Boundary Problems in Conformal Field Theory

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Abstract

This thesis is concerned with various aspects of two dimensional conformal quantum field theory with a boundary. It is organised as follows: first the necessary formalism and calculational techniques are explained. Then it is shown how to obtain the structure constants characterising a boundary conformal field theory. Finally as an application, the structure constants are used to compare conformal field theory calculations to the thermodynamic Bethe Ansatz.

In two dimensions the symmetry algebra of a conformal field theory is infinite dimensional and so in certain cases, organises the Hilbert space of the quantum theory into finitely many representations. This thesis is concerned with a class of such theories: the minimal models.

When introducing a boundary into the system, two new ingredients appear: conformal boundary conditions and boundary fields. The way in which boundary fields transform under conformal transformations is studied and the result is used to compute how the shape of the boundary affects the short distance expansion of a bulk field in terms of boundary fields.

In minimal models conformal symmetry fixes the correlation functions up to finitely many unknowns, given in terms of the structure constants. The structure constants have to fulfil a set of polynomial equations, the so-called sewing constraints. By solving these constraints, explicit solutions for bulk and boundary structure constants are calculated in the case of A– and D–series minimal models.

These constants form a vital ingredient in two approximation methods. One is conformal perturbation theory, which gives an expansion around the unperturbed theory. The other is the truncated conformal space approach, in which the perturbed Hamiltonian is truncated at a certain energy and diagonalised numerically.

As an example the boundary Lee-Yang model is considered. This model has an off-critical description in terms of integrable scattering. It thus allows for an investigation through the thermodynamic Bethe Ansatz and the outcome is compared to results from the previous two methods.

To summarise, this thesis provides important additions to the growing tool box of boundary conformal field theory. One interesting application is the quantitative investigation of conjectured relationships to other descriptions of boundary integrable systems.

Declaration

This dissertation is the result of my studies at the Department of Mathematics, King's College London, from October 1997 to August 2000. Apart from the contributions of my collaborators this work contains only my own original research unless otherwise stated in the text. Specifically, the contents of chapters 2, 4, 5 can be found in the literature, the research presented in chapter 3 was done in collaboration with G. Watts (unpublished) and the results of chapters 6, 7 have been published as:

[Run98] I. Runkel, *Boundary structure constants for the A-series Virasoro minimal models*, Nucl. Phys. B549 (1999) 563–578, `hep-th/9811178`

[Run99] I. Runkel, *Structure constants for the D-series Virasoro minimal models*, Nucl. Phys. B579 (2000) 561–589 , `hep-th/9908046`

[DRTW99] P. Dorey, I. Runkel, R. Tateo and G. Watts, *g-function flow in perturbed boundary conformal field theories*, Nucl. Phys. B578 (2000) 85–122, `hep-th/9909216`.

This dissertation is not substantially the same as any other that I have submitted for a degree or diploma or other qualification at this or any other university.

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Chapter 1.

Introduction

There are many reasons to study two-dimensional conformal field theory (CFT). Let us briefly look at three of them.

For an elementary particle theorist, a good motivation might be to think of CFT as a model for a model. Currently the best theory in the realm of elementary particle physics is the Standard Model. Due to its complicated structure the predictions it makes are as yet only accessible from perturbative expansions around a free massive theory. One might hope to learn about the behaviour of interacting quantum field theories (QFTs) from models that can be solved explicitly. CFT in two dimensions has enough symmetries to allow for such an exact solution, i.e. the computation of all correlation functions in the quantum field theory. CFTs provide the renormalisation group fixed points in the space of two dimensional QFTs. In some cases these fixed points are connected by a whole line of "integrable theories", so that one can non-perturbatively follow the flow from the UV to the IR. The ultimate aim is to understand, in some sense, the space of 2d-QFTs and draw conclusions for four dimensional interacting QFTs.

If one's interest is in condensed matter physics, the merits of CFT are more direct, as it describes two-dimensional classical systems at a second order phase transition. The central theme here is universality, i.e. that the properties of a system close to the critical point are independent of its microscopic realisation and are characterised by so-called universality classes. At the point of phase transition the system is supposed to be scale invariant and to possess a description in terms of a CFT. The requirement of conformal symmetry is so restrictive that one can hope to find a list of all such theories and thus to classify all universality classes. As it stands this hope may be too naive, but for example all unitary CFTs with a certain parameter, called central charge, between zero and one are known. They are part of the minimal models, the main subject of this thesis. Two prominent members of the list of minimal models are the universality classes of the Ising model and the three state Potts model.

CFT is also a crucial ingredient in string theory. It governs the behaviour of

the fields on the string world sheet that determine the position of the string. If the string is moving in a flat background the CFT is just a free massless theory, but for curved backgrounds more complicated CFTs arise. A theory including open strings naturally requires the CFT to be defined on surfaces with boundaries. Objects of special interest in the theory of open strings are hyper surfaces in the target space on which open strings are allowed to end, so-called D-branes. Classifying all boundary conditions of the CFT on the world sheet consistent with conformal invariance gives information about the possible D-brane configurations in a given string theory. D-branes can a priori have any dimension and it is interesting how the inherently two dimensional CFT characterises higher dimensional objects.

We start in chapter 2 by describing some aspects of CFT defined on the full complex plane and on the upper half plane. The way in which boundary fields transform under conformal transformations is studied in chapter 3 and the result is used to compute how the shape of the boundary affects the short distance expansion of bulk and boundary fields.

In certain cases the symmetry algebra of the conformal field theory organises the Hilbert space into finitely many representations. When this happens one can attempt to find the correlators with the following two steps. The first step (chapter 4) uses the representation theory of the symmetry algebra. The aim is to construct a set of functions, called conformal blocks, and a set of matrices describing the transformation behaviour of conformal blocks. In the second step (chapter 5) the conformal blocks are combined to form the physical correlators of the theory. The precise combination of conformal blocks is fixed by the structure constants, which appear in the short distance expansion of nearby fields. The structure constants are determined by requiring consistency between the various limits one can take in a correlator and the short distance expansion.

In chapter 6 we apply this method to minimal models. They are characterised by the property that the algebra of conformal symmetry by itself is enough to finitely decompose the Hilbert space. We calculate explicit solutions for bulk and boundary structure constants in the case of A– and D–series minimal models.

In chapter 7 we examine the example of the perturbed boundary Lee-Yang model. The two methods used on the CFT side are conformal perturbation theory and the truncated conformal space approach. The Lee-Yang model also admits an off-critical investigation through the thermodynamic Bethe Ansatz and the outcome is compared to the previous two methods.

Chapter 2.

Boundary Conformal Field Theory

It is beyond the scope of this thesis to present a full summary of conformal field theory. Furthermore, it would be quite superfluous, as there already exist many excellent reviews on the subject. A selection of the author's personal favourites, without any particular order, is

[YBk] The book by DiFrancesco, Mathieu and Sénéchal, by now a standard text about conformal field theory.

[Gin88] The Les Houches lecture notes by Ginsparg, an excellent introduction to CFT.

[Gab99] An overview of CFT centered on the role of the symmetry generating chiral algebra.

[AGS89a] An introduction by Alvarez-Gaume, Sierra and Gomez, written with an emphasis on the connection to knots and quantum groups.

[Car89a] In the Les Houches lecture by Cardy the focus is on statistical mechanics. Geometries with boundaries are also discussed.

In the following we will introduce the areas of CFT which are relevant for this thesis. In some cases we will just state the results and it is understood that additional information can be found in the introductory articles just mentioned.

2.1 CFT on the full complex plane

Conformal field theory unfolds its full power only in two-dimensional space. The reason is that the algebra of infinitesimal conformal transformations, i.e. those that preserve angles but not necessarily length, is finite in dimensions greater than two and too restrictive in less than two.

The two dimensional space on which the theory is defined can be equipped with a Minkowski or a Euclidean metric. It is the Euclidean case which has direct appli-

cations to statistical mechanics or in computations of amplitudes in string theory. In this thesis we only consider flat two-dimensional Euclidean space.

The formalism takes its easiest form after the identification of the two dimensional space, with Cartesian coordinates (x, y), and the complex plane:

$$z = x + iy \qquad \bar{z} = x - iy \; . \tag{2.1}$$

Analytic functions automatically preserve angles, and a general infinitesimal conformal transformation can be written as $z \mapsto z + \varepsilon(z)$, where $\varepsilon(z)$ is analytic. We see that there are infinitely many independent such transformations.

In this thesis we will use z^* and $\bar{z}$. The notation z^* always refers to the complex conjugate of z. When writing $\bar{z}$ we want to think of it as an independent variable which will eventually be set to z^*. We will find later that correlators can be written in terms of holomorphic functions f, g as $\langle \phi(z, \bar{z}) \cdots \rangle = \sum f(z) g(\bar{z})$. From this point of view it makes sense to speak of the "chiral half" of a correlator as its z–dependence while keeping $\bar{z}$ fixed.

Note that the group of conformal transformations on the Riemann sphere is finite dimensional and consists only of Möbius transformations

$$z \mapsto \frac{az + b}{cz + d} \; , \tag{2.2}$$

with $a, b, c, d \in \mathbb{C}$. They form the Lie group $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$. I.e. in two and more dimensions the *group* of global conformal transformations is finite dimensional. However the *algebra* of infinitesimal conformal transformations is infinite in two but finite in more than two space dimensions.

To the fields $\phi(z, \bar{z})$ in the theory we can associate a scaling dimension Δ and a spin J (also sometimes denoted with x and s), according to their transformation behaviour under global rescaling and rotation[1], i.e. under the map $w = re^{i\theta} z$ for some fixed real r, θ we have

$$\phi(z, \bar{z}) \longrightarrow r^{\Delta} e^{i\theta J} \phi(w, \bar{w}) \; . \tag{2.3}$$

Fields with J=0 are called spinless or diagonal fields.

Every conformal transformation $w = f(z)$ looks locally like a combined rescaling and rotation. The CFT will contain some fields, called primary fields which only see

[1] More precisely among the fields in the theory we can pick a basis with the desired transformation property. This is necessary since linear combinations of fields in general transform in more complicated ways. It should also be mentioned that (2.3), while being true for all CFTs considered in this thesis, might still fail in more general situations, e.g. in logarithmic CFTs [Gur93, Gab99].

this local behaviour, i.e. whose transformation properties depend only on the first derivative of f. Let $h, \bar{h}$ be s.t. $\Delta = h + \bar{h}$, $J = h - \bar{h}$. Then, for a primary field $\phi(z, \bar{z})$

$$\phi(z, z^*) \xrightarrow{w=f(z)} f'(z)^h (f'(z)^*)^{\bar{h}} \phi(f(z), f(z)^*) \tag{2.4}$$

h and $\bar{h}$ are called the conformal weights of the primary field $\phi(z, \bar{z})$.

We can formulate an infinitesimal version of (2.4). Under the conformal mapping $z \mapsto z + \varepsilon(z)$ a primary field changes according to

$$\delta\phi(z, z^*) = \big(h\varepsilon'(z) + \varepsilon(z)\partial_z + \bar{h}\varepsilon'(z)^* + \varepsilon(z)^*\partial_{\bar{z}}\big)\phi(z, z^*) \; . \tag{2.5}$$

The stress-tensor

Another way to arrive at the properties of correlation functions under infinitesimal transformations is to use the energy momentum or stress tensor[2]. Even though we will never explicitly need an action it is sometimes helpful to think that there is a path integral formulation of the theory. We take the response of the action to an arbitrary infinitesimal coordinate transformation $\varepsilon(x, y)$ as a definition of the stress tensor

$$\delta S = -\int d^2x \, \partial_\mu \varepsilon_\nu(\vec{x}) \, T^{\mu\nu}(\vec{x}) \; . \tag{2.6}$$

This relation is valid for any infinitesimal transformation $\varepsilon(\vec{x})$, not just those describing a symmetry. For a collection of fields X the path integral formulation gives the relation

$$\int d\Phi \, X \, \delta S \, e^{-S[\Phi]} = \int d\Phi \, \delta X \, e^{-S[\Phi]} \; , \tag{2.7}$$

where δS and δX are the changes of the action and the fields under the transformation $\varepsilon(\vec{x})$.

Now consider (2.7) for a collection of primary fields $X = \phi(z_1, z_1^*) \dots \phi(z_n, z_n^*)$. We use a continuous transformation $\varepsilon(x)$ that is conformal in a small disc around each z_k and otherwise arbitrary in a compact region containing all the discs. Outside this region we set $\varepsilon(x)$ to zero, as depicted in fig. 2.1. Let D be the collection of all n discs. Since $\varepsilon(x)$ is a symmetry of the action inside D, we have $\delta S\big|_D = 0$.

[2] To be precise the name energy-momentum tensor refers to Minkowski space time. After all one needs a time dimension to speak of momentum, whereas the term stress tensor refers to the elastic properties of materials. In a slight abuse of notation we will use both names.

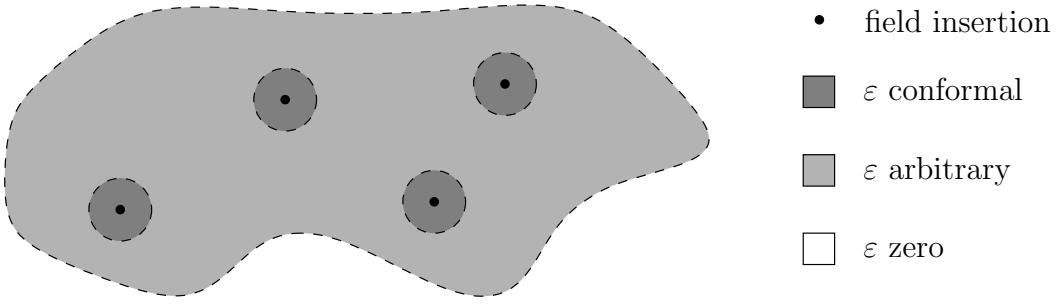


Figure 2.1: Infinitesimal transformation used to deduce properties of the stress tensor on the complex plane.

Putting together (2.5) and (2.7) we get the relation

$$\begin{aligned} &-\int_{\mathbb{R}^2-D} d^2x(\partial_\mu\varepsilon_\nu(x))\langle T^{\mu\nu}(x)\phi(z_1,z_1^*)\dots\phi(z_n,z_n^*)\rangle \\ &\quad=\sum_{k=1}^{n}\big(h_k\varepsilon'(z_k)+\varepsilon(z_k)\partial_{z_k}+\bar{h}\varepsilon'(z_k)^*+\varepsilon(z_k)^*\partial_{\bar{z}}\big)\langle\phi(z_1,z_1^*)\dots\phi(z_n,z_n^*)\rangle \end{aligned} \tag{2.8}$$

As a first consequence we can use Gauß's theorem to reduce the lhs of (2.8) to a surface integral and a term containing $\varepsilon_\nu(x)\partial_\mu T^{\mu\nu}$ integrated over $\mathbb{R}^2-D$. Now $\varepsilon(x)$ could be chosen arbitrarily on any compact region in $\mathbb{R}^2-D$. Since the rhs is independent of the values of ε outside D, this is consistent only if T is conserved.

The reason why we went through this procedure at such length is to compare it later to an analogous argument on the upper half plane. In this case, in addition to the conservation of T, we will obtain the boundary condition for T.

Altogether the properties of the stress tensor are, when its position does not coincide with the insertion points of other fields:

$$T_{xy}=T_{yx}\ ;\ \ T_{xx}{+}T_{yy}=0\ ;\ \ \partial_xT_{xx}{+}\partial_yT_{yx}=0\ ;\ \ \partial_xT_{xy}{+}\partial_yT_{yy}=0 \tag{2.9}$$

The relations orginate, in this order, from invariance under rotations and rescaling and conservation of the stress-tensor.

Using the fact that $T_{\mu\nu}$ is conserved we arrive at the second consequence of (2.8), the Ward identity:

$$\begin{aligned} &-\int_{\partial D}\varepsilon_\mu(x)n_\nu(x)\langle T^{\mu\nu}(x)\phi(z_1,z_1^*)\dots\phi(z_n,z_n^*)\rangle \\ &\quad=\sum_{k=1}^{n}\big(h_k\varepsilon'(z_k)+\varepsilon(z_k)\partial_{z_k}+\bar{h}\varepsilon'(z_k)^*+\varepsilon(z_k)^*\partial_{\bar{z}}\big)\langle\phi(z_1,z_1^*)\dots\phi(z_n,z_n^*)\rangle \end{aligned} \tag{2.10}$$

Here $n_\nu(x)$ is an inward pointing vector normal to the boundary of D.

