, D_{k-1}\right) < \min\left\{ \frac{D_j}{18}\tilde{\delta}_j\left(D_j, \dots, D_{k-1}\right), \frac{D_j^3}{36} \right\} < \frac{D_j^2}{9} \qquad \text{and} \\ &\delta'_j\left(D_j, \dots, D_{k-1}\right) < \frac{D_j}{9}\delta_j\big(\mathbf{IHC}_{k-1,\ell}(\eta, D_{k-1}, \delta'_{k-1}(D_{k-1}), D_{k-2}, \dots, \delta'_{j+1}(D_{j+1} \dots, D_{k-1}), D_j)\big). \end{aligned} \tag{55}$$

(Observe that the right-hand side of the last inequality is a function in the variables $D_j, \dots, D_{k-1}$.) Similarly, we set $r'\left(B_1, \boldsymbol{D}\right)$ so that

$$\begin{aligned} &r'\left(B_1, \boldsymbol{D}\right) \ge \tilde{r}\left(B_1, \boldsymbol{D}\right), \qquad \text{and} \\ &r'\left(B_1, \boldsymbol{D}\right) \ge r\big(\mathbf{IHC}_{k-1,\ell}\left(\eta, D_{k-1}, \delta'_{k-1}(D_{k-1}), D_{k-2}, \dots, \delta'_3(D_3 \dots, D_{k-1}), D_2\right)\big) \end{aligned} \tag{56}$$

where, without loss of generality, we may assume that the functions given in (55) and (56) are monotone. Corollary 27 then yields the integer constants n_k and L_k. Next we define the constants promised by Lemma 29 as follows

$$\begin{aligned} &\tilde{c}_j = \frac{1}{2^{\ell+2}L_k^k} \text{ for } j = 2, \dots, k-1, \quad \tilde{\boldsymbol{c}} = (\tilde{c}_2, \dots, \tilde{c}_{k-1}), \\ &\tilde{L}_k = \binom{\ell}{k-1} L_k^{k-1} \prod_{j=2}^{k-1} \left(\frac{1}{\tilde{c}_j}\right)^{\binom{k-1}{j}} \text{ and } \quad \delta_2 = \frac{\delta'_2(\tilde{\boldsymbol{c}})}{\tilde{L}_k^2}. \end{aligned} \tag{57}$$

Finally, let $m_{k-1,\ell}$ be the integer given by Statement 32 applied to the constants η, $\tilde{c}_{k-1}$, $\delta'_{k-1}(\tilde{c}_{k-1}), \dots, \tilde{c}_2$, $\delta'_2(\tilde{\boldsymbol{c}})$ and set $\tilde{n}_k = \max\{n_k, L_k m_{k-1,\ell}\}$.

5.2. **Getting started.** Let $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$ be an (n, ℓ, k)-complex with $n \ge \tilde{n}_k$. We apply Corollary 27 to $\boldsymbol{\mathcal{G}}$ to obtain a $\big(\mu, \boldsymbol{\delta'}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, r'(\tilde{\boldsymbol{d}})\big)$-equitable $\big(\delta'_k, r'(\tilde{\boldsymbol{d}})\big)$-regular family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(j)}\}_{j=1}^{k-1}$ refining $\boldsymbol{\mathcal{G}}$ (cf. Definition 26), where $\tilde{\boldsymbol{d}} = (\tilde{d}_2, \dots, \tilde{d}_{k-1})$ is the density vector of the family of partitions $\mathscr{R}$. Note that it follows from our choice of σ_j in (54) and part (*ii*) of Corollary 27 that for every $j = 2, \dots, k-1$

$$d_j/\tilde{d}_j \qquad \text{and} \qquad 1/\tilde{d}_j \quad \text{are integers.} \tag{58}$$

We now make a few preparations concerning notation. Having $\tilde{\boldsymbol{d}} = (\tilde{d}_2, \dots, \tilde{d}_{k-1})$ as an outcome of Corollary 27, we derive the constants δ'_j, $\tilde{\delta}_j$ for $j = 2, \dots, k-1$ and r' and $\tilde{r}$ from the functions given in (55) and (56) by setting

$$\delta'_j = \delta'_j(\tilde{d}_j, \dots, \tilde{d}_{k-1}) < \tilde{\delta}_j(\tilde{d}_j, \dots, \tilde{d}_{k-1}) = \tilde{\delta}_j \quad \text{and} \quad r' = r'(a_1, \tilde{\boldsymbol{d}}) \ge \tilde{r}(a_1, \tilde{\boldsymbol{d}}) = \tilde{r}, \tag{59}$$

(the inequalities above follow immediately from (55) and (56)). Moreover, we set $\boldsymbol{\delta'} = (\delta'_2, \dots, \delta'_{k-1})$ and $\tilde{\boldsymbol{\delta}} = (\tilde{\delta}_2, \dots, \tilde{\delta}_{k-1})$.

For any $j = 2, \dots, k-1$ and $\hat{\boldsymbol{y}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$, let $a_j^{\mathrm{reg}}(\hat{\boldsymbol{y}}^{(j-1)})$ be the number of $(\delta'_j, \tilde{d}_j, r')$-regular $(n/a_1, j, j)$-cylinders of $\mathscr{R}^{(j)}$ belonging to $\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{y}}^{(j-1)})$. We then observe that

$$a_j^{\mathrm{reg}}(\hat{\boldsymbol{y}}^{(j-1)}) \le \frac{1}{\tilde{d}_j - \delta'_j} \le \frac{2}{\tilde{d}_j}\,. \tag{60}$$

Finally, we fix the integer vector $\boldsymbol{b} = (b_1, \dots, b_{k-1})$. We set

$$b_1 = a_1 \quad \text{and} \quad b_j \overset{(58)}{=} \frac{1}{\tilde{d}_j} \text{ for } j = 2, \dots, k-1\,. \tag{61}$$

Before constructing the promised perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{b}, \boldsymbol{\psi}) = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ (cf. Definition 28) and the (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$, we proceed with the following simple observation.

Observation regarding 'bad' j-tuples. Since $\mathscr{R}$ is a $(\mu, \boldsymbol{\delta}', \tilde{\boldsymbol{d}}, r')$-equitable (δ'_k, r')-regular partition, all but at most μn^k crossing (w.r.t. $\mathcal{G}^{(1)}$) k-tuples belong to $(\boldsymbol{\delta}', \tilde{\boldsymbol{d}}, r')$-regular $(n/a_1, k, k-1)$-complexes $\boldsymbol{\mathcal{R}}(\hat{\boldsymbol{y}}^{(k-1)})$ (cf. (16)) given by the family of partitions $\mathscr{R}$. We assert that

$$\begin{gathered} \textit{for each } 2 \le j \le k, \textit{ at most } \mu\tbinom{k}{j} n^j \textit{ crossing } j\textit{-tuples belong to} \\ \big((\delta'_2, \dots, \delta'_{j-1}), (\tilde{d}_2, \dots, \tilde{d}_{j-1}), r'\big)\textit{-irregular } (n/a_1, j, j-1)\textit{-complexes of } \mathscr{R}. \end{gathered} \tag{62}$$

Indeed, a j-tuple belonging to an irregular $(n/a_1, j, j-1)$-complex can be extended to $\binom{\ell-j}{k-j} n^{k-j}$ crossing k-tuples and at most $\binom{k}{j}$ such j-tuples can be extended to the same k-tuple. Each such k-tuple necessarily belongs to an irregular $(n/a_1, k, k-1)$-complex. Since there are at most μn^k such "irregular" k-tuples (62) follows.

Itinerary. We define the complex $\tilde{\boldsymbol{\mathcal{G}}}$ and the family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{b}, \boldsymbol{\psi})$ so that $\mathscr{P}$ is a perfect family of partitions refining $\tilde{\boldsymbol{\mathcal{G}}}$. Our plan is to alter the family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ into the family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{b}, \boldsymbol{\psi}) = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$. The families $\mathscr{P}$ and $\mathscr{R}$ will overlap in the regular elements of $\mathscr{R}$. The elements of $\mathscr{R}$ which are not regular are substituted by random cylinders.

We construct $\mathscr{P}^{(j)}$ and $\tilde{\mathcal{G}}^{(j)}$ inductively for $j = 1, \dots, k-1$. First set $\tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$. Since $b_1 = a_1$, we have $A(1, \boldsymbol{a}) = A(1, \boldsymbol{b})$ and $\hat{A}(1, \boldsymbol{a}) = \hat{A}(1, \boldsymbol{b})$. We set $\psi_1 \equiv \varphi_1$ and define $\mathscr{P}^{(1)} = \mathscr{R}^{(1)}$. In other words, both $\mathscr{R}$ and $\mathscr{P}$ split the sets V_λ for $\lambda \in [\ell]$ into the same pieces $V_\lambda = V_{\lambda,1} \cup \dots \cup V_{\lambda,b_1}$.

For $2 \le j < k$, we shall define $\mathscr{P}^{(j)}$ and $\tilde{\mathcal{G}}^{(j)}$ in such a way that the following statement $(\mathcal{C}_j)$ holds:

$(\mathcal{C}_j)$ There is a partition $\mathscr{P}^{(j)} = \mathscr{P}^{(j)}_{\mathrm{orig}} \cup \mathscr{P}^{(j)}_{\mathrm{new}}$ of $K_\ell^{(j)}(V_1, \dots, V_\ell)$ where, for $* \in \{\mathrm{orig}, \mathrm{new}\}$, we define

$$\mathcal{P}^{(j)}_* = \bigcup \Big\{ \mathcal{P}^{(j)} \colon \mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_* \Big\},$$

and an (n, ℓ, j)-cylinder $\tilde{\mathcal{G}}^{(j)} \subseteq K_\ell^{(j)}(V_1, \dots, V_\ell)$ such that (I)–(III) below hold:

(I) $\mathscr{P}^{(j)}_{\mathrm{orig}} = \big\{ \mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}) \colon \boldsymbol{y}^{(j)} \in A(j, \boldsymbol{a})$ and $\boldsymbol{\mathcal{R}}(\boldsymbol{y}^{(j)}) = \{\mathcal{R}^{(h)}(\boldsymbol{y}^{(j)})\}_{h=1}^{j}$ is a $\big((\delta'_2, \dots, \delta'_j), (\tilde{d}_2, \dots, \tilde{d}_j), r'\big)$-regular $(n/a_1, j, j)$-complex$\big\}$,

(II) $$\tilde{\mathcal{G}}^{(j)} = \begin{cases} \mathcal{G}^{(2)} & \text{if} \quad j = 2 \\ \mathcal{G}^{(j)} \cap \mathcal{P}^{(j)}_{\mathrm{orig}} = \mathcal{G}^{(j)} \setminus \mathcal{P}^{(j)}_{\mathrm{new}} & \text{if} \quad 3 \le j < k \end{cases} \quad \text{and}$$

(III) the family of partitions $\mathscr{P}_j = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(j)}\}$ is a perfect $\big((9\delta'_2/\tilde{d}_2, \dots, 9\delta'_j/\tilde{d}_j), (\tilde{d}_2, \dots, \tilde{d}_j), r', \boldsymbol{b}\big)$-family.

Before we give an inductive proof of statement $(\mathcal{C}_j)$, we list a few of its consequences in Fact 42. The properties *(1)*–*(5)* of Fact 42 will be derived directly from $(\mathcal{C}_j)$. They are utilized in our proof, in particular, we use Fact 42 with $j-1$ to establish $(\mathcal{C}_j)$.

Fact 42 (Consequences of $(\mathcal{C}_j)$)**.** *Let $2 \le j \le k-1$ be fixed. If $(\mathcal{C}_{j'})$ holds for $2 \le j' \le j$ and if $\mathscr{P}^{(2)}$ refines $\tilde{\mathcal{G}}^{(2)}$, then the following is true:*

(1) $\tilde{\mathcal{G}}^{(j)} \subseteq \mathcal{G}^{(j)}$,

(*2*) $\tilde{\boldsymbol{\mathcal{G}}}^{(j)} = \{\tilde{\mathcal{G}}^{(h)}\}_{h=1}^{j}$ *is an* (n,ℓ,j)*-complex and for each* $2 \le h \le j$*, i.e.,* $\tilde{\mathcal{G}}^{(h)} \subseteq \mathcal{K}_h\big(\tilde{\mathcal{G}}^{(h-1)}\big)$*,*

(*3*) $\mathscr{P}_j$ *refines the complex* $\tilde{\boldsymbol{\mathcal{G}}}^{(j)}$*,*

(*4*) *for every* $j \le i \le \ell$*,*

$$\Big|\mathcal{K}_i(\tilde{\mathcal{G}}^{(j)}) \triangle \mathcal{K}_i(\mathcal{G}^{(j)})\Big| \le \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i\,, \qquad \textit{and}$$

(*5*) *for every* $\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\boldsymbol{b})$*,*

$$\Big|\mathcal{K}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)\Big| = (1\pm\eta)\prod_{h=2}^{j} \tilde{d}_h^{\binom{j+1}{h}} \times \left(\frac{n}{b_1}\right)^{j+1} > \frac{(n/b_1)^{j+1}}{\ln(n/b_1)}.$$

Proof of Fact 42. part (*1*) follows clearly from (II). We prove (*2*) by induction on j. For $j = 1$ or 2 there is nothing to prove. Let $j \ge 3$. Suppose $(\mathcal{C}_i)$ is true for $2 \le i \le j$ and suppose, by induction, (*2*) holds for $j-1$, i.e, $\tilde{\boldsymbol{\mathcal{G}}}^{(j-1)}$ is an $(n,\ell,j-1)$-complex. We show that every j-tuple $J \in \tilde{\mathcal{G}}^{(j)}$ satisfies $J \in \mathcal{K}_j(\tilde{\mathcal{G}}^{(j-1)})$.

Let $J \in \tilde{\mathcal{G}}^{(j)}$ be fixed. It then follows from (II) of $(\mathcal{C}_j)$ that

$$J \in \mathcal{G}^{(j)} \cap \mathcal{P}^{(j)}_{\mathrm{orig}}\,. \tag{63}$$

We first confirm

$$J \in \mathcal{K}_j(\mathcal{P}^{(j-1)}_{\mathrm{orig}})\,. \tag{64}$$

To that end, since $J \in \mathcal{P}^{(j)}_{\mathrm{orig}}$, it follows from (I) of $(\mathcal{C}_j)$ that there exists $\boldsymbol{y}^{(j)} \in A(j,\boldsymbol{a})$ such that $J \in \mathcal{R}^{(j)}(\boldsymbol{y}^{(j)})$ and the complex $\boldsymbol{\mathcal{R}}(\boldsymbol{y}^{(j)}) = \{\mathcal{R}^{(h)}(\boldsymbol{y}^{(j)})\}_{h=1}^{j}$ is $\big((\delta_2',\dots,\delta_j'),(\tilde{d}_2,\dots,\tilde{d}_j),r'\big)$-regular. Consequently, $J \in \mathcal{K}_j\big(\mathcal{R}^{(j-1)}(\boldsymbol{y}^{(j)})\big)$ and by (I) of $(\mathcal{C}_{j-1})$ we have that $\mathcal{R}^{(j-1)}(\boldsymbol{y}^{(j)}) \subseteq \mathcal{P}^{(j-1)}_{\mathrm{orig}}$. This yields $J \in \mathcal{K}_j(\mathcal{P}^{(j-1)}_{\mathrm{orig}})$ as claimed in (64).

Now from (63) and (64), we infer that $J \in \mathcal{K}_j(\mathcal{G}^{(j-1)} \cap \mathcal{P}^{(j-1)}_{\mathrm{orig}})$ (since $\boldsymbol{\mathcal{G}}$ is a complex), and so by (II) of $(\mathcal{C}_{j-1})$ we have $J \in \mathcal{K}_j(\tilde{\mathcal{G}}^{(j-1)})$. This completes the proof of (*2*).

Next we show part (*3*), again by induction on j. The statement is trivial for $j = 1$. It holds for $j = 2$ by assumption of Fact 42. So let $j \ge 3$ and assume that $\mathscr{P}_{j-1}$ refines $\{\tilde{\mathcal{G}}^{(h)}\}_{j=1}^{j-1}$. We have to show that either $\mathcal{P}^{(j)} \subseteq \tilde{\mathcal{G}}^{(j)}$ or $\mathcal{P}^{(j)} \cap \tilde{\mathcal{G}}^{(j)} = \varnothing$ for every $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}$. So let $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}$ be fixed. If $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_{\mathrm{new}}$, then $\mathcal{P}^{(j)} \cap \tilde{\mathcal{G}}^{(j)} = \varnothing$ by (II) of $(\mathcal{C}_j)$. Therefore, we may assume that $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_{\mathrm{orig}}$. Now, if $\mathcal{P}^{(j)} \cap \mathcal{G}^{(j)} = \varnothing$, then again by (II) of $(\mathcal{C}_j)$ we infer $\mathcal{P}^{(j)} \cap \tilde{\mathcal{G}}^{(j)} = \varnothing$. On the other hand, if $\mathcal{P}^{(j)} \cap \mathcal{G}^{(j)} \ne \varnothing$, then $\mathcal{P}^{(j)} \subseteq \mathcal{G}^{(j)}$ since $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_{\mathrm{orig}}$, (I) of $(\mathcal{C}_j)$, and the fact that the original family of partitions $\mathscr{R}$ refines the complex $\boldsymbol{\mathcal{G}}$. Therefore, $\mathcal{P}^{(j)} \subseteq \mathcal{G}^{(j)} \cap \mathcal{P}^{(j)}_{\mathrm{orig}} = \tilde{\mathcal{G}}^{(j)}$ by (*1*) of Fact 42. This verifies (*3*) of Fact 42.

Next we show (*4*) of Fact 42. From (62) and (I) and (II) of $(\mathcal{C}_j)$ we infer that

$$\big|\mathcal{G}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}\big| = \big|\mathcal{G}^{(j)} \setminus \tilde{\mathcal{G}}^{(j)}\big| \le \mu\binom{k}{j} n^j\,.$$

Consequently, by the choice of μ in (54)

$$\Big|\mathcal{K}_i(\tilde{\mathcal{G}}^{(j)}) \triangle \mathcal{K}_i(\mathcal{G}^{(j)})\Big| \le \mu\binom{k}{j} n^j \times \binom{\ell-j}{i-j} n^{i-j} \le \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i\,,$$

which yields (*4*).

Finally, we note that (*5*) follows from (III) and $\mathbf{IHC}_{k-1,\ell}$ (cf. (18)) since $j \le k-1$. In particular, (*5*) is a consequence of the choice of δ_j' and r' in (55) and (56), (III) of $(\mathcal{C}_j)$, and (18). □

5.3. **Proof of statement** $(\mathcal{C}_j)$**.** As mentioned earlier, we verify $(\mathcal{C}_j)$ by induction on j.