Cartesian coordinates x^a	Complex coordinates z^μ
$x = \frac{1}{2}(z+\bar{z})\,;\; y = \frac{1}{2i}(z-\bar{z})$ $\partial_x = \partial_z + \partial_{\bar{z}}\,;\; \partial_y = i(\partial_z - \partial_{\bar{z}})$ $\frac{\partial x^a}{\partial z^\mu} = P^a{}_\mu = \begin{pmatrix} \partial_z x & \partial_{\bar{z}} x \\ \partial_z y & \partial_{\bar{z}} y \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -i/2 & i/2 \end{pmatrix}$ $g_{ab} = g^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$z = x + iy\,;\; \bar{z} = x - iy$ $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)\,;\; \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ $\frac{\partial z^\mu}{\partial x^a} = Q^\mu{}_a = \begin{pmatrix} \partial_x z & \partial_y z \\ \partial_x \bar{z} & \partial_y \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ $g_{\mu\nu} = P^T P = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\;;\; g^{\mu\nu} = QQ^T = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$
$T_{xx} = T_{zz} + T_{z\bar{z}} + T_{\bar{z}z} + T_{\bar{z}\bar{z}}$ $T_{xy} = i(T_{zz} - T_{z\bar{z}} + T_{\bar{z}z} - T_{\bar{z}\bar{z}})$ $T_{yx} = i(T_{zz} + T_{z\bar{z}} - T_{\bar{z}z} - T_{\bar{z}\bar{z}})$ $T_{yy} = -T_{zz} + T_{z\bar{z}} + T_{\bar{z}z} - T_{\bar{z}\bar{z}}$	$T_{zz} = \frac{1}{4}(T_{xx} - T_{yy}) + \frac{1}{4i}(T_{xy} + T_{yx})$ $T_{z\bar{z}} = \frac{1}{4}(T_{xx} + T_{yy}) - \frac{1}{4i}(T_{xy} - T_{yx})$ $T_{\bar{z}z} = \frac{1}{4}(T_{xx} + T_{yy}) + \frac{1}{4i}(T_{xy} - T_{yx})$ $T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{xx} - T_{yy}) - \frac{1}{4i}(T_{xy} + T_{yx})$ $T(z) = 2\pi T_{zz}(z)\,;\; \bar{T}(\bar{z}) = 2\pi T_{\bar{z}\bar{z}}(\bar{z})$

Table 2.1: Relation between some quantities in Cartesian and complex coordinates

Next we will reformulate the previous relations in terms of the stress tensor expressed in complex coordinates. It is convenient to define $T = 2\pi T_{zz}$ and $\bar{T} = 2\pi T_{\bar{z}\bar{z}}$. Using the relations listed in table 2.1, the properties (2.9) in complex coordinates are:

$$T_{z\bar{z}} = T_{\bar{z}z} = 0\;;\;\; \bar{\partial} T(z,\bar{z}) = 0\;;\;\; \partial \bar{T}(z,\bar{z}) = 0 \tag{2.11}$$

The second equation implies in particular that a correlator involving the field $T(z,\bar{z})$ is an analytic function in z on the full complex plane minus the insertion points of other fields in the correlator. This is often abbreviated by omitting the $\bar{z}$ variable, and one writes $T(z)$. Similarly one writes $\bar{T}(\bar{z})$ instead of $\bar{T}(z,\bar{z})$. This will however lead to great confusion when considering the upper half plane later on, where one has to deal with $T(z^*,z)$ and $\bar{T}(z^*,z)$ as functions of z. So in the following we will use $T(z,\bar{z})$ and, when the meaning is clear also $T(z)$.

In transforming (2.10) to the complex stress tensor one has to be careful with minus signs and factors of 2π. One obtains the following relation, valid for an analytic function $\varepsilon(z)$:

$$\begin{aligned} &\Big(\int_z \circlearrowleft_w \; \varepsilon(z) T(z) + \int_{\bar{z}} \circlearrowleft_w \; \varepsilon(z)^* \bar{T}(z^*) \Big) \phi(w,w^*) \\ &\qquad = \big(h\varepsilon'(w) + \varepsilon(w)\partial_w + \bar{h}\varepsilon'(w)^* + \varepsilon(w)^* \partial_{\bar{w}} \big) \phi(w,w^*) \end{aligned} \tag{2.12}$$

Some remarks are in order. The equation is valid when inserted in any correlator. The notation for contour integrals used above and the rest of the thesis is that

the factor of $(2\pi i)^{-1}dz$ (resp. $-(2\pi i)^{-1}d\bar{z}$) is absorbed into the definition of the integral. The contour is indicated in a schematic picture and the variable integrated over appears in the lower left corner. For some closed contour $\gamma : [0,1] \to \mathbb{C}$, a holomorphic function f and an antiholomorphic function g define

$$\begin{aligned} \int_z \; f(z) &:= \frac{1}{2\pi i}\int_0^1 dt f(\gamma(t))\cdot\gamma'(t)\;, \\ \int_{\bar{z}} \; g(z) &:= \frac{-1}{2\pi i}\int_0^1 dt g(\gamma(t))\cdot\gamma'(t)^* \end{aligned} \tag{2.13}$$

With this definition we have

$$\int_z \; \frac{1}{z} = 1 \qquad \int_{\bar{z}} \; \frac{1}{z^*} = 1\;. \tag{2.14}$$

Looking at (2.12) we note that the separate action of T and $\bar{T}$ can be extracted from this equation. Simply consider (2.12) for a 90° rotated transformation $\tilde{\varepsilon}(z) = i\varepsilon(z)$ and take linear combinations. The result is

$$\int_z \; \varepsilon(z)T(z)\phi(w,w^*) = \big(h\varepsilon'(w) + \varepsilon(w)\partial_w\big)\phi(w,w^*) \tag{2.15}$$

Note that (2.15), while being an identity for correlators, does *not* correspond to a infinitesimal coordinate transformation. It is only in the combination (2.12) that this interpretation is allowed.

Inserting (2.15) in a correlator and using Cauchy's formula one can deduce the conformal Ward identity:

$$\begin{aligned} &\langle T(\zeta)\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle = \\ &\sum_{j=1}^n \Big\{\frac{h}{(\zeta-z_j)^2} + \frac{1}{\zeta-z_j}\frac{\partial}{\partial z_j}\Big\}\langle\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle + \mathrm{reg}(\zeta) \end{aligned} \tag{2.16}$$

where $\mathrm{reg}(\zeta)$ is a regular function on the complex plane. A similar relation holds for $\bar{T}(\bar{z})$.

In particular we see that the operator product expansion (OPE) –or short distance expansion– of the stress tensor with a primary bulk field is

$$T(z)\phi(w,\bar{w}) = \Big(\frac{h}{(z-w)^2} + \frac{1}{z-w}\frac{\partial}{\partial w}\Big)\phi(w,\bar{w}) + \mathrm{reg}(z-w) \tag{2.17}$$

As expounded in more detail in the introductory texts, the most general OPE for T

consistent with associativity is

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \mathrm{reg}(z-w)\ , \tag{2.18}$$

with an analogous expression for $\bar{T}(\bar{z})\bar{T}(\bar{w})$. The OPE of T with $\bar{T}$ has no poles. (2.18) gives rise to the following transformation behaviour under a conformal map $w{=}f(z)$:

$$T(z) \xrightarrow{z\mapsto f(z)} f'(z)^2 T(w) + \tfrac{c}{12}\{f;z\} \tag{2.19}$$

In particular T is not primary. The constant c is called central charge and turns out largely to fix the properties of the conformal field theory. $\{f;z\}$ is called the *Schwarzian derivative* and is defined as

$$\{f;z\} := \frac{f'''(z)}{f'(z)} - \frac{3}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2\ . \tag{2.20}$$

Since the theory is defined on the full complex plane we can actually do better than (2.16). For $\zeta \to \infty$, in any direction, the leading behaviour of the correlator will be given by $\langle T(\zeta)T(0)\rangle = \frac{c}{2}\cdot\zeta^{-4}$. The regular function reg.(ζ) thus goes to zero in any direction on the complex plane, and it has no poles, hence it has to be identically zero. We obtain a much stronger relation, which allows us to remove the energy momentum tensor completely from the correlator:

$$\begin{aligned}&\langle T(\zeta,\zeta^*)\ \phi_1(z_1,z_1^*)\dots\phi_m(z_m,z_m^*)\rangle\\ &= \sum_{j=1}^{m}\Big\{\frac{h_j}{(\zeta-z_j)^2} + \frac{1}{\zeta-z_j}\frac{\partial}{\partial z_j}\Big\}\langle\phi_1(z_1,z_1^*)\dots\phi_m(z_m,z_m^*)\rangle\end{aligned} \tag{2.21}$$

Using the OPE (2.18) we can even extend this to correlators with several insertions of T and $\bar{T}$. The conformal Ward identity on the full complex plane now reads:

$$\begin{aligned}&\langle T(\zeta)\ T(u_1)\dots T(u_k)\ \bar{T}(\bar{v}_1)\dots\bar{T}(\bar{v}_\ell)\ \phi_1(z_1,\bar{z}_1)\dots\phi_m(z_m,\bar{z}_n)\rangle\\ &= \sum_{j=1}^{k}\frac{c/2}{(\zeta-u_j)^4}\ \langle T(u_1)\dots T(u_{j-1})T(u_{j+1})\dots T(u_k)\ \ \bar{T}\text{'s}\ \ \phi\text{'s}\ \rangle\\ &\quad + \Big(\sum_{j=1}^{k}\Big\{\frac{2}{(\zeta-u_j)^2} + \frac{1}{\zeta-u_j}\frac{\partial}{\partial u_j}\Big\} + \sum_{j=1}^{m}\Big\{\frac{h_j}{(\zeta-z_j)^2} + \frac{1}{\zeta-z_j}\frac{\partial}{\partial z_j}\Big\}\Big)\\ &\qquad\cdot\langle T(u_1)\dots T(u_k)\ \bar{T}(\bar{v}_1)\dots\bar{T}(\bar{v}_\ell)\ \phi_1(z_1,\bar{z}_1)\dots\phi_m(z_m,\bar{z}_m)\rangle\end{aligned} \tag{2.22}$$

Together with a similar relation for $\bar{T}$ this makes it possible to express a correlator

with any number of T's and $\bar{T}$'s in terms of correlators involving only primary fields.

Hilbert space formulation

We use radial quantization on the complex plane. This means we pick out a point z_0 on the complex plane and think of the vectors in the Hilbert space as describing states on concentric circles around z_0. For the conformal theories we are considering in this thesis, there is a state field correspondence in the sense that a field inserted at z_0 can be thought of as a state and vice versa. Usually one takes z_0 to be the origin, but let us keep it arbitrary for the moment. On the upper half plane we will have to distinguish between the Hilbert space constructed around a point on the boundary and around a point in the bulk.

We can define the action of the stress tensor T and its anti holomorphic counterpart $\bar{T}$ on the Hilbert space via their modes. Define

$$\begin{aligned} L_m(z_0) &= \int_{\zeta} \; (\zeta - z_0)^{m+1} T(\zeta, \zeta^*) \; , \\ \bar{L}_m(z_0) &= \int_{\bar{\zeta}} \; (\zeta^* - {z_0}^*)^{m+1} \bar{T}(\zeta, \zeta^*) \; . \end{aligned} \tag{2.23}$$

From (2.18) one can deduce that the modes have to fulfil the Virasoro algebra Vir, that is (omitting the z_0 at each L)

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \delta_{m+n,0}\, \tfrac{c}{12}(m-1)m(m+1) \\ [L_m, \bar{L}_n] &= 0 \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \delta_{m+n,0}\, \tfrac{c}{12}(m-1)m(m+1) \; . \end{aligned} \tag{2.24}$$

Furthermore, performing the corresponding contour integral with (2.17), we get the commutation relations

$$\begin{aligned} [L_m(z_0), \phi(w, \bar{w})] &= \left(h(m+1)(w-z_0)^m + (w-z_0)^{m+1}\tfrac{\partial}{\partial w}\right) \phi(w, \bar{w}) \\ [\bar{L}_m(z_0), \phi(w, \bar{w})] &= \left(\bar{h}(m+1)(\bar{w}-\bar{z}_0)^m + (\bar{w}-\bar{z}_0)^{m+1}\tfrac{\partial}{\partial \bar{w}}\right) \phi(w, \bar{w}) \end{aligned} \tag{2.25}$$

Note that, as was the case for T, the conserved charges L_m and $\bar{L}_m$ individually do not correspond to an infinitesimal transformation on the complex plane. It is only in the combinations $L_m + \bar{L}_m$ and $i(L_m - \bar{L}_m)$ with $m \geq -1$ that do. The transformations are $z \to z + \varepsilon \cdot (z - z_0)^{m+1}$ and $z \to z + \varepsilon \cdot i\,(z - z_0)^{m+1}$, respectively.

In particular from (2.25) we see that a primary field $\phi(z_0, z_0^*)$ is annihilated by all $L_m(z_0)$'s and $\bar{L}_m(z_0)$'s with $m > 0$. This is just rephrasing the statement made

earlier, that primary fields only see the first derivative in a change of coordinates, which for $z{=}z_0$ in $z \to z + \varepsilon \cdot (z{-}z_0)^{m+1}$ and $m{>}0$ is $f'(z_0) = 1$. For a primary field this transformation looks like the identity and hence $\phi(z_0, \bar{z}_0)$ and $L_m(z_0)$ commute for $m{>}0$.

The Hilbert space thus decomposes into highest weight representations of $\mathsf{Vir} \otimes \mathsf{Vir}$. Each module is spanned by a highest weight state $|h, \bar{h}\rangle$ and an infinite set of descendent states of the form $L_{m_1} \ldots \bar{L}_{n_1} \ldots |h, \bar{h}\rangle$ with all m and n less than zero. Once we know the central charge c of the theory and the conformal weights $(h, \bar{h})$ of all primary fields we can construct the Hilbert space. As we will see in the next section some care has to be taken in the construction of a basis, since not all products of L's and $\bar{L}$'s are linearly independent.

We denote fields $\alpha(z_0)$ which commute with $L_1(z_0)$ as *quasi–primary fields*. Note that if $\alpha(z_0)$ additionally commutes with $L_2(z_0)$ it is primary, since from (2.24) we see that all L_m with $m{>}0$ can be obtained as commutators of L_1, L_2. The state $|\alpha\rangle$ corresponding to $\alpha(z)$ is also called quasi–primary and has the defining property $L_1|\alpha\rangle{=}0$.

The inner product on the Hilbert spaces is defined by the commutation relations (2.24). For two highest weight states $|i\rangle$, $|j\rangle$ we define

$$\langle i|j\rangle = \delta_{i,j} \cdot a_i \ , \tag{2.26}$$

for some constant a_i. For descendent states we define (we will come back to this in the following section)

$$(L_m)^\dagger = L_{-m} \tag{2.27}$$

and use the Vir–algebra to reduce a general inner product to the form (2.26). E.g. $\langle i|(L_{-1})^\dagger L_{-1}|j\rangle{=}\langle i|L_1 L_{-1}|j\rangle{=}\delta_{i,j} 2h_i \langle i|i\rangle$.

The operator L_0 acts like a grading operator on a Vir–module $M(h, c)$. That is, a state $|\chi\rangle{=}L_{-m_1} \ldots L_{-m_k}|h\rangle$ is an L_0 eigenstate with eigenvalue $L_0|\chi\rangle{=}(h + m_1 + \cdots + m_k)|\chi\rangle$. From (2.25) we see that the combinations $L_0{+}\bar{L}_0$ and $iL_0{-}i\bar{L}_0$ generate rescalings and rotations, respectively. Translations in turn are generated by $L_{-1}{+}\bar{L}_{-1}$ and $iL_{-1}{-}i\bar{L}_{-1}$.

The identity field $\mathbb{1}(z_0)$ corresponds to the vacuum state $|0\rangle$ in the Hilbert space $\mathcal{H}(z_0)$. From definition (2.23) it follows that $L_m|0\rangle{=}0{=}\bar{L}_m|0\rangle$ for $m{\geq}{-}1$. This implies that the vacuum is translation, rotation and scale invariant, and that it is a $\mathsf{Vir}{\otimes}\mathsf{Vir}$–highest weight vector in $\mathcal{H}$. To be precise we should call $|0\rangle$ the $sl(2)$–invariant vacuum, since e.g. for a non–unitary theory on a cylinder it is not the

Figure 2.2: Operator product expansion (OPE): The field outside the disc cannot tell the difference between two fields inside the disc or an appropriate state inserted on its boundary.

state of lowest energy and thus not the real vacuum. It will always be clear from the context whether "vacuum" refers to the state of lowest energy or the $sl(2)$–invariant state $|0\rangle$. Also, throughout this thesis the expressions *correlation function*, *n-point function*, *amplitude* and *vacuum–expectation value* all refer to the (radially ordered) vacuum–expectation value $\langle 0|\dots|0\rangle$ w.r.t. to the $sl(2)$–invariant vacuum. When we talk about physical expectation values, the identity one–point function, or partition function, $\langle 1\rangle$ is divided out explicitly.