5.3.1. *The induction start.* In the immediate sequel, we define $\mathscr{P}^{(2)} = \mathscr{P}^{(2)}_{\text{new}} \cup \mathscr{P}^{(2)}_{\text{orig}}$ of $K^{(2)}_\ell(V_1, \ldots, V_\ell)$. In our construction, we use that our constants satisfy

$$a_1^2 \delta_2 < L_k^2 \delta_2 \overset{(57)}{<} \tilde{L}_k^2 \delta_2 \overset{(57)}{=} \delta_2'(\tilde{\boldsymbol{c}}) \leq \delta_2'(\tilde{\boldsymbol{d}}) \overset{(59)}{=} \delta_2' \overset{(55)}{<} \tilde{d}_2 \leq d_2 \tag{65}$$

and also use that $d_2/\tilde{d}_2$ is an integer (see (58)). Before constructing the partition $\mathscr{P}^{(2)}$, we require some notation.

Notation. Recall that the partition $\mathscr{R}^{(2)} = \{\mathcal{R}^{(2)}(\boldsymbol{y}^{(2)})\colon \boldsymbol{y}^{(2)} \in A(2, \boldsymbol{a})\}$ refines the partition $\mathcal{G}^{(2)} \cup \overline{\mathcal{G}^{(2)}}$ (here, $\overline{\mathcal{G}^{(2)}} = K^{(2)}_\ell(V_1, \ldots, V_\ell) \setminus \mathcal{G}^{(2)}$). Therefore, for each $\hat{\boldsymbol{y}}^{(1)} = \big((\lambda, \lambda'), (\beta, \beta')\big) \in \hat{A}(1, \boldsymbol{a})$, there exist disjoint sets of indices $I_2^{\text{reg}} = I_2^{\text{reg}}(\hat{\boldsymbol{y}}^{(1)})$ and $\overline{I}_2^{\text{reg}} = \overline{I}_2^{\text{reg}}(\hat{\boldsymbol{y}}^{(1)})$ so that $\big\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)\big\}_{\alpha \in I_2^{\text{reg}}}$ and $\big\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)\big\}_{\alpha \in \overline{I}_2^{\text{reg}}}$ are the collections of all $(\delta_2', \tilde{d}_2, 1)$-regular graphs $\mathcal{R}^{(2)}(\boldsymbol{y}^{(2)}) = \mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)$ whose edge sets are subsets of $\mathcal{G}^{(2)}(\boldsymbol{y}^{(1)}) = \mathcal{G}^{(2)}[V_{\lambda,\beta} \cup V_{\lambda',\beta'}]$ and $\overline{\mathcal{G}^{(2)}}(\boldsymbol{y}^{(1)}) = (V_{\lambda,\beta} \times V_{\lambda',\beta'}) \setminus \mathcal{G}^{(2)}$, respectively.

Plan for constructing $\mathscr{P}^{(2)}$. We now outline our plan for constructing $\mathscr{P}^{(2)} = \{\mathcal{P}^{(2)}(\boldsymbol{x}^{(2)})\colon \boldsymbol{x}^{(2)} \in A(2, \boldsymbol{b})\}$. Later we fill in the technical details. With $\hat{\boldsymbol{x}}^{(1)} = \hat{\boldsymbol{y}}^{(1)} = \big((\lambda, \lambda'), (\beta, \beta')\big) \in \hat{A}(1, \boldsymbol{a}) = \hat{A}(1, \boldsymbol{b})$ fixed, we define a partition $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ of $\mathcal{K}_2\big(\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}}^{(1)})\big) = V_{\lambda,\beta} \times V_{\lambda',\beta'}$. More precisely, with $\hat{\boldsymbol{x}}^{(1)} = \hat{\boldsymbol{y}}^{(1)}$ defining a pair of sets $V_{\lambda,\beta}$, $V_{\lambda',\beta'}$, we consider all regular subgraphs of $V_{\lambda,\beta} \times V_{\lambda',\beta'}$ from the partition $\mathscr{R}^{(2)}$ and leave them in the "original part" ($\mathscr{P}^{(2)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(1)})$) of $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$. In other words, for $\hat{\boldsymbol{x}}^{(1)} = \hat{\boldsymbol{y}}^{(1)}$ we set

$$\mathscr{P}^{(2)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(1)}) = \Big\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)\Big\}_{\alpha \in I_2^{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})} \cup \Big\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)\Big\}_{\alpha \in \overline{I}_2^{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})}. \tag{66}$$

This collection of graphs consists of all subgraphs of $V_{\lambda,\beta} \times V_{\lambda',\beta'}$ belonging to $\mathscr{R}^{(2)}$ which are $(\delta_2', \tilde{d}_2, 1)$-regular. In order to simplify the notation, we set

$$\mathcal{P}^{(2)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(1)}) = \bigcup \Big\{\mathcal{P}^{(2)}\colon\ \mathcal{P}^{(2)} \in \mathscr{P}^{(2)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(1)})\Big\}\,.$$

For the construction of the partition of $(V_{\lambda,\beta} \times V_{\lambda',\beta'}) \setminus \mathcal{P}^{(2)}_{\text{orig}}$, we will use the Slicing Lemma to introduce new $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular graphs that do not belong to $\mathscr{R}^{(2)}$. We shall call the collection of those graphs $\mathscr{P}^{(2)}_{\text{new}}(\hat{\boldsymbol{x}}^{(1)})$. We now provide the technical details to the plan described above.

Technical details for constructing $\mathscr{P}^{(2)}$. Let $\hat{\boldsymbol{x}}^{(1)} = \big((\lambda, \lambda'), (\beta, \beta')\big) \in \hat{A}(1, \boldsymbol{a})$ remain fixed. Let

$$\mathcal{G}^{(2)}_{\text{reg}}(\hat{\boldsymbol{x}}^{(1)}) = \mathcal{P}^{(2)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(1)}) \cap \mathcal{G}^{(2)}$$

be the union of all graphs $\mathcal{P}^{(2)} \subseteq \mathcal{G}^{(2)}$ in $\mathscr{P}^{(2)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(1)})$. Similarly, we define

$$\overline{\mathcal{G}^{(2)}_{\text{reg}}}(\hat{\boldsymbol{x}}^{(1)}) = \mathcal{P}^{(2)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(1)}) \cap \overline{\mathcal{G}^{(2)}}\,.$$

Note that while $\overline{\mathcal{G}^{(2)}_{\text{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ and $\mathcal{G}^{(2)}_{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})$ are disjoint, they are not necessarily complements of each other. Moreover, observe that $\mathcal{G}^{(2)}_{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})$ is the union of $\alpha_2^{\text{reg}} = |I_2^{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})| \leq a_2^{\text{reg}}(\hat{\boldsymbol{x}}^{(1)}) \leq 2/\tilde{d}_2$ (see (60)) $(\delta_2', \tilde{d}_2, 1)$-regular graphs. Consequently, $\mathcal{G}^{(2)}_{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})$ is $(2\delta_2'/\tilde{d}_2, \alpha_2^{\text{reg}}\tilde{d}_2, 1)$-regular (cf. Proposition 50). Similarly, $\overline{\mathcal{G}^{(2)}_{\text{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ is $(2\delta_2'/\tilde{d}_2, \overline{\alpha}_2^{\text{reg}}\tilde{d}_2, 1)$-regular, where $\overline{\alpha}_2^{\text{reg}} = |\overline{I}_2^{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})|$.

Since $\mathcal{G}^{(2)}$ is $(\delta_2, d_2, 1)$-regular by the assumption of Lemma 29, we infer that $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) = \mathcal{G}^{(2)}[V_{\lambda,\beta}, V_{\lambda',\beta'}]$ is $(a_1^2\delta_2, d_2, 1)$-regular. Therefore, $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ is $(\delta_2', d_2, 1)$-regular by (65). Consequently, since $2\delta_2'/\tilde{d}_2 + \delta_2' \leq 3\delta_2'/\tilde{d}_2$, we have that $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \setminus \mathcal{G}^{(2)}_{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})$ is $(3\delta_2'/\tilde{d}_2, d_2 - \alpha_2^{\text{reg}}\tilde{d}_2, 1)$-regular. We now apply the Slicing Lemma, Lemma 30, to $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \setminus \mathcal{G}^{(2)}_{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})$.

To this end, recall $d_2/\tilde{d}_2$ is an integer (see (58)) and set $p = \tilde{d}_2(d_2 - \alpha_2^{\text{reg}}\tilde{d}_2)^{-1}$ so that $1/p = d_2/\tilde{d}_2 - \alpha_2^{\text{reg}}$ is an integer. We apply the Slicing Lemma with $\varrho = d_2 - \alpha_2^{\text{reg}}\tilde{d}_2$, $\delta = 3\delta_2'/\tilde{d}_2$, p as above and $r_{\text{SL}} = 1$ to decompose $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \setminus \mathcal{G}^{(2)}_{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})$ into $1/p = d_2/\tilde{d}_2 - \alpha_2^{\text{reg}}$ pairwise edge-disjoint $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular graphs. Denote the family of these bipartite graphs by $\mathscr{P}^{(2)}_{\text{new},\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)})$.

The partition $\mathscr{P}^{(2)}_{\text{new},\overline{\mathcal{G}^{(2)}}}(\hat{\boldsymbol{x}}^{(1)})$ of $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) \setminus \overline{\mathcal{G}^{(2)}_{\text{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ will be defined in a very similar way. Indeed, the graph $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) = (V_{\lambda,\beta}) \times V_{\lambda',\beta'}) \setminus \mathcal{G}^{(2)}$ is $(a_1^2\delta_2, 1-d_2, 1)$-regular since it is the complement of the $(a_1^2\delta_2, d_2, 1)$-regular graph $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$. By (65), the graph $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)})$ is then also $(\delta_2'/\tilde{d}_2, 1-d_2, 1)$-regular. Furthermore, $\overline{\mathcal{G}^{(2)}_{\text{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ is $(2\delta_2'/\tilde{d}_2, \overline{\alpha}_2^{\text{reg}}\tilde{d}_2, 1)$-regular (since $\overline{\mathcal{G}^{(2)}_{\text{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ is the union of $\overline{\alpha}_2^{\text{reg}}$ disjoint $(\delta_2', \tilde{d}_2, 1)$-regular graphs and $\overline{\alpha}_2^{\text{reg}} \le a_2^{\text{reg}} \le 2/\tilde{d}_2$ by (60)). Consequently, $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) \setminus \overline{\mathcal{G}^{(2)}_{\text{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ is $(3\delta_2'/\tilde{d}_2, 1-d_2-\overline{\alpha}_2^{\text{reg}}\tilde{d}_2, 1)$-regular.

We apply the Slicing Lemma with $\varrho = 1 - d_2 - \overline{\alpha}_2^{\text{reg}}\tilde{d}_2$, $\delta = 3\delta_2'/\tilde{d}_2$, $p = \tilde{d}_2/\varrho$ and $r_{\text{SL}} = 1$ to decompose $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) \setminus \overline{\mathcal{G}^{(2)}_{\text{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ into a family $\mathscr{P}^{(2)}_{\text{new},\overline{\mathcal{G}^{(2)}}}(\hat{\boldsymbol{x}}^{(1)})$ of bipartite graphs. We conclude that all of these graphs are $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular. Indeed, note that by (58) we have that $1/p = (1 - d_2 - \overline{\alpha}_2^{\text{reg}}\tilde{d}_2)/\tilde{d}_2$ is an integer.

For $\hat{\boldsymbol{x}}^{(1)} = \big((\lambda,\lambda'),(\beta,\beta')\big)$, set

$$\mathscr{P}^{(2)}_{\text{new}}(\hat{\boldsymbol{x}}^{(1)}) = \mathscr{P}^{(2)}_{\text{new},\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) \cup \mathscr{P}^{(2)}_{\text{new},\overline{\mathcal{G}^{(2)}}}(\hat{\boldsymbol{x}}^{(1)})$$

and

$$\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) = \mathscr{P}^{(2)}_{\text{new}}(\hat{\boldsymbol{x}}^{(1)}) \cup \mathscr{P}^{(2)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(1)})\,.$$

Also set $z(\hat{\boldsymbol{x}}^{(1)}) = |\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})|$. The partition $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ has the following properties:

(A) $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ is a partition of $V_{\lambda,\beta} \times V_{\lambda',\beta'}$.

(B) $z(\hat{\boldsymbol{x}}^{(1)}) = b_2$. Indeed, since all graphs from $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ have density within $\tilde{d}_2 \pm 9\delta_2'/\tilde{d}_2$, we have that

$$\frac{1}{\tilde{d}_2 + 9\delta_2'/\tilde{d}_2} \le z(\hat{\boldsymbol{x}}^{(1)}) \le \frac{1}{\tilde{d}_2 - 9\delta_2'/\tilde{d}_2}. \tag{67}$$

It follows from (55) that $9\delta_2'/\tilde{d}_2 < (\tilde{d}_2/2)^2$ yielding $(\tilde{d}_2 - 9\delta_2'/\tilde{d}_2)^{-1} - (\tilde{d}_2 + 9\delta_2'/\tilde{d}_2)^{-1} < 1$. Consequently, $z(\hat{\boldsymbol{x}}^{(1)}) = b_2$ follows from (61).

(C) $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ refines $\mathcal{G}^{(2)}$ in the sense that for every $\alpha \in [z(\hat{\boldsymbol{x}}^{(1)})]$ either $\mathcal{P}^{(2)}\big((\hat{\boldsymbol{x}}^{(1)},\alpha)\big) \subseteq \mathcal{G}^{(2)}$ or $\mathcal{P}^{(2)}\big((\hat{\boldsymbol{x}}^{(1)},\alpha)\big) \cap \mathcal{G} = \varnothing$.

(D) All graphs from $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ are $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular.

Now, we set

$$\tilde{\mathcal{G}}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) = \bigcup\Big\{\mathcal{P}^{(2)} \in \mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \colon \mathcal{P}^{(2)} \subseteq \mathcal{G}^{(2)}\Big\} \quad\text{and}\quad \tilde{\mathcal{G}}^{(2)} = \bigcup\Big\{\tilde{\mathcal{G}}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \colon \hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1,\boldsymbol{b})\Big\}, \tag{68}$$

and we set

$$\mathscr{P}^{(2)}_{\text{new}} = \bigcup\Big\{\mathscr{P}^{(2)}_{\text{new}}(\hat{\boldsymbol{x}}^{(1)}) \colon \hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1,\boldsymbol{b})\Big\},$$
$$\mathscr{P}^{(2)}_{\text{orig}} = \bigcup\Big\{\mathscr{P}^{(2)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(1)}) \colon \hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1,\boldsymbol{b})\Big\} \quad\text{and}\quad \mathscr{P}^{(2)} = \mathscr{P}^{(2)}_{\text{new}} \cup \mathscr{P}^{(2)}_{\text{orig}}\,.$$

It is left to verify (I)–(III) of the statement $(\mathcal{C}_2)$. Due to (66) and the definition of $I_2^{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})$ and $\overline{I}_2^{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})$, for every $\hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1,\boldsymbol{b})$, we infer that

$$\mathscr{P}^{(2)}_{\text{orig}} = \Big\{\mathcal{R}^{(2)}(\boldsymbol{y}^{(2)}) \colon \mathcal{R}^{(2)}(\boldsymbol{y}^{(2)}) \text{ is } (\delta_2', \tilde{d}_2, 1)\text{-regular}\Big\},$$

which yields (I) of $(\mathcal{C}_2)$. Owing to (C) from above and (68), we have $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$ (which is (II)) and

$$\mathscr{P}^{(2)} \text{ refines } \tilde{\mathcal{G}}^{(2)}\,. \tag{69}$$

Finally, we infer from (B), (D), and (61) that $\mathscr{P}_2 = \{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$ is a perfect $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, r', \boldsymbol{b})$-family (see Definition 28), which gives (III) of $(\mathcal{C}_2)$.

This concludes the construction of $\mathscr{P}^{(2)}$ which satisfies $(\mathcal{C}_2)$ and, therefore, we established the induction start of our construction of $\mathscr{P}$ and $\tilde{\boldsymbol{\mathcal{G}}}$. Also note that we additionally verified (69).

5.3.2. *The inductive step.* We proceed to the inductive step and construct partition $\mathscr{P}^{(j+1)}$ and $(n,\ell,j+1)$-cylinder $\tilde{\mathcal{G}}^{(j+1)}$ which will satisfy (I)–(III) of $(\mathcal{C}_{j+1})$. We assume that $\mathscr{P}^{(h)}$ and $\tilde{\mathcal{G}}^{(h)}$ satisfying $(\mathcal{C}_h)$, $2 \le h \le j$, are given. Moreover, due to (69), we assume Fact 42 holds as well for $2 \le h \le j$.

Our work in constructing $\mathscr{P}^{(j+1)}$ will be quite similar, albeit easier, than our work for constructing $\mathscr{P}^{(2)}$. This is in part because we do not require that $\tilde{\mathcal{G}}^{(j+1)} = \mathcal{G}^{(j+1)}$ for $j \ge 2$. It will be necessary to construct $\mathscr{P}^{(j+1)}$ before constructing $\tilde{\mathcal{G}}^{(j+1)}$ as the partition ends up defining the hypergraph.

Construction of $\mathscr{P}^{(j+1)}$ and $\tilde{\mathcal{G}}^{(j+1)}$. The partition $\mathscr{P}^{(j+1)} = \mathscr{P}^{(j+1)}_{\mathrm{new}} \cup \mathscr{P}^{(j+1)}_{\mathrm{orig}}$ of $K^{(j+1)}_\ell(V_1,\dots,V_\ell)$ will be defined separately for each family $\mathcal{K}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ of $(j+1)$-tuples with $\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\boldsymbol{b})$.

Fix $\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\boldsymbol{b})$. We define the partition $\mathscr{P}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)}) = \mathscr{P}^{(j+1)}_{\mathrm{new}}(\hat{\boldsymbol{x}}^{(j)}) \cup \mathscr{P}^{(j+1)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(j)})$ of $\mathcal{K}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ by distinguishing two cases.

Case 1 **($\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\boldsymbol{b})$ and there exists $1 \le s \le j+1$ so that $\mathcal{P}^{(j)}(\partial_s \hat{\boldsymbol{x}}^{(j)}) \in \mathscr{P}^{(j)}_{\mathrm{new}}$).** We want to apply the Slicing Lemma to $\mathcal{K}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$. For that we first appeal to (5) of Fact 42 for j. Indeed, observe that $\mathcal{K}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ is $(\delta,1,r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$ for any positive δ and integer r. Consequently, we may apply the Slicing Lemma, Lemma 30, with $\varrho = 1$, $p = \tilde{d}_{j+1}$, $\delta = 3\delta'_{j+1}/\tilde{d}_{j+1}$, and $r_{\mathrm{SL}} = r'$ to $\mathcal{F}^{(j+1)} = \mathcal{K}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$. (Observe that $3\delta = 9\delta'_{j+1}/\tilde{d}_{j+1} < \tilde{d}_{j+1} = p\varrho$ by (55).) Since $1/p = 1/\tilde{d}_{j+1} = b_{j+1}$ by (61), we obtain a collection of $1/\tilde{d}_{j+1}$ pairwise edge-disjoint $(9\delta'_{j+1}/\tilde{d}_{j+1}, \tilde{d}_{j+1}, r')$-regular $(n/b_1, j+1, j+1)$-cylinders $\mathcal{P}^{(j+1)}\big(\hat{\boldsymbol{x}}^{(j)},\alpha)\big)$ with $\alpha \in [b_{j+1}]$. Denote by

$$\mathscr{P}^{(j+1)}_{\mathrm{new}}(\hat{\boldsymbol{x}}^{(j)}) = \Big\{\mathcal{P}^{(j+1)}\big((\hat{\boldsymbol{x}}^{(j)},\alpha)\big) \colon\ \alpha \in [b_{j+1}]\Big\}$$

the family of $(n/b_1, j+1, j+1)$-cylinders newly created. Set $\mathscr{P}^{(j+1)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(j)}) = \varnothing$. This concludes our treatment of Case 1.

Case 2 **($\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\boldsymbol{b})$ and $\mathcal{P}^{(j)}(\partial_s \hat{\boldsymbol{x}}^{(j)}) \in \mathscr{P}^{(j)}_{\mathrm{orig}}$ for every $1 \le s \le j+1$).** By the assumption of this case and (I) of $(\mathcal{C}_j)$, we infer that there exists $\hat{\boldsymbol{y}}^{(j)} \in \hat{A}(j,\boldsymbol{a})$ such that $\hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{y}}^{(j)}) = \hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$. Recall the definition of $a^{\mathrm{reg}}_{j+1}(\hat{\boldsymbol{y}}^{(j)})$ (preceding (60)). Without loss of generality, let $\big\{\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big)\big\}_{\alpha\in[a^{\mathrm{reg}}_{j+1}]}$ be an enumeration of the $(\delta'_{j+1}, \tilde{d}_{j+1}, r')$-regular $(n/b_1, j+1, j+1)$-cylinders (regular w.r.t. $\hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{y}}^{(j)}) = \hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$). We set

$$\mathscr{P}^{(j+1)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(j)}) = \Big\{\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big)\Big\}_{\alpha\in[a^{\mathrm{reg}}_{j+1}]} \quad \text{and} \tag{70}$$
$$\mathcal{P}^{(j+1)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(j)}) = \bigcup\Big\{\mathcal{P}^{(j+1)}\colon\ \mathcal{P}^{(j+1)} \in \mathscr{P}^{(j+1)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(j)})\Big\} = \bigcup_{\alpha\in[a^{\mathrm{reg}}_{j+1}]} \mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big)\,.$$

Observe that $\mathcal{P}^{(j+1)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(j)})$ is $(a^{\mathrm{reg}}_{j+1}\delta'_{j+1}, a^{\mathrm{reg}}_{j+1}\tilde{d}_{j+1}, r')$-regular w.r.t. $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$ (cf. Proposition 50) and, as a consequence of (60), also $(3\delta'_{j+1}/\tilde{d}_{j+1}, a^{\mathrm{reg}}_{j+1}\tilde{d}_{j+1}, r')$-regular. Then, $\mathcal{K}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big) \setminus \mathcal{P}^{(j+1)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(j)})$ is $(3\delta'_{j+1}/\tilde{d}_{j+1}, 1 - a^{\mathrm{reg}}_{j+1}\tilde{d}_{j+1}, r')$-regular. We apply the Slicing Lemma to $\mathcal{K}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big) \setminus \mathcal{P}^{(j+1)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(j)})$ with $\varrho = 1 - a^{\mathrm{reg}}_{j+1}\tilde{d}_{j+1}$, $p = \tilde{d}_{j+1}/\varrho$, $\delta = 3\delta'_{j+1}/\tilde{d}_{j+1}$ (yielding $3\delta < p\varrho$ by (55)) and $r_{\mathrm{SL}} = r'$. Note that $1/p = \varrho/\tilde{d}_{j+1} = 1/\tilde{d}_{j+1} - a^{\mathrm{reg}}_{j+1}$ is an integer by (58). We thus obtain a collection $\mathscr{P}^{(j+1)}_{\mathrm{new}}(\hat{\boldsymbol{x}}^{(j)})$ of $1/\tilde{d}_{j+1} - a^{\mathrm{reg}}_{j+1}$ pairwise edge-disjoint $(9\delta'_{j+1}/\tilde{d}_{j+1}, \tilde{d}_{j+1}, r')$-regular $(n/b_1, j+1, j+1)$-cylinders $\mathcal{P}^{(j+1)}\big((\hat{\boldsymbol{x}}^{(j)},\alpha)\big)$. Setting $\mathscr{P}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)}) = \mathscr{P}^{(j+1)}_{\mathrm{new}}(\hat{\boldsymbol{x}}^{(j)}) \cup \mathscr{P}^{(j+1)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(j)})$ yields a partition of $\mathcal{K}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ into $1/\tilde{d}_{j+1} - a^{\mathrm{reg}}_{j+1} + a^{\mathrm{reg}}_{j+1} = 1/\tilde{d}_{j+1} = b_{j+1}$ (by (61)) disjoint $(9\delta'_{j+1}/\tilde{d}_{j+1}, \tilde{d}_{j+1}, r')$-regular $(n/b_1, j+1, j+1)$-cylinders. This concludes our treatment of Case 2.

Now, we set

$$\tilde{\mathcal{G}}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)}) = \bigcup\Big\{\mathcal{P}^{(j+1)} \in \mathscr{P}^{(j+1)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(j)})\colon\ \mathcal{P}^{(j+1)} \subseteq \mathcal{G}^{(j+1)}\Big\}\,,$$
$$\tilde{\mathcal{G}}^{(j+1)} = \bigcup\Big\{\tilde{\mathcal{G}}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)})\colon\ \hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\boldsymbol{b})\Big\}\,, \tag{71}$$

and we set

$$\mathscr{P}^{(j+1)}_{\text{new}} = \bigcup\Big\{\mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)})\colon\ \hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\boldsymbol{b})\Big\}\,,$$

$$\mathscr{P}^{(j+1)}_{\text{orig}} = \bigcup\Big\{\mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})\colon\ \hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\boldsymbol{b})\Big\} \ \text{ and } \ \mathscr{P}^{(j+1)} = \mathscr{P}^{(j+1)}_{\text{new}} \cup \mathscr{P}^{(j+1)}_{\text{orig}}\,.$$

It is left to verify (I)–(III) of statement $(\mathcal{C}_{j+1})$.

Confirmation of $(\mathcal{C}_{j+1})$**.** First we verify (I). To this end, we establish the equality of sets in (I) by decomposing the equality into its respective '$\subseteq$' and '$\supseteq$' parts, and begin by considering the former.

We verify the '$\subseteq$' component of the equality of the sets in (I) of $(\mathcal{C}_{j+1})$. Let $\mathcal{P}^{(j+1)} = \mathcal{P}^{(j+1)}\big((\hat{\boldsymbol{x}}^{(j+1)},\alpha)\big) \in \mathscr{P}^{(j+1)}_{\text{orig}}$. Owing to the construction of $\mathscr{P}^{(j+1)}$ above, $\mathcal{P}^{(j+1)}$ originates from Case 2. By the assumption of Case 2, we know that $\mathcal{P}^{(j)}(\partial_s\hat{\boldsymbol{x}}^{(j+1)}) \in \mathscr{P}^{(j)}_{\text{orig}}$ for every $s\in[j+1]$. Consequently, from (I) of $(\mathcal{C}_j)$ we infer that for each $s\in[j+1]$, there exists $\boldsymbol{y}^{(j)}_s$ such that $\boldsymbol{\mathcal{R}}(\boldsymbol{y}^{(j)}_s) = \big\{\mathcal{R}^{(h)}(\boldsymbol{y}^{(j)}_s)\big\}_{h=1}^{j}$ is a $\big((\delta'_2,\dots,\delta'_j),(\tilde{d}_2,\dots,\tilde{d}_j),r'\big)$-regular $(n/a_1,j,j)$-complex and $\mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}_s) = \mathcal{P}^{(j)}(\partial_s\hat{\boldsymbol{x}}^{(j+1)})$. Clearly,

$$\Bigg\{\bigcup_{s\in[j+1]}\mathcal{R}^{(h)}(\boldsymbol{y}^{(j)}_s)\Bigg\}_{h=1}^{j} \text{ is } \big((\delta'_2,\dots,\delta'_j),(\tilde{d}_2,\dots,\tilde{d}_j),r'\big)\text{-regular} \tag{72}$$

and $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)}) = \bigcup_{s\in[j+1]}\mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}_s)$. Moreover, by the construction in Case 2 and $\mathcal{P}^{(j+1)} \in \mathscr{P}^{(j+1)}_{\text{orig}}$, there exists $\mathcal{R}^{(j+1)} \in \mathscr{R}^{(j+1)}$ such that $\mathcal{P}^{(j+1)} = \mathcal{R}^{(j+1)}$ and $\mathcal{R}^{(j+1)}$ is $(\delta'_{j+1},\tilde{d}_{j+1},r')$-regular with respect to $\bigcup_{s\in[j+1]}\mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}_s) = \hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$. Then (72) yields the '$\subseteq$' component of the equality in (I) of $(\mathcal{C}_{j+1})$.

We now verify the '$\supseteq$' component of the equality in (I). To that end, let $\hat{\boldsymbol{y}}^{(j)} \in \hat{A}(j,\boldsymbol{a})$ and $\alpha\in[a_{j+1}]$ be given so that $\boldsymbol{\mathcal{R}}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big) = \big\{\mathcal{R}^{(h)}\big((\boldsymbol{y}^{(j)},\alpha)\big)\big\}_{h=1}^{j+1}$ is a $\big((\delta'_2,\dots,\delta'_{j+1}),(\tilde{d}_2,\dots,\tilde{d}_{j+1}),r'\big)$-regular complex. Hence, $\mathcal{R}^{(j+1)}(\partial_s\hat{\boldsymbol{y}}^{(j)}) \in \mathscr{P}^{(j)}_{\text{orig}}$ for every $s\in[j+1]$ by the induction assumption (more precisely by (I) of $(\mathcal{C}_j)$). Moreover, the $(n,j+1,j+1)$-cylinder $\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big)$ is $(\delta'_{j+1},\tilde{d}_{j+1},r')$-regular (i.e., $\alpha\in[a^{\text{reg}}_{j+1}(\hat{\boldsymbol{x}}^{(j)})]$) and, consequently, $\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big) \in \mathscr{P}^{(j+1)}_{\text{orig}}$ (cf. (70) in Case 2). This concludes the proof of (I) of $(\mathcal{C}_j)$.

Since $j+1\geq 3$, part (II) follows directly from (71).

In order to verify (III), we appeal to the induction assumption. Observe that we only need to consider $\mathcal{P}^{(j+1)}(\boldsymbol{x}^{(j+1)})$ for $\boldsymbol{x}^{(j+1)} \in A(j+1,\boldsymbol{b})$. It is clear from the construction that in both of cases we partitioned $\mathcal{K}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\boldsymbol{x}^{(j)})\big)$ into b_{j+1} different $(9\delta'_{j+1}/\tilde{d}_{j+1},\tilde{d}_{j+1},r')$-regular $(n/b_1,j+1,j+1)$-cylinders. Consequently, (III) of $(\mathcal{C}_{j+1})$ holds and $(\mathcal{C}_{j+1})$ is verified.

This finishes the inductive proof of statement $(\mathcal{C}_i)$ for $2\leq i\leq k-1$.

5.4. **Finale.** Having inductively defined partitions $\mathscr{P}^{(j)}$ and hypergraphs $\tilde{\mathcal{G}}^{(j)}$, $2\leq j\leq k-1$, we proceed to construct the promised hypergraph $\tilde{\mathcal{G}}^{(k)}$ (see (73) below). Then we shall show that the conclusions of Lemma 29 hold for $\mathscr{P} = \{\mathscr{P}^{(1)},\dots,\mathscr{P}^{(k-1)}\}$ and $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$.

Let $\hat{A}_{\text{reg}}\big(\mathcal{P}^{(k-1)}_{\text{orig}},\mathcal{G}^{(k)},k-1,\boldsymbol{b}\big)$ denote the set of all $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1,\boldsymbol{b})$ for which $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)}) \subseteq \mathcal{P}^{(k-1)}_{\text{orig}}$ and $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k,\tilde{r})$-regular with respect to $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$. We set

$$\tilde{\mathcal{G}}^{(k)} = \bigcup\Big\{\mathcal{G}^{(k)}\cap\mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\colon\ \hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}_{\text{reg}}\big(\mathcal{P}^{(k-1)}_{\text{orig}},\mathcal{G}^{(k)},k-1,\boldsymbol{b}\big)\Big\}\,. \tag{73}$$

It is left to verify that the earlier constructed family of partitions $\mathscr{P} = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ and $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$ satisfy the conclusion of Lemma 29.

Recall that for $2\leq j\leq k-1$, we constructed $\mathscr{P}^{(j)}$ and $\tilde{\mathcal{G}}^{(j)}$ so that $(\mathcal{C}_j)$ and (69) holds. Consequently, by Fact 42 assertions *(1)*–*(5)* hold for every $j=2,\dots,k-1$. The verification of Lemma 29 will rely on these assertions.

We first show that

$$\tilde{\boldsymbol{\mathcal{G}}} \text{ is an } (n,\ell,k)\text{-complex}\,. \tag{74}$$

By (*2*) of Fact 42 for $j = k-1$ we see that $\{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k-1}$ is an $(n, \ell, k-1)$-complex. Now, let $K \in \tilde{\mathcal{G}}^{(k)}$. We have to show that $K \in \mathcal{K}_k(\tilde{\mathcal{G}}^{(k-1)})$. From (73), we infer that $K \in \mathcal{K}_k(\mathcal{G}^{(k-1)} \cap \mathcal{P}^{(k-1)}_{\mathrm{orig}})$ and, consequently, by (II) of $(\mathcal{C}_{k-1})$, we have $K \in \mathcal{K}_k(\tilde{\mathcal{G}}^{(k-1)})$. Therefore, $\tilde{\mathcal{G}}^{(k-1)}$ underlies $\tilde{\mathcal{G}}^{(k)}$ and (74) follows.

Now we show that

$$\tilde{\boldsymbol{d}} \text{ is componentwise bigger than } \tilde{\boldsymbol{c}}. \tag{75}$$

Suppose $\tilde{d}_j \leq \tilde{c}_j$ for some $2 \leq j \leq k-1$. Recall, that $\tilde{\boldsymbol{d}}$ was given by Corollary 27 as the density vector of $\mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$. Moreover, $L_k \geq |A(k-1, \boldsymbol{a})|$ and hence $|\hat{A}(j-1, \boldsymbol{a})| < 2^\ell L_k^k$ for $j = 2, \ldots, k$. Therefore, the assumption $\tilde{d}_j \leq \tilde{c}_j = 1/(2^{\ell+2} L_k^k)$ (see (57)) implies that the number of j-tuples in $(\delta'_j, \tilde{d}_j, r')$-regular polyads of $\mathscr{R}$ is at most $2^\ell L_k^k(\tilde{d}_j + \delta'_j) n^j \leq 2^{\ell+1} L_k^k \tilde{c}_j n^j = n^j/2$. On the other hand, by (62), all but at most $\mu\binom{k}{j} n^j$ crossing j-tuples belong to $(\delta'_j, \tilde{d}_j, r')$-regular polyads of $\mathscr{R}$. Since $(1/2 + \mu\binom{k}{j}) n^j \leq \binom{\ell}{j} n^j$ the assumption $\tilde{d}_j \leq \tilde{c}_j$ must be wrong and we infer that $\tilde{d}_j > \tilde{c}_j$ for every $2 \leq j \leq k-1$, as claimed in (75).

Using (III) of $(\mathcal{C}_{k-1})$ combined with (55) and (56) yields that

$$\mathscr{P} = \mathscr{P}_{k-1} \text{ is a perfect } (\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})\text{-family of partitions}. \tag{76}$$

Moreover, (*3*) of Fact 42 for $j = k-1$ states that

$$\mathscr{P} = \mathscr{P}_{k-1} \text{ refines } \tilde{\boldsymbol{\mathcal{G}}}. \tag{77}$$

From (74)–(77) we infer that it is left to show (*i*)–(*vi*) of Lemma 29, only. We observe that (*i*) is immediate from the construction of $\tilde{\mathcal{G}}^{(k)}$ in (73). Also, due to (73), (II) of $(\mathcal{C}_j)$ for $j = 2, \ldots, k-1$ (see also (*1*) of Fact 42), and the definition of $\tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$ we have (*ii*) of Lemma 29.

Now we verify (*iii*) of Lemma 29. For $3 \leq j < k$ it is given by part (*4*) of Fact 42. For $j = k$, we recall the definition of $\tilde{\mathcal{G}}^{(k)}$ in (73) and consider $\mathcal{G}^{(k)} \setminus \tilde{\mathcal{G}}^{(k)}$. There are two reasons for a k-tuple $K \in \mathcal{G}^{(k)}$ to be in $\mathcal{G}^{(k)} \setminus \tilde{\mathcal{G}}^{(k)}$. Either $K \notin \mathcal{K}_k(\mathcal{P}^{(k-1)}_{\mathrm{orig}})$ or K belongs to a polyad $\hat{\mathcal{P}}^{(k-1)}$ such that $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-irregular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$.

Consider a k-tuple of the first type, i.e., $K \notin \mathcal{K}_k(\mathcal{P}^{(k-1)}_{\mathrm{orig}})$. Owing to (I) of $(\mathcal{C}_{k-1})$ we see that K belongs to a $\big((\delta'_2, \ldots, \delta'_{k-1}), (\tilde{d}_2, \ldots, \tilde{d}_{k-1}), r'\big)$-irregular $(n/a_1, k, k-1)$-complex of the original family of partitions $\mathscr{R}$. Consequently, by (62) (with $j = k$) there are at most μn^k k-tuples K of the first type ($K \notin \mathcal{K}_k(\mathcal{P}^{(k-1)}_{\mathrm{orig}})$).

Now consider a k-tuple K, which is not of the first type, but of the second type. In particular, $K \in \mathcal{K}_k(\mathcal{P}^{(k-1)}_{\mathrm{orig}})$ and $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-irregular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$, the underlying polyad of K in the family of partitions $\mathscr{P}$. From (I) of $(\mathcal{C}_{k-1})$ we infer that $\hat{\mathcal{P}}^{(k-1)}$ corresponds to k different $(n, k-1, k-1)$-cylinders, which are all elements of $\mathscr{R}^{(k-1)}$. Since $\mathscr{R}$ is a (δ'_k, r')-regular partition w.r.t. $\mathcal{G}^{(k)}$ and $\tilde{\delta}_k \geq \delta'_k$ and $\tilde{r} \leq r'$ (cf. (54) and (59)), there are at most $\delta'_k n^k$ k-tuples $K \in \mathcal{G}^{(k)} \cap \mathcal{K}_k(\mathcal{P}^{(k-1)}_{\mathrm{orig}})$ so that $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-irregular w.r.t. to the underlying polyad $\hat{\mathcal{P}}^{(k-1)}$ of K.