Let $|\phi\rangle\in\mathcal{H}$ be the highest weight vector corresponding to the primary field $\phi(z,\bar{z})$. The $\mathsf{Vir}\otimes\mathsf{Vir}$ module built on $|\phi\rangle$, i.e. $|\phi\rangle$ and all its descendents, is called the *conformal family* $[\phi]$. By definition it closes under repeated infinitesimal conformal transformations.

It is much more difficult to check that the set of fields in the theory is also closed with respect to the OPE. From the Hilbert space point of view the OPE is really a statement about completeness. As shown in fig. 2.2 a field outside the circle cannot tell the difference between two fields inside the circle or an appropriate state put on the circle.

Now that we know all the states in the Hilbert space, and thus all the primary and descendent fields, we can write down the OPE of two primary fields as

$$\begin{aligned}\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_k C_{ij}{}^k\,(z-w)^{h_k-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}(\phi_k(w,\bar{w})+\\ + c_1(z-w)\cdot(L_{-1}\phi_k)(w,\bar{w}) + c_2(\bar{z}-\bar{w})\cdot(\bar{L}_{-1}\phi_k)(w,\bar{w})\\ + c_3(z-w)(\bar{z}-\bar{w})\cdot(L_{-1}\bar{L}_{-1}\phi_k)(w,\bar{w})\\ + c_4(z-w)^2\cdot(L_{-2}\phi_k)(w,\bar{w})+\dots)\end{aligned} \tag{2.28}$$

The coefficients c_n depend on the primary fields $c_n\equiv c_n(i,j,k)$. The notation $(L_{-m}A)(w,\bar{w})$ is a shorthand for $[L_{-m}(w),A(w,\bar{w})]$, i.e. $(\zeta-w)^{-m+1}T(\zeta)A(w,\bar{w})$ being integrated around a tight circle centered at w. The sum runs over all primary fields in the theory. The functional form and the constants $c_1,c_2,\dots$ in front of the descendent fields are determined by requiring that the lhs and rhs of (2.28)

describe the same state in the Hilbert space, i.e. $\langle v|lhs\rangle = \langle v|rhs\rangle$ for all states v. An example calculation can be found in appendix A.1. The constant ${C_{ij}}^k$ is called a *structure constant* and is the only nontrivial[3] input in the OPE. It is the coefficient in front of the most relevant contribution of a particular conformal family $[\phi_k]$, i.e. all descendent states vanish faster as $z \to w$.

By the commutation relation (2.24), (2.25) of the L's one can always express an amplitude involving descendent fields as a differential operator acting on an amplitude involving only primary fields. In particular, if a primary field is not present on the rhs of (2.28) none of its descendent will appear in the OPE either. This is a particularity of the Virasoro algebra not necessarily shared by more general symmetry algebras; we will briefly come back to that point in chapter 5.

As mentioned in the beginning, all global conformal transformations are of the form (2.2). These allow us to map three arbitrary points in the complex plane to three fixed points, $0, 1, 2$, say. Thus the form of the $1, 2, 3$-pt correlators is entirely fixed by the behaviour under conformal transformations:

$$\begin{aligned}
&\langle \phi_i(z,\bar z)\rangle = \delta_{h_i,0}\cdot A_i \ , \\
&\langle \phi_i(z,\bar z)\phi_j(w,\bar w)\rangle = \delta_{h_i,h_j}\delta_{\bar h_i,\bar h_j}\cdot B_{ij}\cdot (z-w)^{-2h_i}(\bar z-\bar w)^{-2\bar h_i} \ , \\
&\langle \phi_i(u,\bar u)\phi_j(v,\bar v)\phi_k(w,\bar w)\rangle = C_{ijk}\cdot (u{-}v)^{h_k-h_i-h_j}(u{-}w)^{h_j-h_i-h_k}(v{-}w)^{h_i-h_j-h_k} \\
&\qquad\qquad \cdot (\bar u{-}\bar v)^{\bar h_k-\bar h_i-\bar h_j}(\bar u{-}\bar w)^{\bar h_j-\bar h_i-\bar h_k}(\bar v{-}\bar w)^{\bar h_i-\bar h_j-\bar h_k} \ ,
\end{aligned} \tag{2.29}$$

for some constants A_i, B_{ij}, C_{ijk}. Note that we have some freedom in fixing B_{ij} by redefining the fields $\tilde\phi_i(z,\bar z){=}\lambda_i\cdot\phi_i(z,\bar z)$.

In the rest of the thesis we will assume that the identity is the only bulk field with conformal weights $(0,0)$. Furthermore we choose a basis of primary fields s.t. $B_{ij}{=}\delta_{i,j}\cdot B_i$, for some new constant B_i. An obvious choice would be $B_i{=}1$, as this simplifies the expressions somewhat. But in the way the structure constants are computed in chapter 6 it is more convenient to leave the normalisation of the two-point couplings arbitary at this stage. With these assumptions, the constants in (2.29) are linked to the OPE-coefficients (2.28) as[4]

$$A_i = \langle 1\rangle = \langle 0|0\rangle \ , \quad B_{ij} = \delta_{ij}\,{C_{ii}}^1\,\langle 1\rangle \ , \quad C_{ijk} = {C_{jk}}^i\,{C_{ii}}^1\,\langle 1\rangle \ . \tag{2.30}$$

[3] Nontrivial in the sense that the structure constants are not directly fixed by the representation theory of the Virasoro algebra. See chapters 4 and 5 for details.

[4] No summation is implied in these formulas. Sums over field labels are always written out explicitly.

More on the inner product

The state field correspondence introduced in the previous section links the normalisation of states $|\alpha\rangle$ and fields $\alpha(z,\bar{z})$:

$$|\alpha\rangle = \alpha(0,0)|0\rangle \ . \tag{2.31}$$

In this section we want to clarify the connection between fields and in/out states as well as between two-point functions and the inner product in the Hilbert space.

The aim is to construct an inner product $(a,b)=\langle a|b\rangle$ that is hermitian, i.e. $(a,b)^*=(b,a)$. Following the description in [Gab99] we take the correlators to have the property

$$\big(\ \langle\phi_1(z_1,z_1^*)\dots\phi_n(z_n,z_n^*)\rangle\ \big)^* = \langle\bar{\phi}_1(z_1^*,z_1)\dots\bar{\phi}_n(z_n^*,z_n)\rangle \tag{2.32}$$

for some fields $\bar{\phi}$. We further take the fields to be either purely real or purely imaginary in the sense that $\bar{\phi}(z,z^*)=\sigma(\phi)(-1)^{J_\phi}\cdot\phi(z,z^*)$ for some sign $\sigma(\phi)$. Recall that J_ϕ is the spin of the field ϕ. Let U be the operator implementing the Möbius transformation $z\to z^{-1}$. It has the property $U=U^{-1}$ and, since U is a global symmetry, for any collection X of fields $\langle X\rangle=\langle UXU\rangle$. Define the inner product as

$$(a,b) = \lim_{\varepsilon\to 0}\langle U\bar{a}(\varepsilon)Ub(\varepsilon)\rangle \ . \tag{2.33}$$

This is hermitian due to (2.32) and $SL(2,\mathbb{C})$–invariance:

$$(a,b)^* = \lim_{\varepsilon\to 0}\langle Ua(\varepsilon)U\bar{b}(\varepsilon)\rangle = \lim_{\varepsilon\to 0}\langle UUa(\varepsilon)U\bar{b}(\varepsilon)U\rangle = (b,a) \ . \tag{2.34}$$

The hermitian conjugate of a field $a(z,\bar{z})$ is given by $a(z,\bar{z})^\dagger = U\bar{a}(\bar{z},z)U$. For a quasi primary field $\alpha(z,\bar{z})$ this results in

$$\alpha(z,\bar{z})^\dagger = \sigma(\alpha)\bar{z}^{-2h_\alpha}z^{-2\bar{h}_\alpha}\alpha(\bar{z}^{-1},z^{-1}) \ . \tag{2.35}$$

In particular the out states corresponding to primary fields are given by

$$\langle i| = \lim_{\varepsilon\to 0}\langle 0|\phi_i(\varepsilon)^\dagger = \sigma(i)\lim_{\varepsilon\to 0}\varepsilon^{-2\Delta_\phi}\langle 0|\phi_i(\varepsilon^{-1}) \ . \tag{2.36}$$

We can employ this together with (2.29) to obtain the connection between the inner product and the two-point structure constants:

$$\langle i|i\rangle = \sigma(i)\lim_{\varepsilon\to 0}\varepsilon^{-2\Delta_\phi}\langle 0|\phi_i(\varepsilon^{-1})\phi_i(0)|0\rangle = \sigma(i)\,C_{ii}{}^1\cdot\langle 0|0\rangle \ . \tag{2.37}$$

By an appropriate choice of sign it is possible to have negative two-point functions even in a unitary theory. Changing the sign $\sigma(\phi)$ corresponds to redefining the field ϕ as $i \cdot \phi$. After a little algebra one can show that with the above definitions

$$(\, C_{ij}{}^{k}\,)^* = (-1)^{J_i+J_j+J_k} \cdot \sigma(i)\,\sigma(j)\,\sigma(k) \cdot \, C_{ij}{}^{k} \ . \tag{2.38}$$

The signs $\sigma(\phi)$ can thus be necessary if one wants to have real bulk structure constants and a unitary inner product in the presence of fields with spin.

Suppose we have a quasi–primary chiral field $J(z)$, i.e. a field that does not depend on $\bar{z}$, with integer weight h_J. By relating the mode expansion of $J(z)$ and $J(z)^\dagger$ we find

$$J_m{}^\dagger = \sigma(J) \cdot J_{-m} \ . \tag{2.39}$$

In particular, for the stress tensor T we have $\sigma(T)$=1 and $L_m{}^\dagger$=L_{-m}.

In this thesis we will treat unitary and non-unitary theories on the same footing, and it will cause no problems that the inner product $\langle i|i\rangle$ can be negative. To simplify notation we will choose all signs $\sigma(i)$=1, keeping in mind that we might end up with a non-unitary inner product even for a unitary field theory.

Torus partition function and modular invariance

Consider the map $z = f(w) = \exp(\frac{2\pi i}{R}w)$ from a cylinder of circumference R (constructed from $\mathbb{C}$ by identifying $w{\equiv}w{+}R$) to the complex plane (with coordinate z). The Hamiltonian on the cylinder is given by

$$H_{cyl} = \int_0^R T_{yy} dx \tag{2.40}$$

With the relations in table 2.1 we can express $T_{yy} = -2\pi(T + \bar{T})$ and using (2.19) we obtain the action of H_{cyl} in the Hilbert space of the complex plane

$$H_{cyl} = \frac{2\pi}{R}\Big(L_0 + \bar{L}_0 - \frac{c}{12}\Big) \tag{2.41}$$

The term $c/12$ has its origin in the Schwarzian derivative. (2.41) also implies that for the Hamiltonian to be bounded below, the Hilbert space can only consist of highest weight representations of Vir⊗Vir.

Next suppose we identify two sections of the cylinder to turn it into a torus of

length L. The partition function of this system is

$$Z(R,L) = \text{tr } e^{-L\cdot H_{cyl}} = \text{tr } q^{L_0+\bar{L}_0-c/12} \tag{2.42}$$

where $q = \exp(-2\pi L/R)$. Note that the partition function is unaltered if we change the overall size of the cylinder. Since conformal field theory is designed to describe a scale invariant system, this is what we expect[5].

The partition function can be expressed through the characters of the representations of Vir present in the Hilbert space. The character of a Vir-module $M(h,c)$ of highest weight h and central charge c is defined as

$$\chi_h(q) = q^{-c/24} \cdot \text{tr}_M(q^{L_0}) \ . \tag{2.43}$$

Recalling that L_0 gives the level of a state we see that the character will contain a term $k \cdot q^{-c/24+h+n}$ if the representation contains k states of level n. In terms of characters the partition function becomes

$$Z(R,L) = \sum_k \chi_{h_k}(q)\chi_{\bar{h}_k}(q^*) \ . \tag{2.44}$$

The sum runs over all primary bulk fields present in the theory.

Had we chosen instead the Hamiltonian $\tilde{H}_{cyl} = \int_0^L T_{xx}dy$ and used the map $\tilde{f}(w) = \exp(\frac{2\pi}{L}w)$, we would have arrived at the expression

$$Z(R,L) = \text{tr}e^{-R\cdot H_{cyl}} = \text{tr}\tilde{q}^{L_0+\bar{L}_0-c/12} = \sum_k \chi_{h_k}(\tilde{q})\chi_{\bar{h}_k}(\tilde{q}^*) \tag{2.45}$$

with $\tilde{q} = \exp(-2\pi R/L)$.

Requiring the two expressions for the partition function to coincide places severe constraints on the possible bulk field content of the theory. More generally one characterises a torus by its two periods on the complex plane. One period can be fixed to one by rescaling, and the other is $\tau = iL/R$ (in the first case). All values of τ leading to conformally equivalent tori (i.e. such that are related by rescaling) are generated by the two operations $\tau \to \tau + 1$ and $\tau \to -1/\tau$. The torus partition function should be independent of the parametrisation of the torus. This requirement is called *modular invariance* [Car86].

We have seen that different ways to calculate the torus amplitude gives con-

[5] Some care has to be taken at this point. The partition function of a CFT defined on a disc of radius r for example will in general depend on r. Roughly speaking, this is because the boundary curvature introduces a scale to the system. The scale dependence of CFT partition functions in different geometries has been considered in [CPe88].

straints on the spectrum of the theory in the bulk. As discussed shortly, different ways to calculate the amplitude of a cylinder with finite length (i.e. with boundaries) tells us about the possible conformal boundary conditions.

2.2 Minimal models

This section is a selection of facts about a class of conformal field theories called minimal models. More details can be found for example in the original work by Belavin, Polyakov and Zamolodchikov [BPZ84] or in the choice of texts mentioned in the beginning.

For certain values of the central charge the OPE of fields closes, even if the theory contains only finitely many primary fields. The only values of the central charge for which this happens are:

$$c = 1 - 6(t + t^{-1} - 2) \qquad ; \; t \in \mathbb{Q}^+ - \mathbb{Z}^+ - 1/\mathbb{Z}^+ \; . \tag{2.46}$$

In other words the central charge is parametrised by a rational number $t{=}p/q$ with p, q integers ≥ 2 that have no common divisor. The irreducible highest weight representations of the Virasoro algebra which are present in minimal models can be organised in a $(p{-}1) \times (q{-}1)$ table called the Kac-table. Let r run over $1 \ldots p{-}1$ and s over $1 \ldots q{-}1$. Then the highest weights of the corrresponding representations are

$$h_{r,s} = \frac{1}{4t}\big((r{-}st)^2 - (1{-}t)^2\big) \; . \tag{2.47}$$

These representations are degenerate, i.e. they contain null vectors. A null vector is an element of the Vir-module other than the highest weight state that is annihilated by all L_m with positive m. To obtain an irreducible representation these have to be divided out. For example in the $h_{1,2}$ representation the state $|\eta\rangle{=}(L_{-2}{-}\frac{3}{2(2h+1)}L_{-1}L_{-1})|h_{1,2}\rangle$ has this property. Note that since any L_m can be obtained from commuting L_1's and L_2's it is enough to verify $L_1|\eta\rangle = 0$ and $L_2|\eta\rangle = 0$, reducing the amount of work considerably.

For the degenerate representations just mentioned the character formulas are (e.g. [YBk])

$$\chi_{r,s}(q) = \frac{q^{h_{rs}-c/24}}{\varphi(q)} \cdot \sum_{n\in\mathbb{Z}} \big\{ q^{n(npq+qr-ps)} - q^{n(npq+qr+ps)+rs} \big\} \qquad ; \varphi(q) = \prod_{n=1}^{\infty}(1{-}q^n) \tag{2.48}$$

One checks that for $0<r<p$, $0<s<q$ the exponents entering the sum are always non–negative. It is also possible to work out the behaviour of the characters under the modular transformation $q=e^{2\pi i\tau} \to \tilde{q}=e^{-2\pi i/\tau}$. We have $\chi_k(\tilde{q}) = \sum_\ell {S_k}^\ell \chi(q)$ with

$$S_{rs}{}^{r's'} = 2^{3/2}(pq)^{-1/2}(-1)^{1+rs'+sr'} \cdot \sin(\pi rr'/t) \cdot \sin(\pi ss't) \tag{2.49}$$

The matrix S squares to one.

Explicit knowledge of S makes it possible to check if a given field content is modular invariant. If $M_{k\ell}$ is the multiplicity of a primary field with weights (h_k, h_ℓ), then modular invariance demands $SM = MS$. The solution to this condition[6] has been found by Cappelli, Itzykson and Zuber [CIZ87] leading to an ADE-type classification of all modular invariant minimal models.

The null vectors of degenerate representations give rise to differential equations on the correlation functions. For example, with the null state $|\eta\rangle$ previously mentioned we have

$$0 = \langle i|\phi_j(1,1)\phi_k(z,\bar{z})\big(L_{-2}-\tfrac{3}{2(2h+1)}L_{-1}L_{-1}\big)|h_{1,2}\rangle \ . \tag{2.50}$$

To get the differential operator one commutes the L's past the fields till they annihilate at the out state. The result is a second order differential operator of the form $\mathcal{D}(z)\langle i|\phi_j(1,1)\phi_k(z,\bar{z})|h_{1,2}\rangle=0$. For $h_{1,2}$ it actually has a general solution in terms of hypergeometric functions, given later in eqn. (4.54).

Note that the differential operator acts only on z and not on $\bar{z}$. Replacing the L's with $\bar{L}$ we get another operator, $\bar{\mathcal{D}}(\bar{z})$, acting only on the anti holomorphic part of the correlator. This implies that the correlator can be written as $\sum_{k,\ell} f_k(z)g_\ell(\bar{z})$, where $\mathcal{D}(z)f_k(z)=0$ and $\bar{\mathcal{D}}(\bar{z})g_\ell(\bar{z})=0$. The solutions to the (anti–) holomorphic differential equations are called *conformal blocks* and we will encounter them again in a more general setting in chapter 4.

2.3 CFT on the upper half plane

CFT on the upper half plane was initiated by Cardy and Cardy and Lewellen in a series of articles [Car84, Car89b, CLe91]. Apart from the original work, an introduction can be found in Cardy's Les Houches lecture notes [Car89a]. In this section some aspects of their work are described.

Once we understand CFT on the UHP, we can also calculate correlators in more

[6] It turns out that the constraint arising from $\tau \to 1+\tau$ merely multiplies the character (2.43) by a phase, resulting in the condition $h - \bar{h} \in \mathbb{Z}$. The difficult constraint arises from $\tau \to -1/\tau$. We also demand that the bulk vacuum is unique, i.e. $M_{00} = 1$.

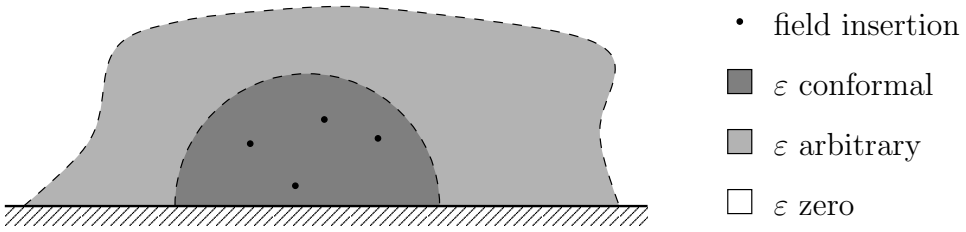


Figure 2.3: Infinitesimal transformation used to deduce properties of the stress tensor on the upper half plane.

general geometries. This is accomplished by a conformal mapping back to the UHP. One can find such a map for any simply connected proper subset of the complex plane (Riemann mapping theorem). So studying CFT on the UHP is in this sense not a restriction, but covers the general case as well.

Conformal boundary conditions

On the full complex plane we used infinitesimal conformal transformations to relate different correlators on the plane. This led to condition (2.21), which the correlators in a conformally invariant theory have to satisfy.

If the theory is defined on the UHP, conformal transformations serve two purposes. Firstly, transformations that change the geometry (e.g. from the UHP to the unit disc) make it possible to express correlators in different geometries in terms of those on the UHP. Secondly, transformations that map the UHP back to itself place constraints on the correlators on the UHP.

For the moment, we are interested in determining the correlators on the UHP, that is we need to consider infinitesimal transformations that map the UHP to itself. On the full complex plane, starting from (2.5), (2.7) we were led to the Ward identities (2.22). On the UHP the same argument leads to a slightly different answer. To see the effect of the boundary consider a collection of primary fields X and an infinitesimal transformation $\varepsilon(x, y)$ with properties (see fig. 2.3)

- $\varepsilon(x, y)$ is a continuous function UHP$\to \mathbb{C}$.
- The transformation $(x, y) \mapsto (x, y) + \varepsilon(x, y)$ maps the UHP to itself, i.e. for all x we have $\varepsilon_y(x, 0) = 0$.
- $\varepsilon(x, y)$ is conformal in a semi-disc D containing all the fields X.
- Let K be a compact set s.t. $D \subset K$. $\varepsilon(x, y)$ is arbitrary in $K - D$.
- $\varepsilon(x, y)$ is zero on $\mathbb{R}^2 - K$.

We want to apply relation (2.7),

$$\int d\Phi\, X\, \delta S\, e^{-S[\Phi]} = \int d\Phi\, \delta X\, e^{-S[\Phi]}\ , \tag{2.51}$$

to the infinitesimal transformation $\varepsilon(x,y)$. Note that up to now we did not say anything about the boundary condition along the real line. To proceed we demand that the boundary condition is conformally invariant. More precisely we demand that $\delta S\big|_D$=0. That is the action, including a segment of the boundary, does not change under the conformal transformation. δX is again given by (2.5) and as before, the rhs is independent of the values of $\varepsilon(x,y)$ outside D. Integrating the lhs by parts gives, in analogy with (2.10),

$$\int d\Phi \delta X e^{-S[\Phi]} = -\int_{\partial(K-D)} n_\nu(\vec{x})\varepsilon_\mu(\vec{x})\langle T^{\mu\nu}(\vec{x})X\rangle + \int_{K-D} \varepsilon_\mu(\vec{x})\partial_\nu\langle T^{\mu\nu}(\vec{x})X\rangle\ . \tag{2.52}$$

Here $n_\nu(\vec{x})$ is an outward pointing normal to the boundary $\partial(K-D)$. From the second term we conclude that the stress tensor is conserved away from the insertion points of other fields $\partial_\mu T^{\mu\nu} = 0$. The first term, however, gives a new condition. Each portion of the integral along the real line amounts to $\int_a^b \varepsilon_x(x,0)\langle T^{xy}(x,y)X\rangle dx$. Demanding this be independent of $\varepsilon_x(x,0)$ leads to the boundary condition for the stress tensor: $T^{xy}(x,0) = 0$.

This condition can be interpreted as the absence of energy flow across the boundary. For this point of view we take time to run along the boundary. The quantity $T_{\mu\nu}v^\nu$ describes the energy-momentum flow in direction $\vec{v}$. At the boundary this flow is always orthogonal to the boundary, i.e. only momentum, but no energy crosses the boundary. That is the case e.g. for elastic particles bouncing off a hard wall. It is however less clear in this picture why this boundary condition should be necessary for conformal invariance.

To summarise we saw that the condition $T^{xy}(x,0) = 0$ is necessary and sufficient to ensure that a boundary condition preserves conformal invariance. After a quick glance at table 2.1 we see that in complex coordinates this corresponds to $T(x,x^*)$=$\bar{T}(x,x^*)$ for all $x\in\mathbb{R}$. Note that this relation does not fix the boundary condition uniquely. It just selects a class of boundary conditions allowed by conformal invariance. The task of finding all boundary conditions consistent with the bulk theory will be addressed briefly in the next section and more thoroughly in section 5.3.

We will now come to the Ward identities on the UHP. First note that in the

bulk the argument separating the action of T and $\bar{T}$ leading to (2.15) goes through in exactly the same way. So correlators involving T and $\bar{T}$ alone are defined unambigously.

The functions $z \mapsto T(z, z^*)$ and $z \mapsto \bar{T}(z^*, z)$ are *both* analytic (i.e. any correlator involving these fields is analytic in z). Furthermore they are equal on the real line, i.e. $T(x)=\bar{T}(x)$. By analytic continuation this has to hold for all $z \in \mathbb{C}$:

$$T(z, z^*) = \bar{T}(z^*, z) \qquad ; \text{ for all } z \in \mathbb{C} \,. \tag{2.53}$$

The general Ward identity (2.16) for a collection of primary fields $\phi_1(z_1, z_1^*) \dots \phi_m(z_m, z_m^*)$ is still valid. However the argument leading to the special form (2.21) required that the only poles of $T(\zeta, \zeta^*)$ be at $\zeta = z_k$ for all k. On the UHP all we can say is that the only poles of $T(\zeta, \zeta^*)$ situated in the UHP are at $\zeta = z_k$. On the other hand we also know that all poles of $\bar{T}(\zeta, \zeta^*)$ are equally located at $\zeta = z_k$. Combining this information with condition (2.53) tells us the location of all poles of T: The correlator $\langle T(\zeta, \zeta^*)\phi_1(z_1, z_1^*) \dots \phi_m(z_m, z_m^*)\rangle_{\text{UHP}}$ has poles only at $\zeta = z_k$ and $\zeta = z_k^*$ for all k. Furthermore it decays as ζ^{-4} for $\zeta \to \infty$ along any direction. We obtain the Ward identity on the UHP:

$$\begin{aligned}
&\langle T(\zeta, \zeta^*)\ \phi_1(z_1, z_1^*) \dots \phi_m(z_m, z_m^*)\rangle_{\text{UHP}} \\
&= \Big(\sum_{j=1}^{m} \Big\{ \frac{h_j}{(\zeta - z_j)^2} + \frac{1}{\zeta - z_j} \frac{\partial}{\partial z_j} \Big\} + \sum_{j=1}^{m} \Big\{ \frac{\bar{h}_j}{(\zeta - z_j^*)^2} + \frac{1}{\zeta - z_j^*} \frac{\partial}{\partial \bar{z}_j} \Big\} \Big) \\
&\qquad \cdot \langle \phi_1(z_1, z_1^*) \dots \phi_m(z_m, z_m^*)\rangle_{\text{UHP}}
\end{aligned} \tag{2.54}$$

There also is a straightforward extension to correlators with several T and $\bar{T}$ insertions in addition to the primary fields ϕ, corresponding to (2.22).

Comparing the full plane and UHP Ward identities (2.21) and (2.54) we see that the two correlators

$$\begin{aligned}
&\langle T(\zeta, \zeta^*)\ \phi_1(z_1, z_1^*) \dots \phi_m(z_m, z_m^*) \bar{\phi}_1(z_1^*, z_1) \dots \bar{\phi}_m(z_m^*, z_m)\rangle_{\mathbb{C}} \,, \\
&\langle T(\zeta, \zeta^*)\ \phi_1(z_1, z_1^*) \dots \phi_m(z_m, z_m^*)\rangle_{\text{UHP}} \,,
\end{aligned} \tag{2.55}$$

have the same singularities in ζ. Here the notation $\bar{\phi}_k(w, \bar{w})$ stands for a primary field with conformal weights $(\bar{h}_k, h_k)$ if the field $\phi_k(w, \bar{w})$ has weights $(h_k, \bar{h}_k)$.

We want to argue that, in a way made precise below, we can think of a correlator on the UHP as a correlator on the full complex plane by reflecting all fields at the real axis. We will denote this procedure, which is due to Cardy [Car84] as the *doubling trick*.

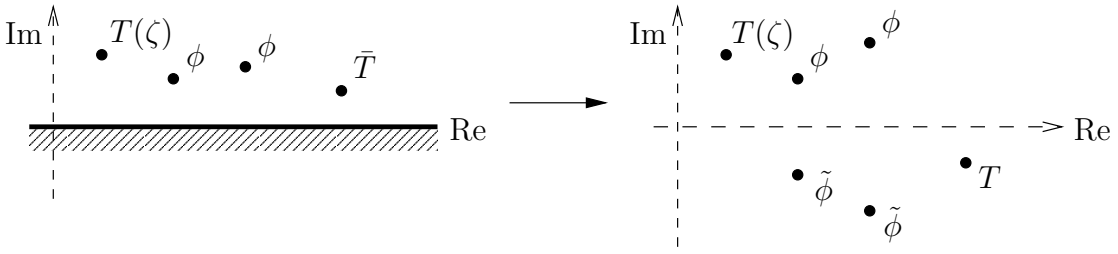


Figure 2.4: Cardy's doubling trick to relate correlators on the UHP and the full complex plane. Both correlators have the same singularities in ζ.

The relation (2.55) extends to correlators with several T and $\bar{T}$ fields as well (fig. 2.4). In this case $\bar{T}$ on the UHP becomes a T on the LHP after reflection. Correspondingly $\bar{L}_{-n}$ reflects to L_{-n} and so forth. Note that the reflection of T itself is the identity, since it is a $(2,0)$ quasiprimary field.

Recall the procedure outlined in section 2.2 that gave rise to differential equations on the correlators. The precise formulation of Cardy's doubling trick is the following: Imagine a correlator of n primary fields in a minimal model on the UHP. Dress one of the primary fields, say ϕ_1, with L_{-n}'s so that it becomes a null state. This gives rise to a differential equation on the UHP correlator. The doubling trick implies that this differential equation is exactly the same as the corresponding equation of the ϕ_1–null state in a full plane correlator with $2n$ primary fields as in (2.55).

We will see in the next section how we can arrive at the same picture by deforming contours on the UHP. The contour of a charge L_n will turn into an $\bar{L}_n$ integral after hitting the boundary.

To obtain the correlators in minimal models we will later construct the conformal blocks, that is all solutions to the null state differential equations. The doubling trick tells us that we can use the same set of functions (in different combinations) to describe both full plane and UHP correlators.

A unifying approach to treat correlation functions on the full plane, the UHP and more general two dimensional surfaces (e.g. non–orientable ones) from the point of view of three dimensional topological field theory has been developed by Felder et al. [FFFS99].

Hilbert space formulation

There are two equally important Hilbert space formulations of the theory on the UHP. One is the bulk description obtained by radial quantisation around a point z_0 with $\mathrm{Im}(z_0) > 0$. From this point of view the boundary is an out-state. In the other formulation the Hilbert space is centered at a point on the boundary. Here

the boundary condition is encoded in the composition of the Hilbert space.

We will address the bulk construction first. Even though $T(z_0, z_0^*)$ and $\bar{T}(z_0, z_0^*)$ are linked via analytic continuation, there is no *local* relation between the two. That is, we cannot express $\bar{T}(z_0, z_0^*)$ as a function of $T(z_0, z_0^*)$ and a *finite* number of derivatives thereof. Correspondingly there is no direct relationship between $L_n(z_0)$ and $\bar{L}_n(z_0)$ acting on fields (or states) at z_0 and we have a situation exactly equivalent to the bulk, with two copies of Vir acting on the Hilbert space:

$$\mathsf{Vir} \otimes \mathsf{Vir} \triangleright \mathcal{H}(z_0) \quad \text{for } \mathrm{Im}(z_0) > 0. \tag{2.56}$$

Intuitively this is what we expect from the statistical mechanics point of view. There the Hilbert space at z_0 describes the possible microscopic configurations on small circles centered at z_0. In this sense the stucture of the Hilbert space is a local property of the theory and should not be influenced by the boundary unless it passes directly through z_0.

To give an example of an out-state that describes a boundary, consider the unit disc, centered at zero. The boundary is the unit circle, and a basis for the corresponding boundary states is given by the so–called Ishibashi states [Ish89]. Let u be the coordinate on the unit disc and z on the UHP. Then $z = f(u) = -i\frac{u-1}{u+1}$ maps the unit disc to the UHP. One verifies that the UHP-boundary condition $T = \bar{T}$ becomes $e^{2i\theta}T(u, u^*) = e^{-2i\theta}\bar{T}(u, u^*)$ for $u = e^{i\theta}$. In particular we see that the condition for a boundary to be conformal depends on the shape of the boundary. Recall the definition (2.23) of the operators L_n. For a contour running on the circular boundary one can check that the T-boundary condition on the unit disc implies $L_n = \bar{L}_{-n}$. We are thus looking for a state $\langle B|$ with the property $\langle B|L_n = \langle B|\bar{L}_{-n}$. The Ishibashi states $\langle\langle i|$ solve this condition and can be constructed as follows: Choose a Vir⊗Vir highest weight state $|i\rangle \in \mathcal{H}$ which has the same left/right conformal weight (h_i, h_i). Let v_k be a basis of the Vir-module $M(h_i, c)$, s.t. each v_k is of the form $L_{-n_1} \dots L_{-n_N}|i\rangle$ and v_0 is the highest weight vector $|i\rangle$. Then the inner product matrix $g_{k\ell} = \langle v_k|v_\ell\rangle/\langle v_0|v_0\rangle$ is real. With $G{=}g^{-1}$ the Ishisbashi state, as an out–state, is given by

$$\langle\langle i| = \sum_{k,\ell} G_{k\ell}\langle v_k \otimes v_\ell| = \langle i| + \dots \tag{2.57}$$

That it has the required property can be seen in two ways. First of all, by direct computation one can show that $\langle\langle i|(L_n - \bar{L}_{-n})|v_m \otimes v_n\rangle = 0$ for all m, n. A more conceptual proof due to Watts [prW] is to understand an Ishibashi state as an intertwiner of two Vir representations. Since the Hilbert space is built from irreducible

representations, by Schur's lemma the left and right representation have to be equal. The space of intertwiners is spanned by projectors on the individual irreps. Written as a vector in the Hilbert space[7] the projector takes the form (2.57).