Summarizing the above, we infer that

$$|\mathcal{G}^{(k)} \triangle \tilde{\mathcal{G}}^{(k)}| = |\mathcal{G}^{(k)} \setminus \tilde{\mathcal{G}}^{(k)}| \leq (\mu + \delta'_k) n^k \overset{(54)}{=} 2\delta'_k n^k.$$

Consequently, by the choice of δ'_k in (54), the following holds for every $k \leq i \leq \ell$,

$$\Big|\mathcal{K}_i(\tilde{\mathcal{G}}^{(k)}) \triangle \mathcal{K}_i(\mathcal{G}^{(k)})\Big| \leq 2\delta'_k n^k \times \binom{\ell - k}{i - k} n^{i-k} \overset{(54)}{\leq} \tilde{\delta}_k \prod_{h=2}^{k} d_h^{\binom{i}{h}} \times n^i,$$

which completes the verification of (*iii*) of Lemma 29.

We further note that (*iv*) of Lemma 29 is an immediate consequence of $b_1 = a_1 \leq \operatorname{rank} \mathscr{R} \leq L_k \leq \tilde{L}_k$ (cf. (57)), $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$ and the assumption of Lemma 29 that $\boldsymbol{\mathcal{G}}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, 1)$-regular complex.

Finally, we show (*v*) of Lemma 29 as follows:

$$\operatorname{rank} \mathscr{P} = |A(k-1, \boldsymbol{b})| = \binom{\ell}{k-1} b_1^{k-1} \prod_{j=2}^{k-1} b_j^{\binom{k-1}{j}} \overset{(61)}{\leq} \binom{\ell}{k-1} L_k^{k-1} \prod_{j=2}^{k-1} \left(\frac{1}{\tilde{d}_j}\right)^{\binom{k-1}{j}}.$$

Then (v) follows from $\tilde{\boldsymbol{c}} \le \tilde{\boldsymbol{d}}$ and the choice of $\tilde{L}_k$ in (57). Finally, (vi) follows from (58). This completes the proof of Lemma 29.

6. Proofs concerning cleaning phase II

We prove Lemma 37 and Lemma 39 in this section. We work in the context of Setup 36, the environment after Cleaning Phase I (after an application of Lemma 29) with the constants summarized in Figure 2. The main objective of this section is to construct the complexes $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ stated in Lemma 37 and Lemma 39. We prove these lemmas in Section 6.2 and Section 6.3, respectively. The following section, Section 6.1, contains some preliminary facts, which are immediate consequences of the choice of constants given in Section 4.4.1 (see Figure 2).

6.1. **Preliminary facts.** We start with the following facts which we apply liberally in the remainder of this section. The first two facts are immediate consequences of $\mathbf{IHC}_{k-1,\ell}$ and the choice of constants in Section 4.4.1 (applied to differing setups).

Fact 43. *For all integers $2 \le j < k$ and $j < i \le \ell$ and every $\Lambda_i \in \binom{[\ell]}{i}$,*

$$\left|\mathcal{K}_i\big(\mathcal{G}^{(j)}[\Lambda_i]\big)\right| = (1 \pm \eta) \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i\,, \tag{78}$$

$$\left|\mathcal{K}_i\big(\tilde{\mathcal{G}}^{(j)}[\Lambda_i]\big)\right| = \big(1 \pm (\eta + \tilde{\delta}_k)\big) \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i\,. \tag{79}$$

Consequently, by the choice of η in (30) and $\tilde{\delta}_k \le 1/8$ in (33),

$$\left|\mathcal{K}_i\big(\tilde{\mathcal{G}}^{(j)}[\Lambda_i]\big)\right| \ge \frac{1 - 1/4 - \tilde{\delta}_k}{1 + 1/4}\left|\mathcal{K}_i\big(\mathcal{G}^{(j)}[\Lambda_i]\big)\right| \ge \frac{1}{2}\left|\mathcal{K}_i\big(\mathcal{G}^{(j)}[\Lambda_i]\big)\right|. \tag{80}$$

Proof. Due to the choice of $\boldsymbol{\delta} = (\delta_2, \dots, \delta_{k-1})$ and r (cf. (31), (37), and (38)) for $2 \le j < k$, the complex $\boldsymbol{\mathcal{G}}^{(j)} = \{\mathcal{G}^{(h)}\}_{h=1}^{j}$ satisfies the assumption of $\mathbf{IHC}_{k-1,\ell}$. As such, we conclude that (78) holds. Since $\tilde{\boldsymbol{\mathcal{G}}}$ is given by Lemma 29, it satisfies (iii) of that lemma and (79) follows. □

In the following fact, $\check{\boldsymbol{\mathcal{H}}}^{(j-1)}$ represents an arbitrary regular $(n/b_1, i, j-1)$-complex arising from an application of Lemma 29 (i.e., the complex $\check{\boldsymbol{\mathcal{H}}}^{(j-1)}$ is "built from blocks" of the partition $\mathscr{P}$).

Fact 44. *If $1 \le j-1 < k$ and $j \le i \le \ell$ and $\check{\boldsymbol{\mathcal{H}}}^{(j-1)} = \{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j-1}$ is a $\big((\tilde{\delta}_2, \dots, \tilde{\delta}_{j-1}), (\tilde{d}_2, \dots, \tilde{d}_{j-1}), \tilde{r}\big)$-regular $(n/b_1, i, j-1)$-complex, then*

$$\left|\mathcal{K}_i\big(\check{\mathcal{H}}^{(j-1)}\big)\right| = (1 \pm \tilde{\eta}) \prod_{h=2}^{j-1} \tilde{d}_h^{\binom{i}{h}} \times \left(\frac{n}{b_1}\right)^i. \tag{81}$$

In particular, due to Lemma 29, for every $1 \le j-1 < k$ and every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{b})$, we have

$$\left|\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)\right| = (1 \pm \tilde{\eta}) \prod_{h=2}^{j-1} \tilde{d}_h^{\binom{j}{h}} \times \left(\frac{n}{b_1}\right)^j. \tag{82}$$

Proof. Similarly as in the proof of Fact 43, by the choice of $\tilde{\eta}$ and $\tilde{\boldsymbol{\delta}} = (\tilde{\delta}_2, \dots, \tilde{\delta}_{k-1})$ and $\tilde{r}$ (cf. (33) and (35)), we infer that $\check{\boldsymbol{\mathcal{H}}}^{(j-1)}$ for $2 \le j-1 \le k-1$ satisfies the assumptions of $\mathbf{IHC}_{k-1,\ell}$ and, consequently, (81) of Fact 44 holds. □

Recall that $\boldsymbol{\mathcal{G}}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular complex where by (ii) and (iii) of Lemma 29 (with $i = j$)

$$\mathcal{G}^{(1)} = \tilde{\mathcal{G}}^{(1)},\ \mathcal{G}^{(2)} = \tilde{\mathcal{G}}^{(2)} \text{ and } \left|\mathcal{G}^{(j)} \setminus \tilde{\mathcal{G}}^{(j)}\right| \le \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{j}{h}} \times n^j \text{ for } 3 \le j \le k\,. \tag{83}$$

Since $\tilde{\delta}_k$ is significantly smaller than δ_j, $3 \le j \le k$ (cf. Figure 2), we infer the following fact by a standard argument.

Fact 45. *The (n,ℓ,k)-complex $\tilde{\boldsymbol{\mathcal{G}}}$ is $(2\boldsymbol{\delta},\boldsymbol{d},r)$-regular.*

Proof. From $\tilde{\mathcal{G}}^{(2)}=\mathcal{G}^{(2)}$ (see Lemma 29 (*ii*)) we infer that $\tilde{\mathcal{G}}^{(2)}$ is $(\delta_2,d_2,1)$-regular w.r.t. $\tilde{\mathcal{G}}^{(1)}=\mathcal{G}^{(1)}$.

We now show that $\tilde{\mathcal{G}}^{(j)}$ is $(2\delta_j,d_j,r)$-regular w.r.t. $\tilde{\mathcal{G}}^{(j-1)}$ for each $j\geq 3$. Let j and $\Lambda_j\in\binom{[\ell]}{j}$ be fixed. Let $\boldsymbol{\mathcal{Q}}^{(j-1)}=\left\{\mathcal{Q}_s^{(j-1)}\right\}_{s\in[r]}$ be a family of subhypergraphs of $\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]\subseteq\mathcal{G}^{(j-1)}[\Lambda_j]$ such that

$$\left|\bigcup_{s\in[r]}\mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right|\geq 2\delta_j\left|\mathcal{K}_j\left(\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]\right)\right|.$$

From (78) and (80), we then infer

$$\left|\bigcup_{s\in[r]}\mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right|>\delta_j\left|\mathcal{K}_j\left(\mathcal{G}^{(j-1)}[\Lambda_j]\right)\right|\geq\delta_j(1-\eta)\prod_{h=2}^{j-1}d_h^{\binom{j}{h}}\times n^j. \tag{84}$$

Since $\boldsymbol{\mathcal{Q}}^{(j-1)}$ is a family of subhypergraphs of $\mathcal{G}^{(j-1)}[\Lambda_j]$ and since $\mathcal{G}^{(j)}[\Lambda_j]$ is (δ_j,d_j,r)-regular with respect to $\mathcal{G}^{(j-1)}[\Lambda_j]$, we see

$$\left|\mathcal{G}^{(j)}[\Lambda_j]\cap\bigcup_{s\in[r]}\mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right|=(d_j\pm\delta_j)\left|\bigcup_{s\in[r]}\mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right|. \tag{85}$$

On the other hand, (83) and (85) imply

$$\begin{aligned}\left|\tilde{\mathcal{G}}^{(j)}[\Lambda_j]\cap\bigcup_{s\in[r]}\mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right|&\overset{(83)}{=}\left|\mathcal{G}^{(j)}[\Lambda_j]\cap\bigcup_{s\in[r]}\mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right|\pm\tilde{\delta}_k\prod_{h=2}^{j}d_h^{\binom{j}{h}}\times n^j\\&\overset{(85)}{=}(d_j\pm\delta_j)\left|\bigcup_{s\in[r]}\mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right|\pm\tilde{\delta}_k\prod_{h=2}^{j}d_h^{\binom{j}{h}}\times n^j\\&=(d_j\pm 2\delta_j)\left|\bigcup_{s\in[r]}\mathcal{K}_j\left(\mathcal{Q}_s^{(j-1)}\right)\right|,\end{aligned}$$

where the last equality uses (84) and $\tilde{\delta}_k d_j\leq\delta_j^2(1-\eta)$ for $j\geq 3$. □

6.2. **Proof of Lemma 37.** The proof of Lemma 37 will take place in stages. Setting $\mathcal{H}^{(1)}=\tilde{\mathcal{G}}^{(1)}$ and $\mathcal{H}^{(2)}=\tilde{\mathcal{G}}^{(2)}$ satisfies part (a) of Lemma 37. We prove part (b) of Lemma 37 in Section 6.2.1 and part (c) in Section 6.2.3.

6.2.1. *Proof of property (b) of Lemma 37.* We prove part (b) by induction on j.

Induction Start. Recall that we set $\mathcal{H}^{(2)}=\tilde{\mathcal{G}}^{(2)}$. Consequently, the symmetric difference considered in part (*b2*) of Lemma 37 is empty. Hence, (*b2*) holds trivially for $j=2$ and it is left to verify (*b1*). To that end, let $\hat{\boldsymbol{x}}^{(1)}=((\lambda_1,\lambda_2),(\beta_1,\beta_2))\in\hat{A}(\mathcal{H}^{(1)},1,\boldsymbol{b})=\hat{A}(1,\boldsymbol{b})$ be fixed. From part (*iv*) of Lemma 29, we infer $d\big(\tilde{\mathcal{G}}^{(2)}\big|\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}}^{(1)})\big)=d_2\pm\tilde{L}_k^2\delta_2$. From (*ii*) of Definition 28, we then infer

$$\frac{d_2-\tilde{L}_k^2\delta_2}{\tilde{d}_2+\tilde{\delta}_2}\leq\left|I(\hat{\boldsymbol{x}}^{(1)})\right|\leq\frac{d_2+\tilde{L}_k^2\delta_2}{\tilde{d}_2-\tilde{\delta}_2}.$$

Therefore, to verify (*b1*), we may show that the left-hand side of the last inequality is bigger than $d_2b_2-1/2$ and the right-hand side is less than $d_2b_2+1/2$. Consequently, it suffices to verify

$$(\tilde{d}_2+\tilde{\delta}_2)(d_2b_2-1/2)<d_2-\tilde{L}_k^2\delta_2\qquad\text{and}\qquad d_2+\tilde{L}_k^2\delta_2<(d_2b_2+1/2)(\tilde{d}_2-\tilde{\delta}_2). \tag{86}$$

The proofs of both inequalities are similar and we only present the details for the first one here.

We consider the left-hand side of the first inequality in (86) and see

$$(\tilde{d}_2+\tilde{\delta}_2)(d_2b_2-1/2)<\tilde{d}_2d_2b_2-\tilde{d}_2/2+\tilde{\delta}_2d_2b_2\overset{(43)}{=}d_2-\tilde{d}_2/2+\tilde{\delta}_2d_2/\tilde{d}_2\leq d_2-\tilde{d}_2/2+\tilde{\delta}_2/\tilde{d}_2. \tag{87}$$

Therefore, the first inequality of (86), follows from the choice of constants $\tilde{d}_2 \gg \tilde{\delta}_2 \gg \tilde{L}_k^2 \delta_2$ (see Figure 2), by

$$(\tilde{d}_2 + \tilde{\delta}_2)(d_2 b_2 - 1/2) \overset{(87)}{<} d_2 - \tilde{d}_2/2 + \sqrt{\tilde{\delta}_2} < d_2 - \tilde{L}_k^2 \delta_2 \,.$$

Induction Step. Assume that for $2 \le j < k$, part (b) of Lemma 37 holds for $j-1$ with inductively defined complex $\boldsymbol{\mathcal{H}}^{(j-1)} = \{\mathcal{H}^{(h)}\}_{h=1}^{j-1}$. (Here, one may want to recall the notation $\hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ defined in (44).) We define the promised sets $I(\hat{\boldsymbol{x}}^{(j-1)})$, $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right)$, and the promised hypergraph $\mathcal{H}^{(j)}$. We then verify that property (b) of Lemma 37 correspondingly holds.

To define the sets $I(\hat{\boldsymbol{x}}^{(j-1)})$, $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right)$, and the hypergraph $\mathcal{H}^{(j)}$, we first set, for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right)$,

$$J\big(\hat{\boldsymbol{x}}^{(j-1)}\big) = \Big\{\beta \in [b_j] \colon \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \beta)\big) \subseteq \tilde{\mathcal{G}}^{(j)}\Big\} \,. \tag{88}$$

We then construct the sets $I(\hat{\boldsymbol{x}}^{(j-1)})$ for every $\hat{\boldsymbol{x}}^{(j-1)}$ in $\hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ as follows:

- *If* $\left|J(\hat{\boldsymbol{x}}^{(j-1)})\right| > d_j b_j$, *then* $I(\hat{\boldsymbol{x}}^{(j-1)})$ *is defined by removing* $\left|J(\hat{\boldsymbol{x}}^{(j-1)})\right| - d_j b_j$ *arbitrary indices from* $J(\hat{\boldsymbol{x}}^{(j-1)})$ *(recall from* (43) *that* $d_j b_j$ *is an integer,* $j = 2, \dots, k-1$*).*
- *If* $\left|J(\hat{\boldsymbol{x}}^{(j-1)})\right| < d_j b_j$, *then* $I(\hat{\boldsymbol{x}}^{(j-1)})$ *is defined by adding* $d_j b_j - \left|J(\hat{\boldsymbol{x}}^{(j-1)})\right|$ *arbitrary indices of* $[b_j] \setminus J(\hat{\boldsymbol{x}}^{(j-1)})$ *to* $J(\hat{\boldsymbol{x}}^{(j-1)})$.

We then define the promised hypergraph $\mathcal{H}^{(j)}$ as

$$\mathcal{H}^{(j)} = \bigcup \Big\{\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big) \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \wedge \alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})\Big\} \,. \tag{89}$$

It remains to prove property (b) of Lemma 37 (for j fixed by induction), which we recall consists of parts $(b1)$ and $(b2)$. Note, however, that the above definitions of $I(\hat{\boldsymbol{x}}^{(j-1)})$ for $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$, and $\mathcal{H}^{(j)}$ immediately imply property $(b1)$ of Lemma 37. Note, in particular, that the family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{b}, \boldsymbol{\psi})$ refines the (n, ℓ, j)-complex $\boldsymbol{\mathcal{H}}^{(j)} = \{\mathcal{H}^{(h)}\}_{h=1}^{j}$ now defined.

The main burden in establishing the Inductive Step, therefore, consists of proving property $(b2)$, which we recall asserts that for fixed i (where $j \le i \le \ell$),

$$\left|\mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)})\right| \le \delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right) n^i. \tag{90}$$

In the immediate sequel, we present an outline for how we prove (90). Subsequent to this outline, we fill in all remaining details.

Outline for proving (90). We begin our outline with notation. For a j-tuple $J_0 \in \mathcal{K}_j(\mathcal{H}^{(1)}) = \mathcal{K}_j(\tilde{\mathcal{G}}^{(1)})$, write $\hat{\boldsymbol{x}}_{J_0}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{b})$ as the (unique) polyad address for which

$$J_0 \in \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}_{J_0}^{(j-1)})). \tag{91}$$

The following two observations guide most of our outline. For any $I_0 \in \mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)})$, there exists $J_0 \in \binom{I_0}{j}$ with $J_0 \in \mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}$. If, for $I_0 \in \mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)})$, there exists $J_0 \in \binom{I_0}{j}$ for which $\hat{\boldsymbol{x}}_{J_0}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{b}) \setminus \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$, then $J_0 \in \mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}$ (although the converse does not hold). In fact, slightly more is true.

Fact 46. *If* $I_0 \in \mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)})$ *has* $J_0 \in \binom{I_0}{j}$ *with* $\hat{\boldsymbol{x}}_{J_0}^{(j-1)} \notin \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$, *then* $J_0 \in \tilde{\mathcal{G}}^{(j)} \setminus \mathcal{H}^{(j)} \subseteq \mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}$.

Proof. It follows from the definition of $\hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ (cf. (44)) that $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}_{J_0}^{(j-1)}) \not\subseteq \mathcal{H}^{(j-1)}$. As such, $J_0 \notin \mathcal{K}_j(\mathcal{H}^{(j-1)})$ follows from property $(b1)$ of Lemma 37 (the partition $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{b}, \boldsymbol{\psi})$ refines the (n, ℓ, j)-complex $\{\mathcal{H}^{(h)}\}_{h=1}^{j}$ (cf. Definition 26)). Thus, $J_0 \notin \mathcal{H}^{(j)}$ (since $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$), and therefore, $I_0 \notin \mathcal{K}_i(\mathcal{H}^{(j)})$. But now, $I_0 \in \mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)})$ must belong to $\mathcal{K}_i(\tilde{\mathcal{G}}^{(j)})$, and hence, J_0 to $\tilde{\mathcal{G}}^{(j)}$. ☐

We choose to organize our proof of (90) by subdividing the set of all $I_0 \in \mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)})$ according to whether there exists a $J_0 \in \binom{I_0}{j}$ for which $\hat{\boldsymbol{x}}_{J_0}^{(j-1)} \notin \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$. To that end, set

$$\mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} = \left\{ I_0 \in \mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)}) \colon \forall\, J_0 \in \binom{I_0}{j}, \hat{\boldsymbol{x}}_{J_0}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \right\} \tag{92}$$

and

$$\begin{aligned} \mathcal{I}_{\hat{A}(j-1,\boldsymbol{b}) \setminus \hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} &= \left(\mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)}) \right) \setminus \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \\ &= \Big\{ I_0 \in \mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)}) \colon \\ &\qquad \exists\, J_0 \in \binom{I_0}{j} \text{ with } \hat{\boldsymbol{x}}_{J_0}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{b}) \setminus \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \Big\}. \end{aligned}$$

Then,

$$\left| \mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)}) \right| = \left| \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \right| + \left| \mathcal{I}_{\hat{A}(j-1,\boldsymbol{b}) \setminus \hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \right|, \tag{93}$$

and so to prove (90), we will prove

$$\left| \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \right| \le \frac{2}{3} \delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}} \right) n^i \tag{94}$$

and

$$\left| \mathcal{I}_{\hat{A}(j-1,\boldsymbol{b}) \setminus \hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \right| \le \frac{1}{3} \delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}} \right) n^i. \tag{95}$$

Since proving (95) is fairly immediate, we proceed to do so now. We then continue our outline by discussing how we prove the inequality (94).

Proof of (95). We first observe that

$$\mathcal{I}_{\hat{A}(j-1,\boldsymbol{b}) \setminus \hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \subseteq \mathcal{K}_i(\tilde{\mathcal{G}}^{(j-1)}) \setminus \mathcal{K}_i(\mathcal{H}^{(j-1)}) \subseteq \mathcal{K}_i(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j-1)}). \tag{96}$$

Indeed, any $I_0 \in \mathcal{I}_{\hat{A}(j-1,\boldsymbol{b}) \setminus \hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}$ contains $J_0 \in \binom{I_0}{j}$ for which $J_0 \notin \mathcal{K}_j(\mathcal{H}^{(j-1)})$. As such, $I_0 \notin \mathcal{K}_i(\mathcal{H}^{(j-1)})$. On the other hand,

$$I_0 \in \mathcal{I}_{\hat{A}(j-1,\boldsymbol{b}) \setminus \hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \subseteq \mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)}) \subseteq \mathcal{K}_i(\mathcal{H}^{(j)}) \cup \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)}) \subseteq \mathcal{K}_i(\mathcal{H}^{(j-1)}) \cup \mathcal{K}_i(\tilde{\mathcal{G}}^{(j-1)}),$$

so that $I_0 \in \mathcal{K}_i(\tilde{\mathcal{G}}^{(j-1)}) \setminus \mathcal{K}_i(\mathcal{H}^{(j-1)})$.

Now, to prove (95), we use (96) and the induction assumption on (*b2*), with j replaced by $j-1$, to conclude

$$\left| \mathcal{K}_i\big(\mathcal{H}^{(j-1)}\big) \triangle \mathcal{K}_i\big(\tilde{\mathcal{G}}^{(j-1)}\big) \right| \le \delta_{j-1}^{1/3} \left(\prod_{h=2}^{j-1} d_h^{\binom{i}{h}} \right) n^i \le \frac{1}{3} \delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}} \right) n^i$$

(where the last inequality follows from $\delta_{j-1} \ll \delta_j, d_j$ as guaranteed in Figure 2). □

We now focus on proving (94), and to that end, need a few preparatory considerations. Fix $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right)$. By our definition of the set $I(\hat{\boldsymbol{x}}^{(j-1)})$ and our definition of $\mathcal{H}^{(j)}$ in (89), we have

$$\mathcal{H}^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = \bigcup_{\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big).$$

Similarly, from our definition (88) of the set $J(\hat{\boldsymbol{x}}^{(j-1)})$, observe that

$$\tilde{\mathcal{G}}^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = \bigcup_{\alpha \in J(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big).$$

As such, it follows that

$$\big(\mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}\big) \cap \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = \bigcup_{\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)}) \triangle J(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big). \tag{97}$$

The identity in (97) prompts the following observation.

Fact 47. *For each $I_0 \in \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}$, there exists $J_0 \in \binom{I_0}{j}$ and $\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)}_{J_0}) \triangle J(\hat{\boldsymbol{x}}^{(j-1)}_{J_0})$ for which $J_0 \in \mathcal{P}^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}_{J_0}, \alpha)$.*

Proof. Fix $I_0 \in \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \subseteq \mathcal{K}_i(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i(\tilde{\mathcal{G}}^{(j)})$ and let $J_0 \in \binom{I_0}{j}$ be any j-tuple for which $J_0 \in \mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}$. Then $J_0 \in \big(\mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}\big) \cap \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}_{J_0})\big)$. Since $I_0 \in \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}$, the vector $\hat{\boldsymbol{x}}^{(j-1)}_{J_0}$ necessarily belongs to $\hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$. As such, by (97), there exists an $\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)}) \triangle J(\hat{\boldsymbol{x}}^{(j-1)})$ with $J_0 \in \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$. □

To prove (94), we subdivide $\mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}$. For a fixed I_0, J_0 and α as in Fact 47, we shall want to know whether the index set $I(\hat{\boldsymbol{x}}^{(j-1)}_{J_0}) \triangle J(\hat{\boldsymbol{x}}^{(j-1)}_{J_0})$ hosting α is a 'large' set or a 'small' set. To that end, we define

$$\hat{B}^{(j-1)} = \left\{ \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \colon \left| I(\hat{\boldsymbol{x}}^{(j-1)}) \triangle J(\hat{\boldsymbol{x}}^{(j-1)}) \right| > \sqrt{\delta_j} d_j b_j \right\}. \tag{98}$$

We then set

$$\mathcal{I}^{\mathrm{big}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} = \left\{ I_0 \in \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \colon \exists\ J_0 \in \binom{I_0}{j} \text{ so that } \ \hat{\boldsymbol{x}}^{(j-1)}_{J_0} \in \hat{B}^{(j-1)} \right\} \tag{99}$$

and

$$\begin{aligned} \mathcal{I}^{\mathrm{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} &= \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \setminus \mathcal{I}^{\mathrm{big}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \\ &= \left\{ I_0 \in \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \colon \forall\ J_0 \in \binom{I_0}{j}\ ,\ \hat{\boldsymbol{x}}^{(j-1)}_{J_0} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)} \right\}. \end{aligned}$$

The subdivision above then gives

$$\left| \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \right| = \left| \mathcal{I}^{\mathrm{big}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \right| + \left| \mathcal{I}^{\mathrm{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \right|. \tag{100}$$

To prove (94), we shall prove each of

$$\left| \mathcal{I}^{\mathrm{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \right| \le \frac{1}{3} \delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}} \right) n^i \tag{101}$$

and

$$\left| \mathcal{I}^{\mathrm{big}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \right| \le \frac{1}{3} \delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}} \right) n^i. \tag{102}$$

The inequality (94) now follows from (100) - (102). The proof of (102) relies on the fact that 'not many' vectors $\hat{\boldsymbol{x}}^{(j-1)}$ belong to $\hat{B}^{(j-1)}$. More precisely, we will show the following.

Claim 48. $\left| \hat{B}^{(j-1)} \right| < 2\sqrt{\delta_j} \prod_{h=2}^{j-1} (d_h b_h)^{\binom{j}{h}} \times b_1^j$.

We defer the proof of Claim 48 until Section 6.2.2. For proving (90), we now only have to verify (101) and (102) and Claim 48.

Proof of (101). We first define a set $\check{\mathcal{I}}^{\mathrm{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}$ and will then observe that it contains $\mathcal{I}^{\mathrm{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}$. Set

$$\begin{aligned} \check{\mathcal{I}}^{\mathrm{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} = \Bigg\{ I_0 \in \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \colon \exists\ J_0 \in \binom{I_0}{j} &\text{ and } \alpha \in I(\hat{\boldsymbol{x}}^{(j-1)}_{J_0}) \triangle J(\hat{\boldsymbol{x}}^{(j-1)}_{J_0}) \\ \text{so that } \ \hat{\boldsymbol{x}}^{(j-1)}_{J_0} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)} &\text{ and } J_0 \in \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}_{J_0}, \alpha)\big) \Bigg\}. \end{aligned}$$

Note that the set $\check{\mathcal{I}}^{\mathrm{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}$ doesn't insist that $\hat{\boldsymbol{x}}^{(j-1)}_{J_0} \notin \hat{B}^{(j-1)}$ for every $J_0 \in \binom{I_0}{j}$. The containment $\mathcal{I}^{\mathrm{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \subseteq \check{\mathcal{I}}^{\mathrm{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}$ is clear from Fact 47. To prove (101), we show $|\check{\mathcal{I}}^{\mathrm{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}| \le \frac{1}{3}\delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}} \right) n^i$.

To that end, let $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)}$ and $\alpha \in J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})$ be fixed and consider the $(n/b_1, j, j)$-complex implicitly given by $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$. By property *(b1)*, there are

$$\binom{\ell-j}{i-j} b_1^{i-j} \left(\prod_{h=2}^{j-1} (d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\right) b_j^{\binom{i}{j}-1} = d_j^{1-\binom{i}{j}} \binom{\ell-j}{i-j} b_1^{i-j} \prod_{h=2}^{j} (d_h b_h)^{\binom{i}{h}-\binom{j}{h}}$$

ways to complete any $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$ to a $\big((\tilde{\delta}_2, \ldots, \tilde{\delta}_j), (\tilde{d}_2, \ldots, \tilde{d}_j), \tilde{r}\big)$-regular $(n/b_1, i, j)$-complex $\check{\boldsymbol{\mathcal{H}}}^{(j)} = \{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j}$ in such a way that $\{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j-1}$ is a subcomplex of $\boldsymbol{\mathcal{H}}^{(j-1)}$. For each such $\check{\mathcal{H}}^{(j)}$, (81) of Fact 44 yields

$$\left|\mathcal{K}_i(\check{\mathcal{H}}^{(j)})\right| \le (1+\tilde{\eta}) \prod_{h=2}^{j} \tilde{d}_h^{\binom{i}{h}} \times \left(\frac{n}{b_1}\right)^i .$$

As such, summing over all choices $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)} \subseteq \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ and $\alpha \in J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})$ gives

$$\begin{aligned} \left|\check{\mathcal{I}}^{\text{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}\right| \le \left|\hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})\right| \;\times\; \max\left\{ \left|J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})\right| : \hat{\boldsymbol{x}}^{(j-1)} \notin \hat{B}^{(j-1)} \right\} \times \\ \times\; d_j^{1-\binom{i}{j}} \binom{\ell-j}{i-j} b_1^{i-j} \left(\prod_{h=2}^{j} (d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\right) \;\times\; (1+\tilde{\eta}) \left(\prod_{h=2}^{j} \tilde{d}_h^{\binom{i}{h}}\right) \left(\frac{n}{b_1}\right)^i . \end{aligned} \tag{103}$$

By property *(b1)*, we have

$$\left|\hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})\right| = \binom{\ell}{j} \left(\prod_{h=2}^{j-1} (d_h b_h)^{\binom{j}{h}}\right) b_1^j .$$

By the definition of $\hat{B}^{(j-1)}$ in (98), we have, for each $\hat{\boldsymbol{x}}^{(j-1)} \notin \hat{B}^{(j-1)}$, $\left|J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})\right| \le \sqrt{\delta_j} d_j b_j$. Consequently, the right-hand side of (103) is less than

$$d_j^{1-\binom{i}{j}} \binom{\ell}{j} \binom{\ell-j}{i-j} \sqrt{\delta_j} (1+\tilde{\eta}) \left(\prod_{h=2}^{j} (d_h b_h \tilde{d}_h)^{\binom{i}{h}}\right) n^i$$

Now, using (43) and the choice of $\tilde{\eta}$ and $\delta_j \ll d_j$ yields

$$\left|\check{\mathcal{I}}^{\text{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}\right| \le \frac{1}{3} \delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right) n^i$$

and (101) follows from $\check{\mathcal{I}}^{\text{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \supseteq \mathcal{I}^{\text{small}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}$. □

Proof of (102). In view of the definition of $\mathcal{I}^{\text{big}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}$ in (99), let $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{B}^{(j-1)}$ be fixed. By property *(b1)*, there are $\binom{\ell-j}{i-j}\big(\prod_{h=2}^{j-1}(d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\big) b_1^{i-j}$ different ways to complete $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ to a $\big((\tilde{\delta}_2, \ldots, \tilde{\delta}_{j-1}), (\tilde{d}_2, \ldots, \tilde{d}_{j-1}), \tilde{r}\big)$-regular $(n/b_1, i, j-1)$-subcomplex $\check{\boldsymbol{\mathcal{H}}}^{(j-1)} = \{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j-1}$ of $\boldsymbol{\mathcal{H}}^{(j-1)}$. For each such $\check{\boldsymbol{\mathcal{H}}}^{(j-1)}$, (81) of Fact 44 yields

$$\left|\mathcal{K}_i(\check{\mathcal{H}}^{(j-1)})\right| \le (1+\tilde{\eta}) \left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{i}{h}}\right) \left(\frac{n}{b_1}\right)^i \le 2 \left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{i}{h}}\right) \left(\frac{n}{b_1}\right)^i .$$

Using Claim 48 in the second inequality below, we therefore see

$$\begin{aligned} \left|\mathcal{I}^{\text{big}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}\right| &\le \left|\hat{B}^{(j-1)}\right| \times \binom{\ell-j}{i-j} \left(\prod_{h=2}^{j-1} (d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\right) b_1^{i-j} \times 2 \left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{i}{h}}\right) \left(\frac{n}{b_1}\right)^i \\ &\le 4 \binom{\ell-j}{i-j} \sqrt{\delta_j} \left(\prod_{h=2}^{j-1} (d_h b_h)^{\binom{i}{h}}\right) \left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{i}{h}}\right) n^i \le 4 \binom{\ell-j}{i-j} \sqrt{\delta_j} \left(\prod_{h=2}^{j-1} (d_h b_h \tilde{d}_h)^{\binom{i}{h}}\right) n^i . \end{aligned}$$

By (43) and the choice of $\delta_j \ll d_j$, we have the upper bound

$$\left|\mathcal{I}^{\mathrm{big}}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}\right| \le 4\binom{\ell-j}{i-j}\sqrt{\delta_j}\left(\prod_{h=2}^{j-1} d_h^{\binom{i}{h}}\right)n^i \le \frac{1}{3}\delta_j^{1/3}\left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right)n^i\,,$$

which is (102). □

6.2.2. *Proof of Claim 48.* The proof is rather straightforward. We first split the set $\hat{B}^{(j-1)}$ into two parts $\hat{B}^{(j-1)}_+$ and $\hat{B}^{(j-1)}_-$ as follows:

$$\begin{aligned}\hat{B}^{(j-1)}_+ &= \left\{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big)\colon\ \left|J\big(\hat{\boldsymbol{x}}^{(j-1)}\big)\right| > \left(1+\sqrt{\delta_j}\right)d_j b_j\right\}\,,\\ \hat{B}^{(j-1)}_- &= \left\{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big)\colon\ \left|J\big(\hat{\boldsymbol{x}}^{(j-1)}\big)\right| < \left(1-\sqrt{\delta_j}\right)d_j b_j\right\}\,.\end{aligned} \tag{104}$$

We prove the following claim which in view of Fact 45 is a slightly stronger statement than Claim 48.

Claim 48′. *If for some $* \in \{+,-\}$,*

$$\left|\hat{B}^{(j-1)}_*\right| \ge \sqrt{\delta_j}\left(\prod_{h=2}^{j-1}(d_h b_h)^{\binom{j}{h}}\right)b_1^j\,,$$

then $\tilde{\mathcal{G}}^{(j)}$ is not $(2\delta_j, d_j, r)$-regular w.r.t. $\tilde{\mathcal{G}}^{(j-1)}$.

Proof. We prove the case $* = -$ only with the other case very similar. We assume there exists an ordered set $\Lambda_j \in \binom{[\ell]}{j}_<$ such that

$$\left|\hat{B}^{(j-1)}_-[\Lambda_j]\right| \ge \frac{\sqrt{\delta_j}}{\binom{\ell}{j}}\left(\prod_{h=2}^{j-1}(d_h b_h)^{\binom{j}{h}}\right)b_1^j \tag{105}$$

where $\hat{B}^{(j-1)}_-[\Lambda_j]$ is the set of $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0,\dots,\hat{\boldsymbol{x}}_{j-1}) \in \hat{B}^{(j-1)}_-$ such that $\hat{\boldsymbol{x}}_0 = \Lambda_j$.

We show that (105) implies that $\tilde{\mathcal{G}}^{(j)}$ is irregular. Note that the polyad addresses $\hat{\boldsymbol{x}}^{(j-1)}$ in $\hat{B}^{(j-1)}_-[\Lambda_j]$ considered in (105) correspond to subhypergraphs of $\mathcal{H}^{(j-1)}$ and not necessarily to subhypergraphs of $\tilde{\mathcal{G}}^{(j-1)}$. The set $\hat{\Gamma}^{(j-1)}_-[\Lambda_j]$ which we define below is the subset of those polyad addresses of $\hat{B}^{(j-1)}_-[\Lambda_j]$ which correspond to subhypergraphs of $\tilde{\mathcal{G}}^{(j-1)}$ as well. Only those addresses are useful to verify Claim 48′. We therefore set

$$\hat{\Gamma}^{(j-1)}_-[\Lambda_j] = \hat{B}^{(j-1)}_-[\Lambda_j] \cap \hat{A}\big(\tilde{\mathcal{G}}^{(j-1)}, j-1, \boldsymbol{b}\big) \tag{106}$$

and

$$\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)} = \left\{\hat{\mathcal{P}}^{(j-1)}\big(\hat{\boldsymbol{x}}^{(j-1)}\big)\colon\ \hat{\boldsymbol{x}}^{(j-1)} \in \hat{\Gamma}^{(j-1)}_-[\Lambda_j]\right\} = \left\{\hat{\mathcal{Q}}^{(j-1)}_1,\dots,\hat{\mathcal{Q}}^{(j-1)}_t\right\}\,,$$

where $t = \left|\hat{\Gamma}^{(j-1)}_-[\Lambda_j]\right|$. In what follows, we show

$$\left|\bigcup_{s\in[t]}\mathcal{K}_j\big(\hat{\mathcal{Q}}^{(j-1)}_s\big)\right| > 2\delta_j\left|\mathcal{K}_j\left(\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]\right)\right| \tag{107}$$

and

$$d\left(\tilde{\mathcal{G}}^{(j)}\big|\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}\right) < d_j - 2\delta_j\,. \tag{108}$$

From (40), we see that $r \ge \left|\hat{A}(j-1,\boldsymbol{b})\right| \ge \left|\hat{\Gamma}^{(j-1)}_-[\Lambda_j]\right| = t$. Therefore, establishing (107) and (108) proves Claim 48′.