Two facts are worth pointing out: Firstly, by construction there is exactly one Ishisbashi state for every diagonal primary bulk field. Secondly, from the above consideration it seems that the set of conformal boundary conditions forms a vector space. However, as we will see later in section 5.3, there are additional constraints which reduce this continuum to a discrete set of possible conformal boundary conditions.

Let us now proceed by constructing the Hilbert space $\mathcal{H}(x_0)$ around a point x_0 on the real axis. From the previous section we know that on the UHP $T(z,z^*) = \bar{T}(z^*,z)$ holds for all $z\in\mathbb{C}$. Define the action of $L_n(x_0)$ and $\bar{L}_n(x_0)$ by contour integration as in (2.23). Consider the contour $\gamma(t) = R\exp(it)$ to be a full circle around x_0, which does not intersect any other field insertions. Values of T and $\bar{T}$ for the half of the contour which lie outside the UHP are obtained by analytic continuation. By reversing the direction of integration one quickly checks that, for any n and any $z_0 \in \mathbb{C}$,

$$\begin{aligned} L_n(z_0) &= \frac{1}{2\pi i}\int_0^{2\pi} (\zeta - z_0)^{n+1} T(\zeta,\zeta^*)\Big|_{\zeta=\gamma(t)} \gamma'(t)dt \\ &= \frac{-1}{2\pi i}\int_0^{2\pi} (\zeta - z_0)^{n+1} \bar{T}(\zeta^*,\zeta)\Big|_{\zeta=\gamma(-t)} \gamma'(-t)dt = \bar{L}_n(z_0^*)\ , \end{aligned} \tag{2.58}$$

since we have $\gamma(-t) = \gamma(t)^*$. In particular this implies that for $x_0 \in \mathbb{R}$ the identity $L_n(x_0) = \bar{L}_n(x_0)$ holds inside any correlator. We conclude that $L_n(x_0)$ and $\bar{L}_n(x_0)$ describe the same operator on $\mathcal{H}(x_0)$. Thus, in contrast to the bulk (2.56), only one copy of Vir acts on a Hilbert space centered on the boundary:

$$\mathsf{Vir} \triangleright \mathcal{H}(x_0) \quad \text{for } x_0 \in \mathbb{R}. \tag{2.59}$$

Invoking again the state field correspondence we can interpret the highest weight states of $\mathcal{H}(x_0)$ as primary fields living on the boundary. Correspondingly descendent states become descendent fields. One may question whether the Hilbert space $\mathcal{H}(x_0)$ is composed of highest weight representations. However just as in the bulk, the highest weight property is equivalent to the energy being bounded from below. To see this, consider the conformal transformation $z = f(w) = \exp(\frac{i\pi}{R}w)$ from an

[7] Actually the Ishibashi state $\langle\langle i|$ is not really a state, but only an element of $\mathrm{End}M(h_i,c)$ (to be seen as the completion of $M(h,c)\otimes M(h,c)^*$). As a formal element in the completion of $M(h,c)\otimes M(h,c)$ it would have infinite norm. In this thesis we will nonetheless think of it as a state, keeping in mind this subtlety.

infinite strip of width R (coordinate w, left boundary is the imaginary axis, the right boundary is the imaginary line through $R\in\mathbb{R}$) to the UHP (coordinate z). As for the infinite cylinder (2.40) we obtain the Hamiltonian as

$$H_{strip}(R) = \int_0^R T_{yy}(x,y)dx = \frac{\pi}{R}\Big(L_0 - \frac{c}{24}\Big)\ . \tag{2.60}$$

In this expression the two semicircular integration contours of T and $\bar{T}$ on the UHP originating from $\int T_{yy}$ have been combined into a full circular integration of T, represented by L_0. Whenever an L_n appears without argument it is a shorthand for $L_n(0)$.

The Hilbert space depends on precisely which conformal boundary condition we have on either side of the strip. We label the boundary conditions with a, b and the Hilbert space with $\mathcal{H}_{ab}$.

Back on the UHP this corresponds to having the conformal boundary condition a on the positive real axis and b on the negative real axis. The fields corresponding to the states in $\mathcal{H}_{ab}$ are now interpreted as *boundary (condition) changing fields*, which we will denote by $\psi^{(ab)}(x)$.

From the point of view of correlation functions the existence of boundary changing fields can be inferred as follows: As a bulk field $\phi(z,z^*)$ approaches the boundary the correlator will display a singular behaviour characteristic for the boundary condition at that part of the real line. If the boundary condition changes at zero, then the correlator will in general display a different singular behaviour for $z\to 0-\varepsilon$ and $z\to 0+\varepsilon$. The only places where we allow the correlator to have discontinuities/singularities is at the insertion points of other fields or at boundaries. In the present situation, on top of the singularity induced by the boundary, we have a discontinuity in the $z\to 0\pm\varepsilon$ behaviour. This we interpret as a boundary changing field.

If the boundary condition is a on both sides of the strip, then on the UHP the whole real line carries the same boundary condition a. In this case we interpret the fields corresponding to the states in the Hilbert space $\mathcal{H}_{aa}$ as boundary fields living on the a–boundary, denoted by $\psi^{(aa)}(x)$.

Formally fields living on a given boundary and boundary changing fields are treated on the same footing. The only difference is in interpretation, since it is more intuitive to think of $\psi^{(aa)}(x)$ as a degree of freedom of the a–boundary, than as a field changing the a–boundary condition to the a–boundary condition. In the following, the term "boundary field" can refer to either of the two possibilities.

Consider a semi disc D on the UHP centered at zero. As in the bulk we demand

the Hilbert space $\mathcal{H}_{ab}$ to be complete in the sense that a (bulk or boundary) field outside D cannot tell the difference between a collection of fields inside D or an appropriate state $|\chi\rangle$ placed at zero. Placing either a bulk field or two boundary fields inside D gives rise to two new OPEs.

We can expand a bulk field $\phi(x+iy)$ (scaling dimension $\Delta = h+\bar{h}$) close to a boundary with boundary condition a in terms of boundary fields $\psi_k^{(aa)}(x)$ as follows:

$$\phi(x+iy) = \sum_k {}^aB_\phi{}^k\,(2y)^{h_k-\Delta}\big(\psi_k^{(aa)}(x) + c_1 y(L_{-1}\,\psi_k^{(aa)})(x) + \dots\big) \tag{2.61}$$

This defines the bulk-boundary couplings ${}^\alpha B_\phi{}^k$. For two boundary fields $\psi_i^{(ab)}(x)$ and $\psi_j^{(bc)}(y)$, with $x>y$, we get the short distance expansion

$$\psi_i^{(ab)}(x)\,\psi_j^{(bc)}(y) = \sum_k C_{ij}^{(abc)k}\,(x-y)^{h_k-h_i-h_j}\big(\psi_k^{(aa)}(y) + c_1(x-y)(L_{-1}\,\psi_k^{(ac)})(y) + \dots\big)\,, \tag{2.62}$$

defining the boundary structure constants $C_{ij}^{(abc)k}$.

In chapter 6 we will see how the three sets of structure constants $C_{ij}{}^k$, ${}^aB_i{}^k$, $C_{ij}^{(abc)k}$ describing a conformal field theory on the upper half plane can be calculated in the case of minimal models.

As in the bulk some simple amplitudes are already fixed by global conformal invariance, i.e. by Möbius transformations (2.2) with $a,b,c,d\in\mathbb{R}$. The resulting correlators are

$$\begin{aligned}
&\langle\,\psi_i^{(aa)}(x)\rangle_{\text{UHP}}^a = \delta_{i,0}\langle 1\rangle_{\text{UHP}}^a\,,\\
&\langle\,\psi_i^{(ab)}(x)\,\psi_j^{(ba)}(y)\rangle_{\text{UHP}}^a = \delta_{i,j}\,C_{ii}^{(aba)1}\,\langle 1\rangle_{\text{UHP}}^a\cdot(x-y)^{-2h_i} && ;\ x>y\,,\\
&\langle\phi_\ell(x+iy)\rangle_{\text{UHP}}^a = \delta_{J_\ell,0}\,{}^aB_\ell{}^1\,\langle 1\rangle_{\text{UHP}}^a\cdot(2y)^{-\Delta_\ell} && ;\ y>0\,,\\
&\langle\,\psi_i^{(ab)}(u)\,\psi_j^{(bc)}(v)\,\psi_k^{(ca)}(w)\rangle_{\text{UHP}}^a = C_{jk}^{(bca)i}\;C_{ii}^{(aba)1}\,\langle 1\rangle_{\text{UHP}}^a\\
&\qquad\cdot(u-v)^{h_k-h_i-h_j}(u-w)^{h_j-h_i-h_k}(v-w)^{h_i-h_j-h_k} && ;\ u>v>w\,,\\
&\langle\,\psi_i^{(aa)}(r)\phi_\ell(x+iy)\rangle_{\text{UHP}}^a = {}^aB_\ell{}^i\;C_{ii}^{(aaa)1}\,\langle 1\rangle_{\text{UHP}}^a\\
&\qquad\cdot(2y)^{-\Delta_\ell}\big((x-r)^2+y^2\big)^{-h_i}\Big(\frac{x-r-iy}{x-r+iy}\Big)^{J_\ell} && ;\ y>0\,.
\end{aligned} \tag{2.63}$$

Here Δ_ℓ and J_ℓ are the scaling dimension and the spin of the bulk field ϕ_ℓ. An important assumption we made when writing down (2.63) is that only the identity on a given boundary $\mathbb{1}^a$ has scaling dimension $h=0$. This is only true for a special set of boundary conditions, which will be called "pure". All other allowed boundary

conditions are superpositions of pure ones, and for these the assumption fails. We will come back to that in section 5.3. With the assumption that the identity is unique, one can always pick a basis of boundary fields, s.t. the boundary two point functions take the above form. The label a in the UHP correlator $\langle \ldots \rangle^a_{\text{UHP}}$ stands for the boundary condition at infinity. When writing down correlators we assume that there is no field insertion at infinity and hence the correlator is nonzero only if the boundary condition is the same at left/right infinity.

Chapter 3.

OPE on Curved Boundaries

In the previous section we have considered the behaviour of boundary fields under transformations that map the real line to itself, i.e. preserve the boundary on the UHP.

This chapter deals with conformal maps that change the shape of the boundary. There is some freedom in the definition of the transformation behaviour of boundary fields. In the choice we make boundary fields will only see reparametrisations of the boundary, and not deformations leaving the boundary length invariant.

We will only consider sufficiently well behaved boundaries (to be made precise below), essentially we demand the boundary to be smooth. With the above definition for boundary fields we find that the boundary–boundary OPE on a curved boundary is identical to the UHP. The local boundary shape enters in the bulk–boundary OPE through the boundary curvature and derivatives thereof. Global aspects like the boundary length or the presence of other boundaries enter only on the level of one–point functions of boundary fields. This is consistent with the idea that the OPE depends only on local properties of the theory and all global information is encoded in the one–point functions.

In the final section of this chapter two applications of the formalism are presented.

3.1 Transformation behaviour of boundary fields

In the bulk (2.3) two numbers were assigned a field: The scaling dimension Δ describes the behaviour under global rescalings and the spin J the behaviour under global rotations. For a primary bulk field Δ and J are linked to the left/right conformal weights via $\Delta = h+\bar{h}$ and $J = h-\bar{h}$. We will now define similar quantities for boundary fields.

On the UHP let $t(x)$ denote the restriction of the stress tensor $T(z, \bar{z})$ to the boundary. Since $T = \bar{T}$ on the real line there is no ambiguity. A primary boundary

field $\psi(x)$ has the OPE

$$t(x)\psi(y) = \Big(\frac{h_\psi}{(x-y)^2} + \frac{1}{x-y}\frac{\partial}{\partial y}\Big)\psi(y) + \mathrm{reg}(x-y)\ . \tag{3.1}$$

The main difference to the corresponding bulk formula (2.17) is the absence of the complex conjugate term, since only one copy of Vir acts on boundary fields. The transformation behaviour corresponding to (3.1) is, for a conformal map $f(z)$: UHP $\to$ UHP,

$$\psi(x) \xrightarrow{x\mapsto x'=f(x)} f'(x)^{h_\psi}\psi(x')\ . \tag{3.2}$$

In particular this tells us that the scaling dimension of a primary boundary field is h_ψ. In addition we can assign a spin s_ψ to the boundary field and demand the behaviour under a global rotation by angle θ to be $\psi(x) \to e^{is_\psi\theta}\psi(x)$. For a primary boundary field $\psi(x)$ of scaling dimension h_ψ and spin s_ψ the transformation behaviour under an arbitrary analytic function f : UHP $\to \mathbb{C}$ then is

$$\begin{aligned}\psi(x) &\xrightarrow{x\mapsto x'=f(x)} |f'(x)|^{h_\psi} e^{is_\psi \arg f'(x)}\psi(x') \\ &= f'(x)^{\frac{h_\psi+s_\psi}{2}}(f'(x)^*)^{\frac{h_\psi-s_\psi}{2}}\psi(x')\ .\end{aligned} \tag{3.3}$$

We will now investigate two implications of the transformation behaviour (3.3). First consider the unit disc with a bulk field inserted at zero and a boundary field at $e^{i\theta}$. The disc (coordinate u) can be transformed back to the UHP using a Möbius transformation f : disc $\to$ UHP

$$\begin{aligned} z = f(u) &= -i\frac{u-1}{u+1} & f(e^{i\theta}) &= \tan\tfrac{\theta}{2} \\ f'(u) &= \frac{-2i}{(u+1)^2} & f'(e^{i\theta}) &= \frac{e^{-i(\theta+\pi/2)}}{2(\cos\frac{\theta}{2})^2}\ . \end{aligned} \tag{3.4}$$

The one-bulk one-boundary field correlator on the UHP is given in (2.63). For the correlator on the disc we get

$$\begin{aligned}\langle\phi(0)\psi(e^{i\theta})\rangle_{\mathrm{disc}} &= f'(0)^{h_\phi}(f'(0)^*)^{\bar h_\phi} f'(x)^{\frac{h_\psi+s_\psi}{2}}(f'(x)^*)^{\frac{h_\psi-s_\psi}{2}}\frac{\langle\phi(i)\psi(\tan\frac{\theta}{2})\rangle_{\mathrm{UHP}}}{\langle 1\rangle_{\mathrm{UHP}}}\cdot\langle 1\rangle_{\mathrm{disc}} \\ &= \tilde C e^{-i\theta(s+J)}\ ,\end{aligned} \tag{3.5}$$

for some constant $\tilde C$. First of all we note that the correlator (3.5) will be single

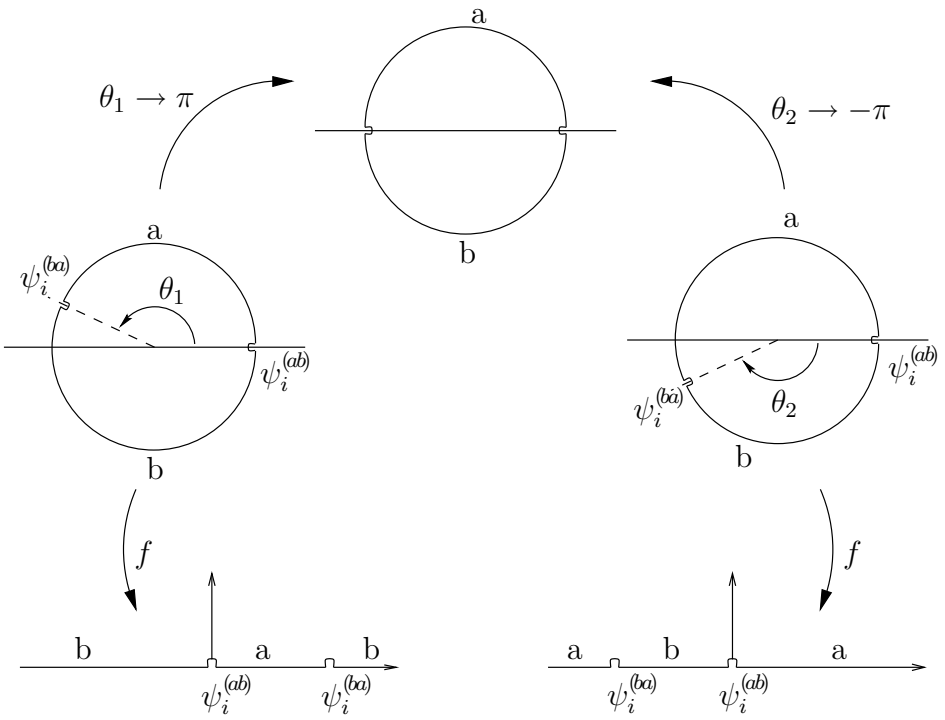


Figure 3.1: Two limits of the disc two-point function and their corresponding situations on the UHP[2]

valued only for

$$s + J \in \mathbb{Z} \, . \tag{3.6}$$

One might imagine situations involving disorder fields in which it is desirable to pick up a sign, say, when taking the boundary field once around the disc. This thesis will, however, only be concerned with local fields and we demand all correlators to be single valued.