We first verify (107). Observe that due to the definition of $\hat{\Gamma}^{(j-1)}_-[\Lambda_j]$ in (106),

$$\begin{aligned}\hat{B}^{(j-1)}_-[\Lambda_j] \setminus \hat{\Gamma}^{(j-1)}_-[\Lambda_j] &\subseteq \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big)\Big\backslash \hat{A}\big(\tilde{\mathcal{G}}^{(j-1)}, j-1, \boldsymbol{b}\big)\\ &\subseteq \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big) \triangle \hat{A}\big(\tilde{\mathcal{G}}^{(j-1)}, j-1, \boldsymbol{b}\big)\end{aligned} \tag{109}$$

and since $\mathscr{P}^{(j-1)}$ respects $\mathcal{H}^{(j-1)}$ (cf. part (*b1*)) and $\mathscr{P}^{(j-1)}$ respects $\tilde{\mathcal{G}}^{(j-1)}$ (cf. Setup 36),

$$\bigcup\Big\{\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big):\ \hat{\boldsymbol{x}}^{(j-1)}\in\hat{A}\big(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b}\big)\triangle\hat{A}\big(\tilde{\mathcal{G}}^{(j-1)},j-1,\boldsymbol{b}\big)\Big\} \\ =\mathcal{K}_j\big(\mathcal{H}^{(j-1)}\big)\triangle\mathcal{K}_j\big(\tilde{\mathcal{G}}^{(j-1)}\big)\,. \tag{110}$$

Combining (109) and (110) with the induction hypothesis on (*b2*) for $j-1$ yields

$$\Big|\bigcup\Big\{\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big):\ \hat{\boldsymbol{x}}^{(j-1)}\in\hat{B}_-^{(j-1)}[\Lambda_j]\setminus\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\Big\}\Big| \\ \le\Big|\mathcal{K}_j\big(\mathcal{H}^{(j-1)}\big)\triangle\mathcal{K}_j\big(\tilde{\mathcal{G}}^{(j-1)}\big)\Big|<\delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1}d_h^{\binom{j}{h}}\right)n^j\,.$$

Consequently, we have

$$\Big|\bigcup_{s\in[t]}\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big|=\Big|\bigcup\Big\{\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big):\ \hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\Big\}\Big| \\ \ge\sum\Big\{\Big|\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)\Big|:\ \hat{\boldsymbol{x}}^{(j-1)}\in\hat{B}_-^{(j-1)}[\Lambda_j]\Big\}-\delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1}d_h^{\binom{j}{h}}\right)n^j\,.$$

Applying (82) of Fact 44 to each term in the sum above yields the further lower bound

$$\Big|\hat{B}_-^{(j-1)}[\Lambda_j]\Big|\,(1-\tilde{\eta})\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j-\delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1}d_h^{\binom{j}{h}}\right)n^j\,.$$

Finally, from our assumption (105) and (43), we infer

$$\Big|\bigcup_{s\in[t]}\Big\{\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big\}\Big|\ge\binom{\ell}{j}^{-1}(1-\tilde{\eta})\sqrt{\delta_j}\left(\prod_{h=2}^{j-1}\Big(d_hb_h\tilde{d}_h\Big)^{\binom{j}{h}}\right)n^j-\delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1}d_h^{\binom{j}{h}}\right)n^j \\ =\left(\binom{\ell}{j}^{-1}(1-\tilde{\eta})\sqrt{\delta_j}-\delta_{j-1}^{1/3}\right)\left(\prod_{h=2}^{j-1}d_h^{\binom{j}{h}}\right)n^j\ge\delta_j^{3/4}\left(\prod_{h=2}^{j-1}d_h^{\binom{j}{h}}\right)n^j \tag{111}$$

where the last inequality follows from the choice of $\tilde{\eta}$, and $\delta_j\gg\delta_{j-1}$. Now, (107) follows from (111) combined with (79) of Fact 43 for $j-1$ and $i=j$.

It is left to verify (108). First, observe that from the definition of $\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}$ that we have

$$\Big|\tilde{\mathcal{G}}^{(j)}\cap\bigcup_{s\in[t]}\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big|=\sum\Big\{\Big|\tilde{\mathcal{G}}^{(j)}\cap\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)\Big|:\ \hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\Big\} \\ =\sum\sum\Big\{\Big|\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)},\beta)\big)\Big|:\ \hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j],\ \beta\in J(\hat{\boldsymbol{x}}^{(j-1)})\Big\}. \tag{112}$$

Recall that by Definition 28, part (*ii*), every $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)},\beta)\big)$ is $(\tilde{\delta}_j,\tilde{d}_j,\tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$, $\hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j]$ and $\beta\in J(\hat{\boldsymbol{x}}^{(j-1)})$. Consequently, from (82) of Fact 44, we note

$$\Big|\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)},\beta)\big)\Big|\le\Big(\tilde{d}_j+\tilde{\delta}_j\Big)(1+\tilde{\eta})\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j$$

for every $\hat{\boldsymbol{x}}^{(j-1)}$ and β considered in (112). Consequently, we may bound (112), using that for every $\hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\subseteq\hat{B}_-^{(j-1)}$, $|J(\hat{\boldsymbol{x}}^{(j-1)})|<(1-\sqrt{\delta_j})d_jb_j$ (cf. (104)), as

$$\Big|\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\Big|\times\Big(1-\sqrt{\delta_j}\Big)d_jb_j\times\Big(\tilde{d}_j+\tilde{\delta}_j\Big)(1+\tilde{\eta})\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j\,. \tag{113}$$

On the other hand, we infer again from (82) of Fact 44 that

$$\left|\bigcup_{s\in[t]}\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right| = \sum\left\{\left|\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)\right|:\ \hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\right\}$$

$$\geq \left|\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\right| \times (1-\tilde{\eta})\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j . \tag{114}$$

Comparing (113) and (114) yields

$$d\left(\tilde{\mathcal{G}}^{(j)}\big|\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}\right) < d_j\frac{\big(1-\sqrt{\delta_j}\big)\left(b_j\tilde{d}_j+b_j\tilde{\delta}_j\right)(1+\tilde{\eta})}{1-\tilde{\eta}} .$$

From (43) and $\tilde{\eta}\ll\delta_j$ (observe $j>2$ here), we infer

$$d\left(\tilde{\mathcal{G}}^{(j)}\big|\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}\right) < d_j\left(1-\delta_j^{3/4}\right)\left(1+\tilde{\delta}_j/\tilde{d}_j\right) . \tag{115}$$

Finally, we observe that (108) follows from (115) and the choice of constants $\delta_j\gg\tilde{d}_j\gg\tilde{\delta}_j$. This completes the proof of Claim 48′. □

6.2.3. *Proof of property (c) of Lemma 37.* In this section, we define the promised hypergraph $\mathcal{H}^{(k)}$ and confirm the properties (*c1*) and (*c2*).

We begin by mentioning that the hypergraph $\tilde{\mathcal{G}}^{(k)}$ 'almost' satisfies the properties of the promised $\mathcal{H}^{(k)}$. In particular, due to Lemma 29 (i), the hypergraph $\tilde{\mathcal{G}}^{(k)}$ is $(\tilde{\delta}_k,\tilde{r})$-regular w.r.t. every polyad $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ for $\hat{\boldsymbol{x}}^{(k-1)}\in\hat{A}(k-1,\boldsymbol{b})$. However, the relative density $d\big(\tilde{\mathcal{G}}^{(k)}|\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)$ of $\tilde{\mathcal{G}}^{(k)}$ w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ may be 'wrong' (that is, differ substantially from d_k) for some $\hat{\boldsymbol{x}}^{(k-1)}\in\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})$. To construct $\mathcal{H}^{(k)}$, we intend to replace $\tilde{\mathcal{G}}^{(k)}$ on those 'aberrantly dense' polyads, but otherwise, take $\mathcal{H}^{(k)}$ to coincide with $\tilde{\mathcal{G}}^{(k)}$.

To make the plan above precise, we define

$$\hat{B}^{(k-1)} = \left\{\hat{\boldsymbol{x}}^{(k-1)}\in\hat{A}\big(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b}\big):\ \left|d\big(\tilde{\mathcal{G}}^{(k)}\big|\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\big)-d_k\right| > \sqrt{\delta_k}\right\}.$$

Using $\hat{B}^{(k-1)}$, we define $\mathcal{H}^{(k)}$ 'polyad-wise' as follows. For fixed $\hat{\boldsymbol{x}}^{(k-1)}\in\hat{A}\big(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b}\big)$, set

$$\mathcal{H}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big) = \begin{cases}\tilde{\mathcal{G}}^{(k)}\cap\mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\big) & \text{if } \hat{\boldsymbol{x}}^{(k-1)}\in\hat{A}\big(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b}\big)\setminus\hat{B}^{(k-1)},\\ \mathcal{S}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big) & \text{if } \hat{\boldsymbol{x}}^{(k-1)}\in\hat{B}^{(k-1)} .\end{cases}$$

where, for each $\hat{\boldsymbol{x}}^{(k-1)}\in\hat{B}^{(k-1)}$, the k-graph $\mathcal{S}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$ is obtained by applying the Slicing Lemma, Lemma 30, to $\mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)$ with parameters $m=n/b_1$, $p=d_k$, $\varrho=1$, $\delta=\tilde{\delta}_k/3$ and $r_{\mathrm{SL}}=\tilde{r}$. In particular, the Slicing Lemma guarantees

$$\mathcal{S}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\subseteq\mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\big) \text{ is } (\tilde{\delta}_k,d_k,\tilde{r})\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big). \tag{116}$$

We then set

$$\mathcal{H}^{(k)} = \bigcup\left\{\mathcal{H}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}):\ \hat{\boldsymbol{x}}^{(k-1)}\in\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})\right\}$$

to be the k-graph promised in Lemma 37. It remains to prove the corresponding properties (*c1*) and (*c2*).

With $\mathcal{H}^{(k)}$ defined above, we claim that property (*c1*) of Lemma 37 is immediately satisfied. Indeed, the Slicing Lemma (cf. (116)) guarantees that property (*c1*) is satisfied whenever $\hat{\boldsymbol{x}}^{(k-1)}\in\hat{B}^{(k-1)}$. To see that property (*c1*) is satisfied otherwise, fix $\hat{\boldsymbol{x}}^{(k-1)}\in\hat{A}\left(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b}\right)\setminus\hat{B}^{(k-1)}$ so that $\mathcal{H}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)=\mathcal{H}^{(k)}\cap\mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\big)=\tilde{\mathcal{G}}^{(k)}\cap\mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\big)$. By property (i) of Lemma 29, $\tilde{\mathcal{G}}^{(k)}$ is $(\tilde{\delta}_k,\tilde{r})$-regular with respect to $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$. Moreover, since $\hat{\boldsymbol{x}}^{(k-1)}\notin\hat{B}^{(k-1)}$, our construction of $\mathcal{H}^{(k)}$ gives

$$d\left(\mathcal{H}^{(k)}\Big|\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\right) = d\left(\tilde{\mathcal{G}}^{(k)}\Big|\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\right) = d_k\pm\sqrt{\delta_k} .$$

Thus, property (*c1*) is satisfied with $\mathcal{H}^{(k)}$ as defined above.

The remainder of this section is therefore devoted to the proof of property $(c2)$, the statement of which asserts

$$\left|\mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big)\triangle\,\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)\right| \le \delta_k^{1/3}\left(\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\right) n^\ell\,. \tag{117}$$

Our proof of (117) is quite similar [3] to our proof of (90) (that is, proving property $(c2)$ is quite similar to proving property $(b2)$). Recall from our proof of (90) that we defined, for $j \le i \le \ell$ and $j < k$, the partition (cf. (92))

$$\mathcal{K}_i\big(\mathcal{H}^{(j)}\big)\triangle\,\mathcal{K}_i\big(\tilde{\mathcal{G}}^{(j)}\big) = \mathcal{I}_{\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})} \cup \mathcal{I}_{\hat{A}(j-1,\boldsymbol{b})\setminus\hat{A}(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b})}.$$

To prove (117), we shall decompose $\mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big)\triangle\,\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)$ in the analogous way. Indeed, for a fixed $K_0 \in \mathcal{H}^{(k)}\triangle\,\tilde{\mathcal{G}}^{(k)}$, we write $\hat{\boldsymbol{x}}_{K_0}^{(k-1)} \in \hat{A}(k-1,\boldsymbol{b})$ as the polyad address for which $K_0 \in \mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\big)$ (the analog of $\hat{\boldsymbol{x}}_{J_0}^{(j-1)}$ from (91)). Set

$$\mathcal{L}_{\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})} = \left\{L_0 \in \mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big)\triangle\,\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)\colon\ \hat{\boldsymbol{x}}_{K_0}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b}) \text{ for all } K_0 \in \binom{L_0}{k}\right\}$$

and

$$\begin{aligned}\mathcal{L}_{\hat{A}(k-1,\boldsymbol{b})\setminus\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})} &= \Big(\mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big)\triangle\,\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)\Big)\setminus\mathcal{L}_{\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})}\\ &= \left\{L_0 \in \mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big)\triangle\,\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)\colon\ \exists\, K_0 \in \binom{L_0}{k} \text{ with } \hat{\boldsymbol{x}}_{K_0}^{(k-1)} \notin \hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})\right\}.\end{aligned}$$

Then,

$$\left|\mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big)\triangle\,\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)\right| = \left|\mathcal{L}_{\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})}\right| + \left|\mathcal{L}_{\hat{A}(k-1,\boldsymbol{b})\setminus\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})}\right|,$$

and so to prove (117), we shall prove each of

$$\left|\mathcal{L}_{\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})}\right| \le \frac{1}{2}\delta_k^{1/3}\left(\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\right) n^\ell \tag{118}$$

and

$$\left|\mathcal{L}_{\hat{A}(k-1,\boldsymbol{b})\setminus\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})}\right| \le \frac{1}{2}\delta_k^{1/3}\left(\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\right) n^\ell. \tag{119}$$

The proof of (119) follows almost immediately by induction. Indeed, one may argue, identically as we did for (96), that

$$\mathcal{L}_{\hat{A}(k-1,\boldsymbol{b})\setminus\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})} \subseteq \mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k-1)}\big)\setminus\mathcal{K}_\ell\big(\mathcal{H}^{(k-1)}\big) \subseteq \mathcal{K}_\ell\big(\mathcal{H}^{(k-1)}\big)\triangle\,\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k-1)}\big).$$

As such, property $(b2)$ of Lemma 37 (with $j = k-1$ and $i = \ell$) ensures

$$\left|\mathcal{L}_{\hat{A}(k-1,\boldsymbol{b})\setminus\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})}\right| \le \left|\mathcal{K}_\ell\big(\mathcal{H}^{(k-1)}\big)\triangle\,\mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k-1)}\big)\right| \le \delta_{k-1}^{1/3}\left(\prod_{h=2}^{k-1} d_h^{\binom{\ell}{h}}\right) n^\ell \le \frac{1}{2}\delta_k^{1/3}\left(\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\right) n^\ell$$

where the last inequality follows from $\delta_{k-1} \ll \delta_k, d_k$ (as given in Figure 2). Thus, to prove (117), it only remains to prove (118).

Similarly as (102) followed from Claim 48, we shall deduce (118) from the following claim.

Claim 49. $\left|\hat{B}^{(k-1)}\right| < 2\sqrt{\delta_k}\prod_{h=2}^{k-1}(d_h b_h)^{\binom{k}{h}} \times b_1^k\,.$

The proof of Claim 49 follows the lines of the proof of Claim 48, so we omit the details here. We mention, however, that in the proof of Claim 48 (concerning $\hat{B}^{(j-1)}$), we used the Counting Lemma for $(j-1)$-uniform hypergraphs (cf. (82) of Fact 44). In a proof of Claim 49, we would do the same with $j-1 = k-1$.

[3] In fact, it will be slightly easier to prove (117). Indeed, in the proof of (101) (part of the proof of (90)), we used the counting lemma for j-uniform hypergraphs. With $j = k$, however, we have to proceed differently, and will therefore allow the weaker statement of densities varying in $d_k \pm \sqrt{\delta_k}$. (We return to this issue in Lemma 39.)

Proof of (118). First we note that

$$\forall L_0 \in \mathcal{L}_{\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})}\,,\ \exists K_0 \in \binom{L_0}{k} \text{ such that } \hat{\boldsymbol{x}}^{(k-1)}_{K_0} \in \hat{B}^{(k-1)}\,. \tag{120}$$

Indeed, let $L_0 \in \mathcal{L}_{\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})}$. Since $L_0 \in \mathcal{K}_\ell\big(\mathcal{H}^{(k)}\big) \triangle\, \mathcal{K}_\ell\big(\tilde{\mathcal{G}}^{(k)}\big)$, there exists $K_0 \in \binom{L_0}{k}$ so that $K_0 \in \mathcal{H}^{(k)} \triangle\, \tilde{\mathcal{G}}^{(k)}$. By definition of $\mathcal{L}_{\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})}$, we are ensured $\hat{\boldsymbol{x}}^{(k-1)}_{K_0} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$. However, our construction of $\mathcal{H}^{(k)}$ guarantees that for any $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$ for which $\big(\mathcal{H}^{(k)} \triangle\, \tilde{\mathcal{G}}^{(k)}\big) \cap \mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\big) \neq \varnothing$, we have $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$. Thus, (120) holds.

Now, let $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$ be fixed. Property (*b1*) implies that there are $\prod_{h=2}^{k-1}(d_h b_h)^{\binom{\ell}{h}-\binom{k}{h}} \times b_1^{\ell-k}$ ways to complete $\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$ to a $\big((\tilde{\delta}_2,\dots,\tilde{\delta}_{k-1}),(\tilde{d}_2,\dots,\tilde{d}_{k-1}),\tilde{r}\big)$-regular $(n/b_1,\ell,k-1)$-subcomplex $\check{\boldsymbol{\mathcal{H}}}^{(k-1)} = \{\check{\mathcal{H}}^{(h)}\}_{h=1}^{k-1}$ of $\mathcal{H}^{(k-1)}$. Moreover, (81) of Fact 44 implies that $\check{\boldsymbol{\mathcal{H}}}^{(k-1)}$ satisfies

$$\Big|\mathcal{K}_\ell(\check{\mathcal{H}}^{(k-1)})\Big| \le (1+\tilde{\eta})\left(\prod_{h=2}^{k-1}\tilde{d}_h^{\binom{\ell}{h}}\right)\left(\frac{n}{b_1}\right)^\ell \le 2\left(\prod_{h=2}^{k-1}\tilde{d}_h^{\binom{\ell}{h}}\right)\left(\frac{n}{b_1}\right)^\ell.$$

Therefore, with Claim 49, we have

$$\begin{aligned}\Big|\mathcal{L}_{\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})}\Big| &\le \Big|\hat{B}^{(k-1)}\Big| \times \left(\prod_{h=2}^{k-1}(d_h b_h)^{\binom{\ell}{h}-\binom{k}{h}}\right) b_1^{\ell-k} \times 2\left(\prod_{h=2}^{k-1}\tilde{d}_h^{\binom{\ell}{h}}\right)\left(\frac{n}{b_1}\right)^\ell \\ &\le 4\sqrt{\delta_k}\left(\prod_{h=2}^{k-1}(d_h b_h \tilde{d}_h)^{\binom{\ell}{h}}\right) n^\ell \overset{(43)}{=} 4\sqrt{\delta_k}\left(\prod_{h=2}^{k-1} d_h^{\binom{\ell}{h}}\right) n^\ell\,.\end{aligned}$$

As such, with $\delta_k \ll d_k$ (as given in Figure 2), we have

$$\Big|\mathcal{L}_{\hat{A}(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b})}\Big| \le \frac{1}{2}\delta_k^{1/3}\left(\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\right) n^\ell\,,$$

proving (118). □

6.3. **Proof of Lemma 39.** Lemma 39 follows from a simple and straightforward application of the Slicing Lemma, Lemma 30. Recall Setup 36 and that $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(h)}\}_{h=1}^k$ is the (n,ℓ,k)-complex given by Lemma 37.