By just looking at (3.3) the choices $s = \pm h$ look quite natural. They correspond to the boundary field transforming with $f'(x)^h$ or $(f'(x)^*)^h$, respectively. In this case correlators would be (anti–) analytic functions in $u{=}f(x)$ even for curved boundaries. In the light of (3.6) it is however clear that neither of the two choices would in general be consistent with single valuedness of the correlator and is unsuitable for local fields.

Another conclusion we can draw from (3.6) is that fermionic[1] bulk fields can only couple to fermionic boundary fields.

As a second implication of the transformation behaviour (3.3) consider the boundary two point function on the unit disc. We take the disc to have two different boundary conditions a and b and the two fields to be the associated boundary

[1] Here fermionic is meant to denote fields of half–integer spin $s \in \mathbb{Z}{+}\frac{1}{2}$.

changing fields (see figure 3.1)[2]. Again we use the transformation (3.5) to relate the disc correlator to the UHP correlator given in (2.63). Due to the requirement $x>y$ in (2.63) there are two domains:

$$\begin{aligned}\theta_1 > 0 :&\langle\, \psi_i^{(ab)}(e^{i\theta_1})\, \psi_i^{(ba)}(1)\rangle_{\text{disc}} = C_{ii}^{(bab)1}\, \langle 1\rangle^b_{\text{disc}} e^{-i\frac{\pi}{2}(s_{ab}+s_{ba})} e^{-i\theta_1 s_{ba}} \big(\sin\tfrac{\theta_1}{2}\big)^{-2h_i} \\ \theta_2 < 0 :&\langle\, \psi_i^{(ab)}(1)\, \psi_i^{(ba)}(e^{i\theta_2})\rangle_{\text{disc}} = C_{ii}^{(aba)1}\, \langle 1\rangle^a_{\text{disc}} e^{-i\frac{\pi}{2}(s_{ab}+s_{ba})} e^{-i\theta_2 s_{ba}} \big(-\sin\tfrac{\theta_2}{2}\big)^{-2h_i}\end{aligned} \tag{3.7}$$

For the correlator to be continuous on the disc we demand the limits as $\theta_1 \to \pi$ and $\theta_2 \to -\pi$ to coincide. It follows that

$$C_{ii}^{(bab)1}\, \langle 1\rangle^b_{\text{disc}} = e^{2\pi i s_{ba}}\, C_{ii}^{(aba)1}\, \langle 1\rangle^a_{\text{disc}}\ . \tag{3.8}$$

This relation is quite interesting since it links boundary structure constants and identity one-point functions on the unit disc. Thinking of path integrals, $\langle 1\rangle^a_{\text{disc}}$ can be interpreted as the partition function of the unit disc with boundary condition a. In chapter 6 it is shown that, independent of the normalisation of the fields, in general one has $C_{ii}^{(bab)1} \neq C_{ii}^{(aba)1}$. In particular it would be inconsistent to set all $\langle 1\rangle^a_{\text{disc}}$ to one.

We will mainly be concerned with minimal models. In this case all bulk fields have integer spin $J \in \mathbb{Z}$. The aim later is to work out the structure constants of local fields. Since neither fermions nor disorder fields will enter our considerations, it is a perfectly good choice to demand

$$s = 0 \tag{3.9}$$

for all boundary fields.

3.2 OPEs on curved boundaries

In this section we will compute the bulk-boundary OPE next to a smoothly curved boundary. In fact we demand the boundary to be analytic in a sense made precise below. The reason for this restriction is that such boundaries look straight when one gets close enough and the leading behaviour of the bulk-boundary OPE remains

[2] This figure also illustrates a problem with the notation. Inside correlators we order boundary fields s.t. more positive insertion points are more to the left. In (2.62), (2.63) we defined the notation s.t. for $x > y$, $\psi^{(ab)}(x)\,\psi^{(bc)}(y)$ is an allowed combination, whereas $\psi^{(ba)}(x)\,\psi^{(cb)}(y)$ is not. When drawing a picture, however, the ordering is inversed, i.e. $\psi^{(ab)}(x)\,\psi^{(bc)}(y)$ appears as $\psi^{(bc)}(y)\,\psi^{(ab)}(x)$, which looks wrong. This point should be kept in mind when translationg correlators into pictures. In schematic illustrations we will sometimes circumvent this problem by having the real axis point to the left, as e.g. in (5.20)–(5.22) or (A.70).

unchanged. This is not true for example for corners on the boundary (see [Car89a]) or fractal boundaries (see [Car98]), which can also modify the leading asymptotics as a bulk field approaches the boundary.

In the previous section we have seen that correlators on any simply connected subset of the plane can be computed by mapping it back to the UHP. However in situations where the map to the UHP is not known explicitly it might still be useful to have a power series expansion of the correlator, and this is provided by the OPE. Consider for example a surface with several disjoined boundaries. The OPE is a local property and does not change if additional boundaries are introduced. The map to the UHP on the other hand might no longer exist. From the point of view of the OPE all global properties are encoded in the one-point functions on the boundary. An example of this will be given in section 3.4.

Descendent fields on curved boundaries

For the calculations that follow it will be useful to introduce the arc-length map γ. Consider a region R of the complex plane with boundary B of length L. Initially the arc-length map $\gamma_R : [0, L] \to B$ is defined as an invertible map from the real interval $[0, L]$ to the boundary B, s.t. $\left|\frac{d}{ds}\gamma_R(s)\right| = 1$. If there exists a complex neighbourhood $U \supset [0, L]$ and an analytic function $\gamma : U \to B$ s.t. $\gamma\big|_{[0,L]} = \gamma_R$, we call B an *analytic boundary*. To fix the direction of the tangent $\dot{\gamma}(s) := \frac{d}{ds}\gamma(s)$ we demand that points z with $\mathrm{Im}(z) > 0$ are mapped to the interior of the region R.

Now let $f : R \to \mathbb{C}$ be a conformal transformation. Then the reparametrisation $\tilde{f}$ of the boundary induced by f can be written as

$$\tilde{f} = \hat{\gamma}^{-1} \circ f \circ \gamma \ . \tag{3.10}$$

Here $\gamma(s)$ is the arc-length map for R and $\hat{\gamma}(s)$ is the arc-length map for $f(R)$. $\tilde{f}(s)$ has the property $\frac{d}{ds}\tilde{f}(s) = |f'(\gamma(s))|$.

In section 2.3 we obtained boundary fields on the UHP via the state field correspondence. The space of fields living on the real line with boundary condition a is given by the conformal families $[\psi_i^{(aa)}]$. Descendents are obtained by contour integration of T, i.e. by action of all $L_{-m}(x)$. A more intrinsic way to define descendents of boundary fields is as follows: Define $t(x)$ to be the stress tensor on the boundary. For the UHP we take it to be the restriction of $T(z, \bar{z})$ to the real line.

On an arbitrary boundary we define the modes of a boundary field $\chi(s)$ as

$$t(s{+}d)\ \chi(s) = \sum_{m=1}^{\infty} (\ell_{-m}\chi)(s) \cdot d^{m+2}\ , \tag{3.11}$$

where d is the oriented distance between t and χ, measured in arc-length. On the UHP one can check that both definitions agree: $\ell_{-m}\chi{=}L_{-m}\chi$.

Up to now we only defined $t(s)$ on the boundary of the UHP. To define it on a general (analytic) boundary we take the transformation behaviour to be

$$t(s) \xrightarrow{s'=\tilde{f}(s)} \tilde{f}'(s)^2 \cdot t(s') + \tfrac{c}{12}\{\tilde{f};s\} \cdot \mathbb{1}\ . \tag{3.12}$$

For a point x on the boundary of the UHP we have $T(x){=}\bar{T}(x){=}t(x)$. The corresponding condition for a general boundary can be obtained by using (3.12) with the arc-length map γ. Let x be a point on the real line and $u{=}\gamma(x)$ a point on the curved boundary. Then

$$t(u) = \gamma'(x)^2 \cdot T(u,u^*) + \tfrac{c}{12}\{\gamma;x\} \cdot \mathbb{1} = (\gamma'(x)^*)^2 \cdot \bar{T}(u,u^*) + \tfrac{c}{12}\big(\{\gamma;x\}\big)^* \cdot \mathbb{1}\ . \tag{3.13}$$

In particular, since the relation between T and $\bar{T}$ is now more complicated, the choice $t(u) = T(u,u^*)$ would not be natural on curved boundaries.

In (3.13) we used the property that, if the real line is mapped to the curved boundary with the arc-length map, no boundary reparametrisation is induced. Hence the boundary stress tensor does not change. Equally, with the choice (3.9) primary boundary fields transform as $\psi{\to}\tilde{f}'(s)^h\psi(s')$. Consider the equation

$$t(s{+}d)\ \psi(s) = \sum_{m=1}^{\infty} (\ell_{-m}\psi)(s) \cdot d^{m+2}\ . \tag{3.14}$$

Let f be a conformal transformation that does not reparametrise the boundary: $\tilde{f}(s){=}s$. Under such a map the lhs of (3.14) does not change. Recursively we see that none of the descendent fields changes under a transformation with $\tilde{f}{\equiv}\mathrm{id}$.

Boundary–boundary OPE

With the choice of basis in the set of boundary fields we made with (3.11) and (3.12), the boundary-boundary OPE $\psi(s{+}d)\ \psi(s)$ is independent of the boundary shape and identical to the corresponding OPE on the UHP. As described in appendix A.1

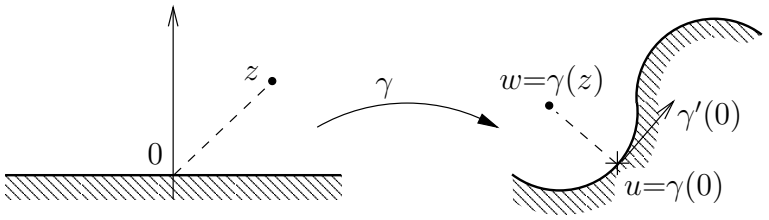


Figure 3.2: Mapping of the UHP to a region with curved boundary

we get

$$\begin{aligned}\psi_i^{(ab)}(s{+}d)\,\psi_j^{(bc)}(s) = \\ \sum_k C_{ij}^{(abc)k}\, d^{h_k-h_i-h_j}\cdot\Big\{\ \psi_k^{(ac)}(s)\ +\ d\cdot\tfrac{h_i-h_j+h_k}{2h_k}\cdot(\ell_{-1}\,\psi_k^{(ac)})(s) \\ +\ d^2\cdot\big(\tfrac{(h_i-h_j+h_k)(h_i-h_j+h_k+1)}{4h_k(2h_k+1)}\cdot(\ell_{-1}\ell_{-1}\,\psi_k^{(ac)})(s)+A_{ij}^{\alpha}\,\psi_\alpha^{(ac)}(s)\big)\ +\ \ldots\Big\}\,.\end{aligned} \tag{3.15}$$

Here $\psi_\alpha^{(ac)}(s)$ denotes the level two quasi–primary state given in table A.1 and the coefficient A_{ij}^{α} can be found in (A.11). The expansion is valid for $d > 0$, i.e. one reaches the field $\psi_i^{(ab)}(s+d)$ by moving from $\psi_j^{(bc)}(s)$ for a distance d along the boundary, in the direction indicated by $\gamma'(s)$.

Bulk–boundary OPE

One way to obtain the bulk-boundary OPE for curved boundaries is to use the arc-length map to transform the OPE from the UHP to the curved boundary. Say we have a bulk field $\phi_i(w)$ and u is the closest point to w on the curved boundary. We can always shift γ such that $\gamma(0){=}u$. Let $z{=}x{+}iy$ be the point that is mapped to w (see fig. 3.2).

When evaluating the OPE on the UHP, there is a slight complication because the bulk field at $z{=}x{+}iy$ has to be expressed in terms of boundary fields at 0, and not x. The expansion in terms of fields at x is given in appendix A.1 and is of the form:

$$\phi_i(x{+}iy) = \sum_k {}^aB_i{}^k\,(2y)^{h_k-\Delta}\sum_N y^N\cdot\psi_{k,N}(x) \tag{3.16}$$

The field $\psi_{k,N}(x)$ can be translated back to the origin using $\psi_{k,N}(x)\ =$

$\exp(xL_{-1})|k,N\rangle$. Expanding out the exponential we find

$$\phi_i(x+iy) = \sum_k {}^aB_i{}^k\,(2y)^{h_k-\Delta} \sum_{N,k} y^N x^k\ \frac{1}{k!}\cdot {L_{-1}}^k|k,N\rangle\ . \tag{3.17}$$

For brevity, let us denote the field corresponding to the state $k!^{-1}{L_{-1}}^k|k,N\rangle$ with $\chi_{N,k}(0)$. Now we apply the conformal transformation γ to both sides of (3.17). The lhs picks up the factor $\gamma'(x+iy)^{h_\phi}(\gamma'(x+iy)^*)^{\bar h_\phi}$. Since the arc-length map does not rescale the boundary, the RHS of (3.17) is not affected by the transformation. Rearranging terms we get

$$\phi_i(w) = \gamma'(x+iy)^{-H}(\gamma'(x+iy)^*)^{-\bar H} \sum_k {}^aB_i{}^k\,(2y)^{h_k-\Delta} \sum_{N,k} y^N x^k\cdot\chi_{N,k}(0)\ . \tag{3.18}$$

The coordinates x and y have to be expressed in terms of the distance to the curved boundary $d{=}|w{-}u|$. The relation is

$$x+iy = \gamma^{-1}\big(\gamma(0)+i\cdot d\cdot\gamma'(0)\big)\ , \tag{3.19}$$

where x and y are real. This has to be substituted back into (3.18) and Taylor–expanded in d. The resulting expression is the bulk-boundary OPE for curved boundaries. To second order in d we get

$$\begin{aligned}\phi(d) = e^{-J\theta}\sum_\psi {}^aB_\phi{}^\psi\,(2d)^{h_\psi-\Delta}\Bigg(&\psi(u) + d\cdot\Big(\frac{iJ}{h_\psi}\cdot\ell_{-1}\psi(u) + \frac{h_\psi+\Delta}{2}\cdot\dot\theta\ \psi(u)\Big)\\ &+ d^2\cdot\Big(\frac{-2J^2-h_\psi}{2h_\psi(2h_\psi+1)}\cdot\ell_{-1}\ell_{-1}\psi(u) + \frac{iJ(h_\psi+\Delta+1)}{2h_\psi}\cdot\dot\theta\ \ell_{-1}\psi(u)\\ &\quad+\frac{3h_\psi^2+6\Delta h_\psi+5h_\psi+3\Delta^2+7\Delta}{24}\cdot\dot\theta^2\ \psi(u) + \frac{iJ}{2}\cdot\ddot\theta\ \psi(u)\\ &\quad+\frac{4(3J^2-\Delta+h_\psi-2\Delta\,h_\psi-{h_\psi}^2)}{c-10h_\psi+2c\,h_\psi+16{h_\psi}^2}\alpha(u)\Big)\ +\cdots\Bigg)\ .\end{aligned} \tag{3.20}$$

Here the sum runs over all primary fields living on the a–boundary. The angle θ denotes the direction of the tangent to the boundary in u, i.e. $\gamma'(0){=}e^{i\theta}$ and $\dot\theta$ is the boundary curvature. The dot stands for the derivative w.r.t. the arc-length, i.e. $\dot f = \frac{d}{ds}f(\gamma(s))$. The field $\alpha = (L_{-2} - \frac{3}{2(2h_\psi+1)}(L_{-1})^2)\psi$ is the level two quasi-primary descendent of ψ given in table A.1.

Another way to arrive at (3.20) is to write down a general expression for the bulk-boundary OPE (3.16), where the coefficients of the fields are taken to depend on the curvature, and require that both sides transform in the same way under infinitesimal conformal transformations. This approach is more conceptual than the

one presented above, but the author has not been able to cast the resulting formulas describing the infinitesimal transformations into a convenient form. This remains for future work.

Some remarks concerning (3.20) are in order. First note that in the case of degenerate representations (3.20) may contain null-states. This manifests itself in singularities of the coefficients, e.g. that of $\ell_{-1}\psi$ for h_ψ=0. In this case the correct thing to do is to leave out all null vectors from the rhs. Second, applying a global rescaling to both sides of (3.20) shows that the term multiplying d^N can only contain combinations of the form $\theta^{(n_1)}\dots\theta^{(n_k)}\cdot\ell_{-m_1}\dots\ell_{-m_\ell}\psi(u)$ where $n_1+\dots+n_k+m_1+\dots+m_\ell=N$. Here $\theta^{(n)}$ denotes the nth derivative of $\theta(s)$ w.r.t. to arc-length. In particular, as opposed to the case of a straight boundary, the primary field $\psi(u)$ can appear in all orders of the expansion.

3.3 Boundary one-point functions on the cylinder

Our aim is to compute a power series expansion of correlators in the presence of curved boundaries. With the various OPEs (2.28), (3.15), (3.20) any correlator can be reduced to a sum of boundary one-point functions. In this section we consider one–point functions on analytic boundaries of simply connected subsets of the complex plane.

It follows from the Riemann mapping theorem that for any simply connected proper subset M of $\mathbb{C}$ we can find an analytic bijection $f : M \to \text{UHP}$. On the UHP the only boundary field with non-vanishing one–point function is the identity (we will come back to this point below). Since under the conformal map f a field in the conformal family $[\psi]$ will be transformed into a linear combination of fields in the same conformal family, only fields in $[\mathbb{1}]$ can have non-vanishing one-point functions[3] on ∂M. The transformation behaviour of arbitrary descendent fields under a general conformal map f can be computed with a procedure due to Gaberdiel [Gab94]. This enables us to express one–point functions on ∂M in terms of $\langle 1\rangle^a_{\text{UHP}}$.

At least in the case of the semi–infinite cylinder it is also possible to obtain the one–point functions without resorting to the UHP. It would be an interesting problem for future research to try to generalize the method to arbitrary curved boundaries.

From now on we will consider the cylinder obtained from the UHP by identifying the imaginary axis with a vertical line passing through the real point L. Note that

[3] Recall from section 2.1 that we were taking n–point functions to be expectation values w.r.t. to the state corresponding to the **1**–operator. In non-unitary theories this will in general not be the state of lowest energy.

this procedure does not influence the OPE on the UHP as this is a local property of the theory.

The map $u = \gamma(w) = \frac{L}{2\pi} e^{2\pi i w/L}$ from the semi–infinite cylinder to a disc of radius $\frac{L}{2\pi}$ is one of the few cases where the arc-length map is actually a bijection. It follows the boundary correlators on the semi-infinite cylinder are identical to those on the disc. At this point it is important to notice that to transform one–point functions we need the coordinate transformation to be one–to–one everywhere on the surface. Whereas to compute the OPE on a curved boundary it was enough to know that the conformal map is defined locally, the one–point functions depend on global information. In particular the arc–length map does in general not extend to the full surface and cannot be used to transform n–point functions of boundary fields.

To motivate the construction on the cylinder recall that on the UHP itself we can obtain the descendent boundary fields via contour integration

$$(\ell_n \chi)(x) = \int_{\zeta \circlearrowleft x} (\zeta - x)^{n+1} T(\zeta) \chi(x) \; . \tag{3.21}$$

To compute the one-point function $\langle (\ell_{-n}\chi)(x) \rangle_{\text{UHP}}$ for $n \geq 1$ one can take the contour off to infinity resulting in

$$\langle (\ell_{-n}\chi)(x) \rangle_{\text{UHP}} = 0 \qquad ; \quad \text{for all } \chi(x) \text{ and } n \geq 1. \tag{3.22}$$

The only primary field with non-zero one-point function is the identity itself: $\langle 1 \rangle_{\text{UHP}} \neq 0$. In the periodic geometry of the cylinder this argument fails. One problem is that the function $(\zeta - x)^{n+1}$ used in (3.21) is not periodic and thus not defined on the cylinder.

For simplicity we will consider fields at $x=0$ in the following. We want a function $h(\zeta)$ that is periodic $h(\zeta) = h(\zeta + L)$ and behaves as $h(\zeta) \sim \zeta$ for ζ close to zero. We choose

$$h(\zeta) = \frac{L}{2\pi i} \big(e^{\frac{2\pi i}{L}\zeta} - 1 \big) \; . \tag{3.23}$$

The next step is to compute the contour integrals taking the role of (3.21) on the cylinder

$$\int_{\zeta \circlearrowleft 0} h(\zeta)^{n+1} T(\zeta) \chi(0) \; . \tag{3.24}$$

In Appendix A.2 the expectations value of these contour integrals are computed,

with the result

$$\int_{\zeta\circlearrowleft 0} h(\zeta)^{n+1}\langle T(\zeta)\chi(0)\rangle_{\text{cyl}} = \begin{cases} -\frac{c}{24}\left(\frac{iL}{2\pi}\right)^n \cdot \langle\chi(0)\rangle_{\text{cyl}} & \text{for } n\leq -2 \text{ and } \chi(0)\in[\mathbb{1}] \\ 0 & \text{for } n\leq -1 \text{ and } \chi(0)\notin[\mathbb{1}] \\ ? & \text{otherwise} \end{cases} \tag{3.25}$$

Finally, to make contact with the descendent fields we can expand $h(\zeta)^{n+1}$ in powers of w

$$h(\zeta)^{-n+1} = \sum_{k=0}^{\infty} a(n)_k \cdot \zeta^{k-n+1} \tag{3.26}$$

where the first few coefficients are

$$a(n)_0{=}1\,,\ a(n)_1{=}\tfrac{2\pi}{iL}\tfrac{n-1}{2}\,,\ a(n)_2{=}\left(\tfrac{2\pi}{iL}\right)^2\tfrac{(n-1)(3n-4)}{24}\,,\ a(n)_3{=}\left(\tfrac{2\pi}{iL}\right)^3\tfrac{(n-1)^2(n-2)}{48}\,.$$

Combining (3.25) and (3.26) yields, for $\beta\in[\mathbb{1}]$ and $n{\geq}2$,

$$-\frac{c}{24}\left(\frac{2\pi}{iL}\right)^n\langle\beta\rangle_{\text{cyl}} = \sum_{k=0}^{\infty} a(n)_k\cdot\langle\ell_{k-n}\beta\rangle_{\text{cyl}} \tag{3.27}$$

For $n=2$ and $\beta=\mathbb{1}$ this gives

$$-\frac{c}{24}\left(\frac{2\pi}{iL}\right)^2\langle\mathbb{1}\rangle_{\text{cyl}} = \langle\ell_{-2}\mathbb{1}\rangle_{\text{cyl}} + a(2)_1\langle\ell_{-1}\mathbb{1}\rangle_{\text{cyl}} + a(2)_2\langle\ell_0\mathbb{1}\rangle_{\text{cyl}} + \dots \tag{3.28}$$

Now the last two terms and all the terms abbreviated by the dots will just be zero, since $L_n|0\rangle{=}0$ for $n\geq -1$. As a first check we can verify that this is exactly what is obtained by directly transforming the stress tensor from the cylinder to the UHP (e.g. use the map (A.27) together with the transformation behaviour of t given in (3.12)).

For the higher descendents the procedure can be iterated. E.g.

$$\begin{aligned}
-\tfrac{c}{24}\left(\tfrac{2\pi}{iL}\right)^3\langle\mathbb{1}\rangle_{\text{cyl}} &= \langle\ell_{-3}\mathbb{1}\rangle_{\text{cyl}} + a(3)_1\langle\ell_{-2}\mathbb{1}\rangle_{\text{cyl}} \\
-\tfrac{c}{24}\left(\tfrac{2\pi}{iL}\right)^4\langle\mathbb{1}\rangle_{\text{cyl}} &= \langle\ell_{-4}\mathbb{1}\rangle_{\text{cyl}} + a(4)_1\langle\ell_{-3}\mathbb{1}\rangle_{\text{cyl}} + a(4)_2\langle\ell_{-2}\mathbb{1}\rangle_{\text{cyl}} \\
-\tfrac{c}{24}\left(\tfrac{2\pi}{iL}\right)^2\langle\ell_{-2}\mathbb{1}\rangle_{\text{cyl}} &= \langle\ell_{-2}\ell_{-2}\mathbb{1}\rangle_{\text{cyl}} + a(2)_1\langle\ell_{-1}\ell_{-2}\mathbb{1}\rangle_{\text{cyl}} \\
&\quad + a(2)_2\langle\ell_0\ell_{-2}\mathbb{1}\rangle_{\text{cyl}} + a(2)_4\langle\ell_2\ell_{-2}\mathbb{1}\rangle_{\text{cyl}}
\end{aligned} \tag{3.29}$$

with resulting expressions

$$\begin{aligned}&\langle \ell_{-2}\rangle_{\rm cyl} = -\tfrac{c}{24}\big(\tfrac{2\pi}{iL}\big)^2\langle 1\rangle_{\rm cyl}\ , \quad \langle \ell_{-3}\rangle_{\rm cyl} = 0\ , \quad \langle \ell_{-4}\rangle_{\rm cyl} = 0\ ,\\ &\langle \ell_{-2}\ell_{-2}\rangle_{\rm cyl} = \tfrac{c}{24}\big(\tfrac{c}{24}+\tfrac{11}{60}\big)\big(\tfrac{2\pi}{iL}\big)^4\langle 1\rangle_{\rm cyl}\ .\end{aligned} \tag{3.30}$$

This result is in agreement with the calculations in [Gab94].

By a residue computation similar to the one presented in appendix A.2, one can show that for $n{>}0$: $a(n{+}2)_n{=}\big(\frac{2\pi}{iL}\big)^n$. This implies recursively that, for $n{\geq}3$, $\langle \ell_{-n}\rangle_{\rm cyl}{=}0$.

3.4 Two examples

OPE on a disc

This is really an example when not to use the above formalism, but it does provide a nontrivial check. We will test the formalism in two cases: the correlator of two boundary fields on a disc of circumference L and the one–point function of a bulk field in the same geometry. In both cases the correlators can be computed exactly via the Möbius transformation to the UHP and as a power series expansion using the OPE on curved boundaries.

For two boundary fields the exact correlator on a disc of circumference L is

$$\begin{aligned}\langle\psi(s{+}d)\psi(s)\rangle^a_{\rm disc,L} &= C^{(aaa)1}_{\psi\psi}\,\langle 1\rangle^a_{\rm disc,L}\cdot\big|\tfrac{L}{\pi}\sin\tfrac{\pi d}{L}\big|^{-2h_\psi}\ ,\\ &= C^{(aaa)1}_{\psi\psi}\,\langle 1\rangle^a_{\rm disc,L}\cdot d^{-2h_\psi}\,\big(1+\tfrac{h_\psi}{12}(Ad)^2+\tfrac{h_\psi(5h_\psi+1)}{1440}(Ad)^4+\dots\big)\ .\end{aligned} \tag{3.31}$$

where $A{:=}\frac{2\pi}{L}$. It is enough to calculate the boundary–boundary OPE restricted to the conformal family of the identity [$\mathbb{1}$], since only these fields can have non-vanishing one–point functions. The result is given in (A.14) and reads

$$\begin{aligned}\psi^{(aa)}(s{+}d)\,\psi^{(aa)}(s) = {}& C^{(aaa)1}_{ii}\,d^{-2h_\psi}\\ &\cdot\Big\{\ \mathbb{1}\ +\ d^2\cdot\tfrac{2h_\psi}{c}\cdot t(s)\ +\ d^3\cdot\tfrac{h}{c}\cdot(\ell_{-1}t)(s)\\ &\quad +\ d^4\cdot\big(\tfrac{3h}{10c}\cdot(\ell_{-1}\ell_{-1}t)(s)-\tfrac{6h(5h+1)}{5c(5c+22)}\cdot\gamma(s)\big)\ +\ \dots\Big\}+\dots\ .\end{aligned} \tag{3.32}$$

Here $t(s)$ is the stress tensor on the boundary and $\gamma(s)$ is the quasi-primary field

$\gamma(s)=\ell_{-4}\mathbb{1}-\frac{5}{3}\cdot\ell_{-2}\ell_{-2}\mathbb{1}$. The boundary one-point functions are given by

$$\begin{aligned}&\langle t(s)\rangle_{\text{disc,L}}=\tfrac{c}{24}\,A^2\langle 1\rangle\ ,\ \langle\gamma(s)\rangle_{\text{disc,L}}=-\tfrac{c}{24}\,\tfrac{5c+22}{72}\,A^4\langle 1\rangle\ ,\\&\langle\ell_{-1}t(0)\rangle_{\text{disc,L}}=\langle\ell_{-1}\ell_{-1}t(0)\rangle_{\text{disc,L}}=0\ .\end{aligned}\tag{3.33}$$

Combining (3.32) and (3.33) reproduces expression (3.31).

Next consider the correlator $\langle\phi(r)\rangle_{\text{disc,L}}$ of a spinless bulk field ϕ at radius r on a disc of circumference L. From the map to the UHP we obtain

$$\begin{aligned}\langle\,\phi\big(\tfrac{L}{2\pi}-d\big)\,\rangle^a_{\text{disc,L}}&={}^aB_\phi{}^1\,\langle 1\rangle^a_{\text{disc,L}}\cdot(2d)^{-\Delta}\big(1-\tfrac{2\pi}{L}d\big)^{-\Delta}\\&={}^aB_\phi{}^1\,\langle 1\rangle^a_{\text{disc,L}}\cdot(2d)^{-\Delta}\cdot\Big(1+\tfrac{\Delta}{2}\cdot Ad+\tfrac{\Delta(\Delta+1)}{8}(Ad)^2\\&\quad+\tfrac{\Delta(\Delta+1)(\Delta+2)}{48}(Ad)^3+\tfrac{\Delta(\Delta+1)(\Delta+2)(\Delta+3)}{384}(Ad)^4+\dots\Big)\ .\end{aligned}\tag{3.34}$$

where $d=\frac{L}{2\pi}-r$ is the distance to the boundary on the disc and we have again $A=\frac{2\pi}{L}$.

The arc-length map for a disc of circumference L is $\gamma(s)=\frac{L}{2\pi}e^{isL/2\pi}$; the boundary curvature is constant: $\dot\theta(s)=\frac{2\pi}{L}$. This implies that all higher derivatives vanish, and expression (3.20) simplifies considerably. To obtain the coupling of the bulk field ϕ to the conformal family of the identity $[\mathbb{1}]$ to fourth order, one can start from the corresponding OPE on the UHP given in (A.25). Altogether the bulk-boundary OPE is, when restricted to $[\mathbb{1}]$ and with $\dot\theta(s)\equiv const$, to fourth order in d:

$$\begin{aligned}\phi(d)=&B_\phi{}^1\cdot(2d)^{-\Delta}\cdot\Big\{\mathbb{1}+d\cdot\tfrac{\Delta}{2}\dot\theta\cdot\mathbb{1}\\&+d^2\cdot\big(\tfrac{\Delta(3\Delta+7)}{24}\dot\theta^2\cdot\mathbb{1}-\tfrac{4\Delta}{c}\cdot t(0)\big)\\&+d^3\cdot\big(\tfrac{\Delta(\Delta+2)(\Delta+5)}{48}\dot\theta^3\cdot\mathbb{1}-\tfrac{2\Delta(\Delta+2)}{c}\dot\theta\cdot t(0)\big)\\&+d^4\cdot\big(\tfrac{\Delta(15\Delta^3+210\Delta^2+845\Delta+938)}{5760}\dot\theta^4\cdot\mathbb{1}-\tfrac{\Delta(3\Delta^2+19\Delta+22)}{6c}\dot\theta^2\cdot t(0)\\&\qquad+\tfrac{2\Delta}{5c}\cdot\ell_{-1}\ell_{-1}t(0)-\tfrac{24\Delta(5\Delta+2)}{5c(5c+22)}\cdot\gamma(0)\big)+\dots\Big\}\end{aligned}\tag{3.35}$$

Substituting (3.33) into (3.35) one verifies that the various powers of Δ combine exactly in the right way to reproduce (3.34).

We see that in this simple case the method of computing an approximation to the correlator through the curved boundary OPE is much more complicated that the direct calculation. It is nonetheless interesting to observe how the OPE and the one-point functions join together to give the correct answer.

Fluctuating Boundaries

In this example the bulk-boundary OPE for curved boundaries is applied to investigate the effect of averaging over many different boundaries. One can think of two kinds of random variables, fast and slow ones. Here the fields appearing in the n-point functions of CFT are the fast variables, while the boundary shape is varying only slowly. This distinction has to be made since we neglect dynamic effects from the moving boundary and treat the setup at each instance in time as quasi-static.