Proof of Lemma 39. Recall that by part (*c*) of Lemma 37, for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, the (n,ℓ,k)-cylinder

$$\mathcal{H}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{H}^{(k)} \cap \mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big) \text{ is } (\tilde{\delta}_k, \bar{d}(\hat{\boldsymbol{x}}^{(k-1)}), \tilde{r})\text{-regular} \tag{121}$$

w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ where $\bar{d}(\hat{\boldsymbol{x}}^{(k-1)}) = d_k \pm \sqrt{\delta_k}$.

Construction of $\boldsymbol{\mathcal{H}}_-$. For $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, apply the Slicing Lemma, Lemma 30, with $\varrho = \bar{d}(\hat{\boldsymbol{x}}^{(k-1)})$, $p = (d_k - \sqrt{\delta_k})/\varrho$, $\delta = \tilde{\delta}_k$ and $r_{\mathrm{SL}} = \tilde{r}$ to $\mathcal{H}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ to obtain a $(3\tilde{\delta}_k, d_k - \sqrt{\delta_k}, \tilde{r})$-regular hypergraph $\mathcal{S}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$.

Note that the assumptions of the Slicing Lemma are satisfied. Indeed, the family of partitions $\mathscr{P}$ is a perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family and, consequently, the polyad $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $\big((\tilde{\delta}_2,\dots,\tilde{\delta}_{k-1}),(\tilde{d}_2,\dots,\tilde{d}_{k-1}),\tilde{r}\big)$-regular. Hence, by (82) of Fact 44 (with $j=k$),

$$\Big|\mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\Big| > \frac{(n/b_1)^k}{\ln(n/b_1)}$$

for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$.

We then set

$$\mathcal{H}_-^{(k)} = \bigcup\left\{\mathcal{S}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \colon\ \hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})\right\}.$$

Obviously, $\mathcal{H}_-^{(k)}$ has the desired properties (α) and (*β1*) by construction.

Construction of $\boldsymbol{\mathcal{H}}_+$. The construction of $\mathcal{H}_+^{(k)}$ is similar and follows by an application of the Slicing Lemma to the complement of $\mathcal{H}^{(k)}$. More precisely, for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, set $\overline{\mathcal{H}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) =$

$\mathcal{K}_k(\hat{\mathcal{P}}(\hat{\boldsymbol{x}}^{(k-1)})) \setminus \mathcal{H}^{(k)}$. Note that, due to (121), $\overline{\mathcal{H}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(\tilde{\delta}_k, 1-\bar{d}(\hat{\boldsymbol{x}}^{(k-1)}), \tilde{r})$-regular. Consequently, we can apply the Slicing Lemma to $\overline{\mathcal{H}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ with $\varrho = 1-\bar{d}(\hat{\boldsymbol{x}}^{(k-1)})$, $p = (1-d_k-\sqrt{\delta_k})/\varrho$, $\delta = \tilde{\delta}_k$ and $r_{\rm SL} = \tilde{r}$ to obtain a $(3\tilde{\delta}_k, 1-d_k-\sqrt{\delta_k}, \tilde{r})$-regular hypergraph $\overline{\mathcal{S}}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. We then set $\mathcal{S}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{K}_k(\hat{\mathcal{P}}(\hat{\boldsymbol{x}}^{(k-1)})) \setminus \overline{\mathcal{S}}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Clearly, $\mathcal{S}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(3\tilde{\delta}_k, d_k+\sqrt{\delta_k}, \tilde{r})$-regular and it contains $cH^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Finally, we define $\mathcal{H}_+^{(k)}$ to be the union of all $\mathcal{S}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ constructed that way.

Construction of $\mathcal{F}$. The construction of $\mathcal{F}^{(k)}$ is only slightly more involved than before (owing to the requirement $\mathcal{H}_-^{(k)} \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{H}_+^{(k)}$).

Let $\mathcal{H}_-^{(k)}$ and $\mathcal{H}_+^{(k)}$ be given as constructed above and for $* \in \{+,-\}$ and $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, let $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{H}_*^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)}))$. Due to ($\beta 1$) and ($\beta 2$), $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(3\tilde{\delta}_k, d_k^*, \tilde{r})$-regular where $d_k^- = d_k - \sqrt{\delta_k}$ and $d_k^+ = d_k + \sqrt{\delta_k}$. Moreover, $\mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \supseteq \mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ and, consequently, $\mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \setminus \mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(6\tilde{\delta}_k, 2\sqrt{\delta_k}, \tilde{r})$-regular. We now apply the Slicing lemma to $\mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \setminus \mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ with $\varrho = 2\sqrt{\delta_k}$, $p = \sqrt{\delta_k}/\varrho = 1/2$, $\delta = 6\tilde{\delta}_k$ and $r_{\rm SL} = \tilde{r}$ to obtain a $(18\tilde{\delta}_k, \sqrt{\delta_k}, \tilde{r})$-regular hypergraph $\mathcal{S}_{\mathcal{F}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Now define $\mathcal{F}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ to be the disjoint union $\mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \cup \mathcal{S}_{\mathcal{F}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Clearly, $\mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \subseteq \mathcal{F}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \subseteq \mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Moreover, it is straightforward (see also Proposition 50 below) to verify that $\mathcal{F}^{(k)}$ is $(21\tilde{\delta}_k, d_k, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ and, consequently,

$$\mathcal{F}^{(k)} = \bigcup \left\{ \mathcal{F}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \colon \hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}) \right\},$$

has the desired properties. □

7. Proof of the union lemma

7.1. **Union of regular hypergraphs.** Below we present some useful facts regarding regularity properties of the union of regular (m,j,j)-cylinders. We distinguish two cases depending on whether the (m,j,j)-cylinders in question have the same underlying polyad or not.

The first proposition says that we may unite disjoint regular (m,j,j)-cylinders of the same density which share the same underlying $(m,j,j-1)$-cylinder without spoiling the regularity too much.

Proposition 50. *Let $j \geq 2$, t and m be positive integers, let δ and d be positive reals and let $\mathcal{P}_1^{(j)}, \dots, \mathcal{P}_t^{(j)}$ be a family of pairwise edge disjoint (m,j,j)-cylinders with the same underlying $(m,j,j-1)$-cylinder $\hat{\mathcal{P}}^{(j-1)}$. If for every $s \in [t]$, the hypergraph $\mathcal{P}_s^{(j)}$ is $(\delta, d, 1)$-regular with respect to $\hat{\mathcal{P}}^{(j-1)}$, then $\mathcal{P}^{(j)} = \bigcup_{s\in[t]} \mathcal{P}_s^{(j)}$ is $(t\delta, td, 1)$-regular with respect to $\hat{\mathcal{P}}^{(j-1)}$.*

Proposition 50 is a straightforward consequence from the definition, Definition 9, and we omit the proof. The next proposition gives us control when we unite hypergraphs having different underlying polyads. Before we make this precise, we define the setup for our proposition.

Setup 51. *Let $j \geq 3$, t and m be fixed positive integers and let δ and d be positive reals. Let $\{\hat{\mathcal{P}}_s^{(j-1)}\}_{s\in[t]}$ be a family of $(m,j,j-1)$-cylinders such that*

$$\mathcal{K}_j\Big(\bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}\Big) = \bigcup_{s\in[t]} \mathcal{K}_j(\hat{\mathcal{P}}_s^{(j-1)}) \qquad (122)$$
$$\text{and} \quad \mathcal{K}_j(\hat{\mathcal{P}}_s^{(j-1)}) \cap \mathcal{K}_j(\hat{\mathcal{P}}_{s'}^{(j-1)}) = \varnothing \quad \text{for } 1 \leq s < s' \leq t\,.$$

In other words, $\bigcup_{s\in[t]} \mathcal{K}_j(\hat{\mathcal{P}}_s^{(j-1)})$ is a partition of the j-cliques of $\bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}$. Let $\{\mathcal{P}_s^{(j)}\}_{s\in[t]}$ be a family of (m,j,j)-cylinders such that $\hat{\mathcal{P}}_s^{(j-1)}$ underlies $\mathcal{P}_s^{(j)}$ for any $s \in [t]$. Set $\hat{\mathcal{P}}^{(j-1)} = \bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}$ and $\mathcal{P}^{(j)} = \bigcup_{s\in[t]} \mathcal{P}_s^{(j)}$.

Proposition 52. *Let $\{\mathcal{P}_s^{(j)}\}_{s\in[t]}$ and $\{\hat{\mathcal{P}}_s^{(j-1)}\}_{s\in[t]}$ satisfy Setup 51. If $\mathcal{P}_s^{(j)}$ is $(\delta, d, 1)$-regular w.r.t. $\hat{\mathcal{P}}_s^{(j-1)}$ for every $s \in [t]$, then $\mathcal{P}^{(j)}$ is $(2\sqrt{\delta}, d, 1)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$.*

Proof. Let $\hat{\mathcal{Q}}^{(j-1)} \subseteq \hat{\mathcal{P}}^{(j-1)}$ be such that

$$\big|\mathcal{K}_j\big(\hat{\mathcal{Q}}^{(j-1)}\big)\big| \geq 2\sqrt{\delta}\big|\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}\big)\big| . \tag{123}$$

For every $s \in [t]$, set $\hat{\mathcal{Q}}_s^{(j-1)} = \hat{\mathcal{Q}}^{(j-1)} \cap \hat{\mathcal{P}}_s^{(j-1)}$. Since $\big\{\mathcal{K}_j\big(\hat{\mathcal{P}}_s^{(j-1)}\big)\big\}_{s\in[t]}$ is a partition of the j-cliques of $\bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}$, we have that $\big\{\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\big\}_{s\in[t]}$ is a partition of the j-cliques of $\hat{\mathcal{Q}}^{(j-1)} = \bigcup_{s\in[t]} \hat{\mathcal{Q}}_s^{(j-1)}$. Hence,

$$\sum_{s\in[t]} \Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| = \Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big| . \tag{124}$$

Define

$$T = \Big\{ s \in [t] \colon \Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| \geq \delta\Big|\mathcal{K}_j\big(\hat{\mathcal{P}}_s^{(j-1)}\big)\Big|\Big\} .$$

Observe that

$$\sum_{s\notin T} \Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| < \delta\,\Big|\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}\big)\Big| \overset{(123)}{\leq} \sqrt{\delta}\,\Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big| . \tag{125}$$

Inequalities (123), (124) and (125) give

$$\sum_{s\in T} \Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| \geq \Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big| - \delta\,\Big|\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}\big)\Big| \geq \big(1-\sqrt{\delta}\big)\,\Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big| . \tag{126}$$

If $s \in T$, then the $(\delta, d, 1)$-regularity of $\mathcal{P}_s^{(j)}$ w.r.t. $\hat{\mathcal{P}}_s^{(j-1)}$ implies

$$\Big|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| = (d\pm\delta)\,\Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| .$$

Consequently,

$$\begin{aligned}
\Big|\mathcal{P}^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big| &= \sum_{s\in[t]} \Big|\mathcal{P}^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| \\
&= \sum_{s\in T} \Big|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| + \sum_{s\notin T} \Big|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| \\
&= (d\pm\delta)\sum_{s\in T} \Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| + \sum_{s\notin T} \Big|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| .
\end{aligned}$$

We then see that

$$(d-\delta)\sum_{s\in T} \Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| \leq \Big|\mathcal{P}^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big| \leq (d+\delta)\,\Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big| + \sum_{s\notin T} \Big|\mathcal{K}_j\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| .$$

In view of (125) and (126), we infer

$$(d-\delta)(1-\sqrt{\delta}) \leq d\big(\mathcal{P}^{(j)}\big|\hat{\mathcal{Q}}^{(j-1)}\big) \leq d+\delta+\sqrt{\delta}\,,$$

from which Proposition 52 follows. □

7.2. **Proof of Lemma 41.** Before proving Lemma 41, we recall some notation. For $* \in \{+,-\}$, let $\boldsymbol{\mathcal{H}}_* = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1} \cup \{\mathcal{H}_*^{(k)}\} = \{\mathcal{H}_*^{(j)}\}_{j=1}^{k}$ be given by Lemma 39. It follows that for each $j = 2,\dots,k-1$, the set $\hat{A}(\mathcal{H}_*^{(j-1)}, j-1, \boldsymbol{b})$ of polyad addresses with $\hat{\mathcal{P}}(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq \mathcal{H}^{(j-1)}$ satisfies that for each $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}_*^{(j-1)}, j-1, \boldsymbol{b})$, there is an index set $I(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq [b_j]$ of size $d_j b_j$ such that

$$\mathcal{H}_*^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = \bigcup_{\alpha\in I(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)},\alpha)\big) .$$

Moreover, $d\big(\mathcal{H}_*^{(k)}\big|\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big) = d_k^*$, where d_k^* is defined in (47). Recall that for $\Lambda_j = (\lambda_1,\dots,\lambda_j) \in \binom{[\ell]}{j}_<$, we denote by $\mathcal{H}_*^{(j)}[\Lambda_j]$ the subhypergraph of $\mathcal{H}_*^{(j)}$ induced on $V_{\lambda_1} \cup \dots \cup V_{\lambda_j}$.

Due to Lemma 37, we know that for $j = 2,\dots,k-1$, for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$, the set $I(\hat{\boldsymbol{x}}^{(j-1)})$ satisfies $|I(\hat{\boldsymbol{x}}^{(j-1)})| = d_j b_j$; moreover, for every $\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})$, the $(n/b_1, j, j)$-cylinder $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)},\alpha)\big)$ is

$(\tilde{\delta}_j, \tilde{d}_j, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$. Inductively on j, we aim to show that $\mathcal{H}_*^{(j)}[\Lambda_j]$, which is the union of all $\mathcal{P}^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}, \alpha)$, with $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_{j-1})$, $\hat{\boldsymbol{x}}_0 = \Lambda_j$ and $\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})$, is regular w.r.t. $\mathcal{H}_*^{(j-1)}[\Lambda_j]$.

Proof of Lemma 41. We only prove the statement about $\boldsymbol{\mathcal{H}}_*$ here. The proof for $\boldsymbol{\mathcal{F}}$ is identical. Consider the following statement:

(S_j) If $\Lambda_j = (\lambda_1, \dots, \lambda_j) \in \binom{[\ell]}{j}_<$ then $\boldsymbol{\mathcal{H}}_*^{(j)}[\Lambda_j]$ is a $\big((\varepsilon', \dots, \varepsilon'), (d_2^*, \dots, d_j^*), 1\big)$-regular (n, j, j)-complex.

Lemma 41 then follows from (S_k).

We prove statement (S_j) by induction on j. Suppose $j = 2$ and let $\Lambda_2 \in \binom{[\ell]}{2}_<$ be given. By (a) of Lemma 37, we have $\tilde{\mathcal{G}}^{(2)} = \mathcal{H}^{(2)} = \mathcal{H}_*^{(2)}$. Consequently, $\mathcal{H}_*^{(2)}[\Lambda_2]$ is $(\delta_2, d_2, 1)$-regular w.r.t. $\mathcal{H}_*^{(1)}[\Lambda_2]$ and (S_2) follows from $\delta_2 \ll \varepsilon'$ (cf. Figure 2).

We now proceed to the induction step. Assume $3 \le j \le k$ and (S_{j-1}) holds. Let $\Lambda_j = (\lambda_1, \dots, \lambda_j) \in \binom{[\ell]}{j}_<$ be arbitrary but fixed. The proof of (S_j) consists of three steps and we begin with the easiest.

Step 1. Let $\hat{X}(\Lambda_j) \subseteq \hat{A}(\mathcal{H}_*^{(j-1)}, j-1, \boldsymbol{b})$ be the set of all polyad addresses $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_{j-1})$ with $\hat{\boldsymbol{x}}_0 = \Lambda_j$. In Step 1, we consider

$$\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}) = \mathcal{H}_*^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) \tag{127}$$

for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$. To that end, fix a $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$. This step splits into two cases, depending on whether $j = k$ or not.

Case 1 $(3 \le j < k)$**.** We apply Proposition 50 to

$$\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}) \qquad \text{and} \qquad \Big\{\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big) \colon \alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})\Big\}$$

with $\delta = \delta_j$, $d = \tilde{d}_j$ (since $j < k$), and $t = \big|I(\hat{\boldsymbol{x}}^{(j-1)})\big| = d_j b_j = d_j^* b_j$. Consequently, for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$,

$$\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}) = \bigcup_{\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big) = \mathcal{H}_*^{(j)} \cap \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)$$

is $\big((d_j^* b_j)\tilde{\delta}_j, (d_j^* b_j)\tilde{d}_j, 1\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$. Since $b_j \tilde{\delta}_j = \tilde{\delta}_j / \tilde{d}_j \ll \tilde{\eta}$ and from (43), each such $\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$ is $\big(\tilde{\eta}, d_j^*, 1\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$.

Case 2 $(j = k)$**.** Here, we infer from (β) of Lemma 39 that $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{H}_*^{(k)} \cap \mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)$ is $(3\tilde{\delta}_k, d_k^*, 1)$-regular with respect to $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$. Hence, $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is also $\big(\tilde{\eta}, d_k^*, 1\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$.

Summarizing the two cases, we infer that for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$, the $(n/b_1, j, j)$-cylinder $\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$, as given in (127), is

$$\big(\tilde{\eta}, d_j^*, 1\big)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\,. \tag{128}$$

Step 2. In this step, we apply the induction assumption (S_{j-1}). For every $i \in [j]$, set

$$\Lambda_j\,(i) = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_j)\,.$$

We apply (S_{j-1}) to the $(n, j-1, j-1)$-complex $\boldsymbol{\mathcal{H}}_*^{(j-1)}[\Lambda_j(i)]$ for every $i \in [j]$. As a result, we infer that

$$\boldsymbol{\mathcal{H}}_*^{(j-1)}[\Lambda_j] = \bigcup_{i \in [j]} \boldsymbol{\mathcal{H}}_*^{(j-1)}[\Lambda_j\,(i)] = \Big\{\bigcup_{i \in [j]} \mathcal{H}^{(h)}[\Lambda_j\,(i)]\Big\}_{h=1}^{j-1}$$

is an $\big((\varepsilon', \dots, \varepsilon'), (d_2^*, \dots, d_{j-1}^*), 1\big)$-regular $(n, j, j-1)$-complex.

Step 3. Finally, as the last step, we show that the disjoint union

$$\bigcup_{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)} \mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}) = \mathcal{H}_*^{(j)}[\Lambda_j]$$

is $\big(\varepsilon', d_j^*, 1\big)$-regular w.r.t. $\mathcal{H}_*^{(j-1)}[\Lambda_j]$. Recall that $\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$ is $(\tilde{\eta}, d_j^*, 1)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ for each $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$, as shown in Step 1 (cf. (128)). Moreover, $\big\{\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big\}_{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)}$ satisfies (122).

Consequently, the assumptions of Proposition 52 are satisfied with $\delta = \tilde{\eta}$, $d = d_j^*$, and $t = |\hat{X}(\Lambda_j)|$ for the families

$$\left\{\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\right\}_{\hat{\boldsymbol{x}}^{(j-1)}\in\hat{X}(\Lambda_j)} \quad \text{and} \quad \left\{\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})\right\}_{\hat{\boldsymbol{x}}^{(j-1)}\in\hat{X}(\Lambda_j)}.$$

Therefore, it follows from Proposition 52 that $\mathcal{H}_*^{(j)}[\Lambda_j] = \bigcup_{\hat{\boldsymbol{x}}^{(j-1)}\in\hat{X}(\Lambda_j)} \mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$ is $(2\sqrt{\tilde{\eta}}, d_j^*, 1)$-regular w.r.t. $\bigcup_{\hat{\boldsymbol{x}}^{(j-1)}\in\hat{X}(\Lambda_j)} \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}) = \mathcal{H}_*^{(j-1)}[\Lambda_j]$. Then (S_j) follows from $2\sqrt{\tilde{\eta}} < \varepsilon'$ (cf. Figure 2). □

8. Proof of the removal lemma

In this section, we apply the hypergraph Regularity Lemma, Theorem 25, together with the Counting Lemma, Theorem 12, to prove Theorem 3. We shall, in particular, use the Counting Lemma in the form of Corollary 15 in the case $\mathcal{F}^{(k)} = K_{k+1}^{(k)}$ (our proof makes use of the concept of a $(\boldsymbol{\delta}, \geq \boldsymbol{d}, r, K_{k+1}^{(k)})$-regular (n, ℓ, k)-complex which, strictly speaking, doesn't appear in the original formulation seen in Theorem 12). The proof presented here is straightforward and follows the lines from [7, 9].

Proof of Theorem 3. Let k and $\varepsilon > 0$ be given as in the hypothesis of Theorem 3. Keeping in mind that we intend to apply the hypergraph Regularity Lemma in conjunction with the Counting Lemma, we introduce the auxiliary constants:

$$\gamma = \frac{1}{2} \qquad \text{and} \qquad d_k' = \frac{\varepsilon}{4}. \tag{129}$$

We also define a constant δ_k' and functions $\delta_j'(D_j, \ldots, D_{k-1})$ for $j = 2, \ldots, k-1$ in variables $D_2, \ldots, D_{k-1}$ in terms of the the Counting Lemma (applied with $\ell = k+1$, k, γ, and $d_k = d_k'$) as follows

$$\begin{aligned} \delta_k' &= \min\left\{\delta_k\big(\text{Cor.15}(d_k')\big), \frac{1}{8}\right\}, \\ \delta_j'(D_j, \ldots, D_{k-1}) &= \min\left\{\delta_j\big(\text{Cor.15}(D_j, \ldots, D_{k-1}, d_k')\big), \frac{D_j}{2}\right\}. \end{aligned} \tag{130}$$

Moreover, we consider functions $r'(A_1, D_2, \ldots, D_{k-1})$ and $n_0(D_2, \ldots, D_{k-1})$ coming from Theorem 12

$$r'(A_1, D_2, \ldots, D_{k-1}) = r\big(\text{Cor.15}(D_2, \ldots, D_{k-1}, d_k')\big), \tag{131}$$

and

$$n_0(D_2, \ldots, D_{k-1}) = n_0\big(\text{Cor.15}(D_2, \ldots, D_{k-1}, d_k')\big). \tag{132}$$

(We may assume, w.l.o.g., that the function $n_0(D_2, \ldots, D_{k-1})$ is monotone in each variable.) We also fix constants

$$\mu = \frac{1}{4} \qquad \text{and} \qquad \ell_{\text{reg}} = \max\left\{\frac{1}{\varepsilon}, \binom{k}{\lfloor k/2 \rfloor} + k\right\}. \tag{133}$$

We recall the quantification of Theorem 25, which for fixed constants ℓ_{reg} (ℓ in Theorem 25), k, δ_k', μ and functions $\delta_2'(D_2, \ldots, D_{k-1}), \ldots, \delta_{k-1}'(D_{k-1})$, and $r'(A_1, D_2, \ldots, D_{k-1})$ defined in (130)–(133) yields constants n_k and L_k. Finally, we define the promised δ and n_0 as

$$\delta = \frac{1}{4} \times \left(\frac{1}{2L_k^k}\right)^{2^{k+1}} \times \left(\frac{d_k'}{\ell_{\text{reg}} L_k}\right)^{k+1}, \tag{134}$$

and

$$n_0 = \max\Big\{\ell_{\text{reg}} n_k,\ \ell_{\text{reg}} L_k n_0\big(\underbrace{(2L_k)^{-k}, \ldots, (2L_k)^{-k}}_{(k-2)\text{-times}}\big)\Big\}. \tag{135}$$

Having all constants determined we are ready to give a proof of Theorem 3. Let $\mathcal{H}^{(k)}$ be a k-uniform hypergraph on $n \geq n_0$ vertices which contains at most δn^{k+1} copies of $K_{k+1}^{(k)}$. We want to apply the Regularity Lemma to $\mathcal{H}^{(k)}$. In the proof of Theorem 12, it was of some notational advantage to state the Regularity Lemma for $(n, \ell_{\text{reg}}, k)$-cylinders. At this place we have to pay a little tribute to this earlier convenience and we first artificially partition the vertex set of $\mathcal{H}^{(k)}$ into ℓ_{reg} classes $V_1, \ldots, V_{\ell_{\text{reg}}}$ of size n/ℓ_{reg} (we may ignore floors and ceilings again, since they have no effect on the arguments). Deleting all non-crossing k-tuples from $\mathcal{H}^{(k)}$ w.r.t. to the artificial vertex partition into $\ell_{\rm reg}$ classes, we obtain an $(n/\ell_{\rm reg}, \ell_{\rm reg}, k)$-cylinder $\mathcal{H}_1^{(k)} \subseteq \mathcal{H}^{(k)}$, where $n/\ell_{\rm reg} \geq n_k$ due to (135). Moreover,

$$\big|\mathcal{H}^{(k)} \setminus \mathcal{H}_1^{(k)}\big| \leq \ell_{\rm reg} \times \binom{n/\ell_{\rm reg}}{2} \times \binom{n}{k-2} \leq \frac{n^k}{2\ell_{\rm reg}} \overset{(133)}{\leq} \frac{\varepsilon}{2} n^k \,. \tag{136}$$

We apply the Regularity Lemma, Theorem 25, to $\mathcal{H}_1^{(k)}$ with constants $\ell_{\rm reg}$, k, δ'_k, and μ and functions $\delta'_2(D_2, \dots, D_{k-1}), \dots, \delta'_{k-1}(D_{k-1})$, and $r'(A_1, D_2, \dots, D_{k-1})$ defined in (130)–(133). Theorem 25 yields a vector of positive real numbers $\boldsymbol{d'} = (d'_2, \dots, d'_{k-1})$ and a family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ such that for

$$\delta'_j = \delta'_j(d'_j, \dots, d'_{k-1}) \quad \text{for} \quad j = 2, \dots, k-1\,, \qquad \boldsymbol{\delta'} = (\delta'_2, \dots, \delta'_{k-1})\,,$$
$$\text{and} \qquad r' = r'(a_1, d'_2, \dots, d'_{k-1})$$

the following holds:

(i) $\mathscr{R}$ is $(\mu, \boldsymbol{\delta'}, \boldsymbol{d'}, r')$-equitable,
(ii) $\mathscr{R}$ is (δ'_k, r')-regular w.r.t. $\mathcal{H}_1^{(k)}$, and
(iii) $\operatorname{rank} \mathscr{R} \leq L_k$.

Next we define a subhypergraph $\mathcal{H}_2^{(k)}$ of $\mathcal{H}_1^{(k)}$ by deleting those edges of $\mathcal{H}_1^{(k)}$ which either belong to irregular or sparse polyads of $\mathscr{R}$. (Note that all edges $\mathcal{H}_1^{(k)}$ are crossing w.r.t. $\mathscr{R}^{(1)}$ since $\mathcal{H}_1^{(k)}$ is $\ell_{\rm reg}$-partite and $\mathscr{R}^{(1)}$ refines that given vertex partition of $\mathcal{H}_1^{(k)}$.) More precisely, let K be a k-tuple in $\mathcal{H}_1^{(k)}$. We then delete K if at least one of the following applies:

(a) $\boldsymbol{\mathcal{R}}(K) = \{\mathcal{R}^{(h)}\}_{h=1}^{k}$ (see (6)) is not a $(\boldsymbol{\delta'}, \boldsymbol{d'}, r')$-regular complex,
(b) $\mathcal{H}_1^{(k)}$ is not (δ'_k, r')-regular w.r.t. $\mathcal{R}^{(k-1)}$, or
(c) $d\big(\mathcal{H}_1^{(k)} \big| \mathcal{R}^{(k-1)}\big) \leq d'_k$.

We bound the number of deleted edges. Due to (i) and Definition 23, at most $\mu \times (n/\ell_{\rm reg})^k$ edges K of the $(n/\ell_{\rm reg}, \ell_{\rm reg}, k)$-cylinder $\mathcal{H}_1^{(k)}$ are deleted because of (a). Moreover, (ii) and Definition 24 give that at most $\delta'_k \times (n/\ell_{\rm reg})^k$ edges fail to be in $\mathcal{H}_2^{(k)}$ due to (b). Finally, at most $d'_k n^k$ edges can be deleted because of (c). Summarizing the above considerations then gives

$$\big|\mathcal{H}^{(k)} \setminus \mathcal{H}_2^{(k)}\big| \leq \big|\mathcal{H}^{(k)} \setminus \mathcal{H}_1^{(k)}\big| + \left(\frac{\mu + \delta'_k}{\ell_{\rm reg}^k} + d'_k\right) n^k \leq \varepsilon n^k\,, \tag{137}$$

where we used (136), (133), (130) and (129) for the last inequality.

Consequently, it suffices to show that $\mathcal{H}_2^{(k)}$ is $K_{k+1}^{(k)}$-free in order to verify the theorem. So assume the contrary and let K' be the vertex set of a copy of $K_{k+1}^{(k)}$ contained in $\mathcal{H}_2^{(k)}$. Then K' witnesses that $\mathcal{H}_2^{(k)}$ contains a $\big(\boldsymbol{\delta'}, \geq\boldsymbol{d'}, r', K_{k+1}^{(k)}\big)$-regular $(n/(\ell_{\rm reg} a_1), k+1, k)$-subcomplex (see Definition 13). Owing to the choice of the functions in (130)–(132) we see that this $\big(\boldsymbol{\delta'}, \geq\boldsymbol{d'}, r', K_{k+1}^{(k)}\big)$-regular $(n/(\ell_{\rm reg} a_1), k+1, k)$-subcomplex satisfies the assumptions of Corollary 15. (Note that it follows from Claim 53 below, the monotonicity of the function $n_0(D_2, \dots, D_{k-1})$ in (132), and the final choice of n_0 in (135), that $n/(\ell_{\rm reg} a_1)$ is sufficiently large.) Consequently, $\mathcal{H}_2^{(k)} \subseteq \mathcal{H}^{(k)}$ contains at least

$$\frac{1}{2} \prod_{h=2}^{k} (d'_h)^{\binom{k+1}{h}} \times \left(\frac{n}{\ell_{\rm reg} a_1}\right)^{k+1}$$

copies of $K_{k+1}^{(k)}$. If we show that

$$\frac{1}{2} \prod_{h=2}^{k-1} (d'_h)^{\binom{k+1}{h}} \times \left(\frac{d'_k}{\ell_{\rm reg} a_1}\right)^{k+1} > \delta\,, \tag{138}$$

we derive a contradiction to the assumption that $\mathcal{H}^{(k)}$ contains at most δn^{k+1} copies of $K_{k+1}^{(k)}$.

Consequently, in order to prove Theorem 3 it is left to verify (138). For that we first observe that $a_1 < |A(k-1,\boldsymbol{a})| = \operatorname{rank}\mathscr{R} \le L_k$. Then, in view of the choice of δ in (134), inequality (138) follows from the next claim. $\square$

Claim 53. *$d'_j > \frac{1}{2L_k^k}$ for every $j = 2,\dots,k-1$.*

(Clearly, the lower bound promised in Claim 53 is rather crude, but it is all that is required to confirm inequality (138)).

Proof. Suppose, on the contrary, that for fixed $2 \le j \le k-1$, we have $d'_j \le 1/(2L_k^k)$. It shall not be difficult to produce a contradiction from this assumption, but for this, we need to establish a few observations.

First, observe that $|\hat{A}(j-1,\boldsymbol{a})| \le L_k^k$. Indeed, with $\ell_{\rm reg} \ge 2k$ from (133), we have (using the formula in (14))

$$\big|\hat{A}(j-1,\boldsymbol{a})\big| = \binom{\ell_{\rm reg}}{j}\prod_{h=1}^{j-1} a_h^{\binom{j}{h}} \le \binom{\ell_{\rm reg}}{k}\prod_{h=1}^{k-1} a_h^{\binom{k}{h}} = \big|\hat{A}(k-1,\boldsymbol{a})\big|.$$

Observe, moreover, that for each $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1,\boldsymbol{a})$, the polyad $\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ identifies $\binom{k}{k-1}$ classes from $\mathscr{R}$. As there are at most $\operatorname{rank}\mathscr{R} \le L_k$ such classes, we see

$$\big|\hat{A}(j-1,\boldsymbol{a})\big| \le \big|\hat{A}(k-1,\boldsymbol{a})\big| \le \binom{L_k}{k} \le L_k^k,$$

as promised.

We now produce the contradiction desired to prove Claim 53. Using $d'_i \le 1$ for any $i < j$, the number of j-tuples in (δ'_j, d'_j, r')-regular polyads is at most

$$(d'_j + \delta'_j) \times \left(\frac{n}{\ell_{\rm reg} a_1}\right)^j \times \big|\hat{A}(j-1,\boldsymbol{a})\big| \le \frac{3d'_j}{2} \times \left(\frac{n}{\ell_{\rm reg} a_1}\right)^j \times L_k^k \le \frac{3}{4}\left(\frac{n}{\ell_{\rm reg}}\right)^j, \tag{139}$$

where the last inequality used the assumption $d'_j < 1/(2L_k^k)$. On the other hand, as we prove momentarily, at most

$$\mu\left(\frac{n}{\ell_{\rm reg}}\right)^j \binom{k}{j} \Big/ \binom{\ell_{\rm reg}-j}{k-j} \overset{(133)}{\le} \mu\left(\frac{n}{\ell_{\rm reg}}\right)^j \tag{140}$$

distinct j-tuples are in irregular $(n/(\ell_{\rm reg}a_1), j, j)$-complexes of the partition $\mathscr{R}$. As such, (139) and (140) combine to give

$$\binom{\ell_{\rm reg}}{j}\left(\frac{n}{\ell_{\rm reg}}\right)^j = \big|K^{(j)}_{\ell_{\rm reg}}(V_1,\dots,V_{\ell_{\rm reg}})\big| \le \left(\frac{3}{4}+\mu\right)\left(\frac{n}{\ell_{\rm reg}}\right)^j \overset{(133)}{<} \left(\frac{n}{\ell_{\rm reg}}\right)^j,$$

a contradiction establishing Claim 53.

To prove (140), one simply double-counts. Every crossing (w.r.t. $\mathscr{R}^{(1)}$) j-tuple which belongs to a

$$((\delta'_2,\dots,\delta'_{j-1}),(d'_2,\dots,d'_{j-1}),r')\text{-irregular } (n/(\ell_{\rm reg}a_1), j, j-1)\text{-complex}$$

can be extended to $\binom{\ell_{\rm reg}-j}{k-j}(n/\ell_{\rm reg})^{k-j}$ k-tuples in $K^{(k)}_{\ell_{\rm reg}}(V_1,\dots,V_{\ell_{\rm reg}})$. Moreover, at most $\binom{k}{j}$ different j-tuples extend to the same k-tuple. Each such k-tuple necessarily belongs to a $(\boldsymbol{\delta}',\boldsymbol{d}',r')$-irregular polyad. Due to (i), there are at most $\mu \times (n/\ell_{\rm reg})^k$ such k-tuples, proving (140). $\square$

9. Concluding remarks

In the course of writing this paper, the authors realized that the methods used here, combined with the result of [40], yields a hypergraph regularity lemma that would be simpler to state and likely more convenient to use. For 3-uniform hypergraphs, such a result immediately follows from Theorem 34 and the RS-Lemma with $k=3$ (equivalently, the FR-Lemma).

Theorem 54. *For every positive real ν and a positive real-valued function $\varepsilon(D_2)$, there exist integers L_3 and n_3 such that for any 3-uniform hypergraph $\mathcal{H}^{(3)}$ on $n \ge n_3$ vertices there exists a 3-uniform hypergraph $\mathcal{F}^{(3)}$ and a $\big(\nu,\varepsilon(d_2),d_2,1\big)$-equitable family of partitions $\mathscr{R} = \mathscr{R}(2,\boldsymbol{a},\boldsymbol{\varphi}) = \{\mathscr{R}^{(1)},\mathscr{R}^{(2)}\}$ such that* (*i*) *for every* $\hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a})$

$$\left|\left(\mathcal{H}^{(3)} \triangle \mathcal{F}^{(3)}\right) \cap \mathcal{K}_3\left(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\right)\right| \leq \nu d_2^3 \left(\frac{n}{a_1}\right)^3, \tag{141}$$

(*ii*) *all but at most* νn^3 *edges of* $K_n^{(3)}$ *belong to some polyad* $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$ *where* $\mathcal{F}^{(3)}$ *is* $(\varepsilon(d_2), 1)$*-regular w.r.t.* $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$*, and*

(*iii*) $\operatorname{rank} \mathscr{R} \leq L_3$.

We omit the details of the proof of Theorem 54 (see [36] for more general results).

There is an important single difference between Theorem 54 and the FR-Lemma (or, equivalently, RS-Lemma with $k = 3$); Theorem 54 provides an environment sufficient for a direct application of the Dense Counting Lemma, Theorem 16. Indeed, unlike the output of the FR-Lemma where one has constants δ_3, d_2, $\delta_2(d_2)$ and r, so that $\delta_3 \gg d_2 \gg \delta_2(d_2), 1/r$, Theorem 54 admits sufficiently small $\varepsilon(d_2)$ with $\varepsilon(d_2) \ll d_2$ and no formulation of r such that both the 3-uniform hypergraph $\mathcal{F}^{(3)}$ and the graphs from the underlying partition $\mathscr{R}^{(2)}$ are $\varepsilon(d_2)$-regular.

The only cost of the cleaner environment Theorem 54 renders is that our original input hypergraph $\mathcal{H}^{(3)}$ is slightly (albeit negligibly) altered to the output hypergraph $\mathcal{F}^{(3)}$.

In [36] we prove a generalization of Theorem 54 to k-uniform hypergraphs. The proof of that generalization is more technical and requires a restructuring of the arguments from [40] and this paper.

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