The quantity we want to investigate is the connected correlator of two bulk fields at a small distance d from the boundary

$$G(s,0) = \frac{\langle\, \phi(s,d)\, \phi(0,d)\, \rangle^a}{\langle 1 \rangle^a} - \frac{\langle \phi(s,d) \rangle^a \, \langle \phi(0,d) \rangle^a}{\langle 1 \rangle^a \langle 1 \rangle^a} \,. \tag{3.36}$$

Suppose the boundary condition a allows only the conformal family of the identity [𝟙] to live on the boundary. Let us consider first the correlator (3.36) on the UHP, i.e. a straight non-fluctuating boundary. Recall the bulk-boundary OPE (3.35) from the previous example. Irrespective of the scaling dimension of the bulk field, the large s behaviour of $G(s,0)$ will be given by the boundary stress tensor $\langle t(s)t(0)\rangle^a_{\text{UHP}} = \frac{c}{2} s^{-4} \langle 1 \rangle^a_{\text{UHP}}$:

$$G(s,0) \sim \left({}^aB_\phi{}^1 \, \tfrac{\Delta}{c} (2d)^{2-\Delta} \right)^2 \cdot \frac{\langle t(s)t(0)\rangle^a_{\text{UHP}}}{\langle 1 \rangle^a_{\text{UHP}}} \;\propto\; s^{-4} \,. \tag{3.37}$$

We will denote the average over the slow variables by a bar $\overline{X}$. Suppose that the boundary is fluctuating s.t. $\overline{\theta(s)} = \overline{\dot\theta(s)} = 0$ and assume further that the fluctuations are scale invariant. We expect the curvature to scale like $\overline{\dot\theta(s)\dot\theta(0)} = A \cdot s^{-2}$ for some constant A. In this case the leading behaviour of the correlator (3.36) is no longer given by the two point function of the stress tensor, but by that of the curvature instead. Making again use of the OPE (3.35) we find

$$G(s,0) \sim \left({}^aB_\phi{}^1 \, \tfrac{\Delta}{4} (2d)^{1-\Delta} \right)^2 \cdot \overline{\dot\theta(s)\dot\theta(0)} \;\propto\; s^{-2} \,. \tag{3.38}$$

In this case the bulk-boundary OPE for curved boundaries simplified the calculation of the average quantities. To use a conformal map to the UHP in the present case, one would have to average over functions and worry about which measure to use.

Chapter 4.

Conformal Blocks

In this and the following two chapters we present a calculational scheme to obtain the correlation functions of a CFT. For the approach to be applicable, the CFT has to fulfil several assumptions. The main one is that the symmetry algebra $\mathcal{A}$ decomposes the Hilbert space into *finitely* many representations. Very loosely speaking, this property reduces the problem of finding the correlators to solving a system of equations with only finitely many unknowns.

The method to compute correlators presented here is only one of many. Other examples are the path integral for the free boson, fermion, ghost– antighost system [YBk], or the Coulomb gas formulation by which Dotsenko and Fateev first found the minimal model bulk structure constants [DFa84]. In Liouville theory one has to deal with continuous families of representations, but the structure constants can nonetheless be calculated, as demonstrated in [PTe99, FFZ00].

The approach we are going to take naturally splits into two parts. The first part, which is covered in this chapter, depends only on the representation theory of the algebra $\mathcal{A}$. The outcome will be a collection of bases of conformal blocks and matrices describing their behaviour under change of basis. Next the conformal blocks have to be put together with the correct coefficients to form the correlators. The second part of the procedure, treated in the next chapter, is the "physical" input, i.e. to fix the coefficients we demand that the various limits one can take in a given correlator is consistent with the corresponding OPEs.

The method outlined below is not applicable to all CFTs. To give an idea of the assumptions that enter the construction and to show how the calculation of structure constants in chapter 6 might generalise to extended symmetry algebras $\mathcal{A}$, this and the next chapter are not specialised to the Virasoro case. In this chapter a selection of notations and results of Moore and Seiberg is presented. We will follow closely the papers [MSb89a, MSb89b].

4.1 Chiral algebra

To motivate the construction of chiral vertex operators, in this chapter we restrict ourselves to CFTs without boundaries.

The formalism that will be introduced below relies heavily on the fact that all correlators can be expressed through operators acting on some Hilbert space[1] $\mathcal{H}$. Our first assumption thus is:

(A1) Let $\mathcal{H}$ denote a vector space, equipped with a definite inner product and an orthogonal basis v_i. There exists an $\mathcal{H}$ s.t. for every field $\phi_i(z,\bar{z})$ in the conformal field theory we can find a linear operator $\hat{\phi}_i(z,\bar{z})$ acting on $\mathcal{H}$ with the property

$$\begin{aligned}&\langle\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle\\ &\quad=\sum_{k_1,..k_n}\frac{\langle v_0|\hat{\phi}_1(z_1,\bar{z}_1)|v_{k_1}\rangle\langle v_{k_1}|\dots|v_{k_n}\rangle\langle v_{k_n}|\hat{\phi}_n(z_n,\bar{z}_n)|v_0\rangle}{\langle v_{k_1}|v_{k_1}\rangle\dots\langle v_{k_n}|v_{k_n}\rangle}\end{aligned}\tag{4.1}$$

where the rhs is supposed to be well defined whenever $|z_1|>\dots>|z_n|$.

With this assumption operators and fields can be thought of as equivalent objects. We will drop the hat and denote both by ϕ.

The holomorphic, or chiral, fields of the CFT under consideration generate symmetries of the theory. For a good introduction into CFT with emphasis on this point one can refer to the review [Gab99] by Gaberdiel.

Every CFT has at least one chiral field, the energy momentum tensor $T(z)$. If there are more such fields, we can restrict our attention to primary chiral fields $J^i(z)$. Define the modes of the field $J^i(z)$ via contour integration

$$J^i_m=\int_{\zeta}\circlearrowleft_0\ \zeta^{h_i+m-1}J^i(\zeta);\,.\tag{4.2}$$

Let $\mathcal{A}$ be the algebra generated by the modes of the energy momentum tensor L_m and of the chiral fields J^i_m. For the following arguments we do not require $\mathcal{A}$ to be the maximal chiral algebra, i.e. only the modes of a proper subset of chiral fields might be included in $\mathcal{A}$. This will be important in the treatment of the D-series minimal models, where only the Virasoro symmetry will be used, even though other chiral fields are present.

The same construction as for $\mathcal{A}$ can be repeated for anti-holomorphic fields $\bar{J}^i(\bar{z})$. We assume the resulting algebra $\bar{\mathcal{A}}$ to be identical to $\mathcal{A}$.

[1] We will call $\mathcal{H}$ an Hilbert space, even though we include non-unitary theories in the treatment given below.

Next we make the assumption that our CFT is rational in the sense that the space of states $\mathcal{H}$ is organised into a finite number of irreducible representations of $\mathcal{A}\otimes\mathcal{A}$, i.e.

(A2) Let $\mathcal{A}$ be the algebra generated by the modes of the energy momentum tensor and the modes J^i_m of finitely many and not necessarily all Vir-primary chiral fields. Let $\bar{\mathcal{A}}$ be the antichiral counterpart of $\mathcal{A}$. We assume $\mathcal{A}\cong\bar{\mathcal{A}}$. Let $\mathcal{H}_p$ denote an irreducible highest weight representation of $\mathcal{A}$. We assume that the action of $\mathcal{A}\otimes\mathcal{A}$ decomposes the space of states into a finite sum of representations:

$$\mathcal{H} = \bigoplus_{p,q=1}^{N} M_{pq}\mathcal{H}_p\otimes\mathcal{H}_q \tag{4.3}$$

Here M_{pq} denotes the multiplicity of the pair $\mathcal{H}_p\otimes\mathcal{H}_q$.

Some relations for the modes J^i_m will be useful later. Let $J(z)$ be a Vir-primary chiral field and $\phi(z,\bar z)$ an arbitrary $\mathcal{A}$-primary field. The OPE of J and ϕ is

$$J(\zeta)\phi(z,\bar z) = \sum_{m=0}^{h_J-1}(\zeta{-}z)^{-h_J+m}\cdot(J_{-m}\phi)(z,\bar z) \;+\; \mathrm{reg}(\zeta{-}z)\;. \tag{4.4}$$

As in section 2.1, the notation $(J_k\phi)(z,\bar z)$ refers to the contour integral being centered at z, i.e.

$$(J_k\phi)(z,\bar z) = \int_\zeta \circlearrowleft_z\; (\zeta{-}z)^{h_i+k-1}\; J(\zeta)\,\phi(z,\bar z)\;. \tag{4.5}$$

From the OPE (4.4) we obtain the commutator of an arbitrary mode J_n with $\phi(z,\bar z)$

$$[J_n,\phi(z,\bar z)] = \sum_{m=0}^{h_J-1}\binom{h_J{+}n{-}1}{h_J{-}m{-}1}\cdot z^{n+m}\cdot(J_{-m}\phi)(z,\bar z)\;. \tag{4.6}$$

Here the binomial is defined via $(1{+}x)^a{=}\sum_{k=0}^\infty\binom{a}{k}x^k$, i.e. $\binom{a}{0}{=}1$ and for $k{>}0$: $\binom{a}{k}{=}(k!)^{-1}\cdot a(a{-}1)(a{-}2)\cdots(a{-}k{+}1)$. Using a contour deformation as in fig. 4.1b we can re-express $(J_{-m}\phi)(z,\bar z)$ through commutators. For $h_J{>}m{\geq}0$ we get

$$(J_{-m}\phi)(z,\bar z) = \sum_{k=0}^{h_J-m-1}\binom{h_J{-}m{-}1}{h_J{-}m{-}k{-}1}(-1)^k\cdot z^k\cdot[J_{-m-k},\phi(z,\bar z)]\;. \tag{4.7}$$

Inserting the result in (4.6) leads to the formula

$$[J_n, \phi(z,\bar z)] = \sum_{r=0}^{h_J-1} \Big\{ \sum_{s=0}^{r} (-1)^{r+s} \binom{h_J+n-1}{h_J-s-1} \binom{h_J-s-1}{h_J-r-1} \Big\} \cdot z^{r+n} \cdot [J_{-r}, \phi(z,\bar z)] \tag{4.8}$$

Note that any commutator $[J_n, \phi]$ can be expressed through commutators with n in the range $n = 1-h_J, \dots, 0$. E.g., for a spin one field $h_J{=}1$ we find $[J_n, \phi]{=}z^n[J_0, \phi]$ and for the stress tensor T: $[L_n, \phi]{=}(n{+}1)z^n[L_0, \phi] - nz^{n+1}[L_{-1}, \phi]$ (c.f. (2.25)).

We will also use the notion of a conjugate representation and of hermitian conjugation. The conjugate representation $i^\vee$ of an irrep i of $\mathcal{A}$ is the unique irrep s.t. the identity (scalar) representation occurs in the tensor product $i^\vee{\otimes}i$. For a state $A \in \mathcal{H}_i$ with $A{=}J^{a_1}_{-m_1} \dots J^{a_k}_{-m_k}|i\rangle$ we define $A^\vee{=}J^{a_1}_{-m_1} \dots J^{a_k}_{-m_k}|i^\vee\rangle$.

Hermitian conjugation was already defined in (2.35). For the more general situation of a extended chiral algebra we modify this definition slightly. For an $\mathcal{A}$–primary field we set $\phi_i(z,\bar z)^\dagger = \bar z^{-2h_i} z^{-2\bar h_i} \cdot \phi_{i^\vee}(\bar z^{-1}, z^{-1})$. On modes we retain formula (2.39): ${J_m}^\dagger = J_{-m}$.

4.2 Chiral vertex operators

Here we will give two ways to think about chiral vertex operators (CVOs): One is constructive, as matrix elements of linear operators, and one is algebraic, as intertwiners between $\mathcal{A}$-representations.

To motivate the construction of CVOs rewrite the following correlator of primary fields ϕ_i, ϕ_j, ϕ_k, ϕ_ℓ in the operator formalism:

$$\lim_{r\to\infty} r^{2(h_i+\bar h_i)} \langle \phi_{i^\vee}(r,r)\phi_j(z,\bar z)\phi_k(w,\bar w)\phi_\ell(0,0)\rangle = \langle i|\phi_j(z,\bar z)\phi_k(w,\bar w)|\ell\rangle \ . \tag{4.9}$$

Denote by $P_p \otimes P_q$ the projector on $\mathcal{H}_p \otimes \mathcal{H}_q$ in $\mathcal{H}$. Then (4.9) can be rewritten as a finite sum:

$$\langle i|\phi_j(z,\bar z)\phi_k(w,\bar w)|\ell\rangle = \sum_{p,q=1}^{N} \langle i|\phi_j(z,\bar z)P_p \otimes P_q\phi_k(w,\bar w)|\ell\rangle \tag{4.10}$$

Let us assume for the moment that the chiral algebra is just the Virasoro algebra $\mathcal{A} = \mathsf{Vir}$ and leave larger chiral algebras for later. If we replace the projectors by a

basis of $\mathcal{H}_p \otimes \mathcal{H}_q$ the correlator (4.10) becomes

$$\sum_{p,q} \sum_{\vec{P},\vec{P}',\vec{Q},\vec{Q}'} a_{\vec{P}\vec{P}'}\, b_{\vec{Q}\vec{Q}'} \cdot \langle i|\, \phi_j(z,\bar{z})\, L_{-\vec{P}} \bar{L}_{-\vec{Q}} |p,q\rangle\ \langle p,q| L_{\vec{P}'} \bar{L}_{\vec{Q}'}\, \phi_k(w,\bar{w})\, |\ell\rangle\ . \tag{4.11}$$

The factors $a_{\vec{P}\vec{P}'}, b_{\vec{Q}\vec{Q}'}$ take care of the fact that the L's acting on the highest weight state $|p,q\rangle$ do not necessarily form an orthonormal set.

The individual three-point functions in (4.11) can be reduced to a three-point function involving only highest weight states by commuting L's. Doing this we see that the result is of the form:

$$\begin{aligned}
&\langle i|\phi_j(z,\bar{z})\phi_k(w,\bar{w})|\ell\rangle \\
&\quad = \sum_{p,q=1}^{N} C_{pq} \cdot \Big(\sum_{\vec{P},\vec{P}'} \tilde{a}_{\vec{P}\vec{P}'}(i,j,k,l) \cdot z^{h_i - h_j - h_p + |\vec{P}|}\, w^{h_p - h_k - h_\ell + |\vec{P}'|} \Big) \\
&\qquad\qquad \cdot \Big(\sum_{\vec{Q},\vec{Q}'} \tilde{b}_{\vec{Q}\vec{Q}'}(i,j,k,l) \cdot \bar{z}^{\bar{h}_i - \bar{h}_j - h_q + |\vec{Q}|}\, \bar{w}^{h_q - \bar{h}_k - \bar{h}_\ell + |\vec{Q}'|} \Big) \\
&\quad = \sum_{p,q=1}^{N} C_{pq} f_p(z,w) g_q(\bar{z},\bar{w})
\end{aligned} \tag{4.12}$$

where $\tilde{a}_{\vec{P}\vec{P}'}$ and $\tilde{b}_{\vec{Q}\vec{Q}'}$ (and hence f_p and g_q) are determined solely by the commutation relations of the L's. The power series defining the functions $f_p(z,w)$ and $g_q(\bar{z},\bar{w})$ converges for $|z| > |w|$ by assumption (A1).

From this point of view the idea behind CVOs is to construct all possible functions that could arise when inserting a sum over projections into a correlator. These functions are called conformal blocks, and the hope (which will be made an assumption later) is that for a given correlator there are only finitely many conformal blocks. As the main goal of CFT is to find the correlator from the symmetry, the above improves our situation dramatically. Whereas before an n-point correlator was an unknown function in $2n$ real variables, it has now been reduced to a finite bilinear combination of functions whose power series expansion follows directly from the symmetry algebra.

We will now define CVOs to have the same commutation relations with the elements of $\mathcal{A}$ as a corresponding field in the CFT. That is, for a field $\phi(z,\bar{z})$ whose chiral half transforms in the highest weight representation $\mathcal{H}_j$ one constructs the CVO

$$\mathrm{V}^i_{jk}(\cdot, z) : \mathcal{H}_j \otimes \mathcal{H}_k \longrightarrow \mathcal{H}_i \tag{4.13}$$

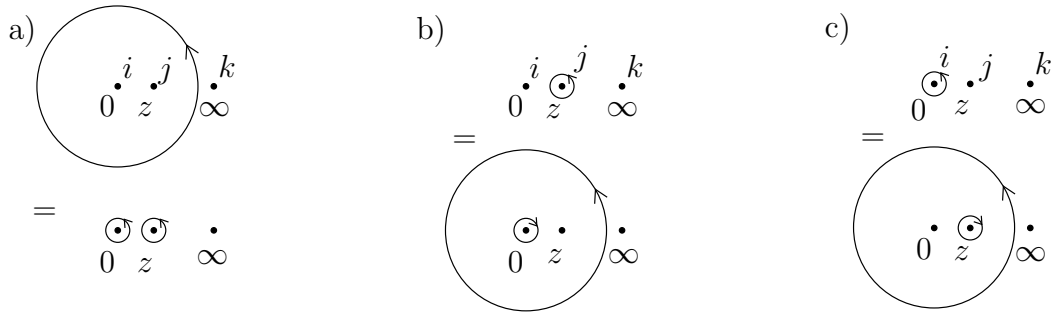


Figure 4.1: Different contour deformations to move modes inside a CVO three-point function

as follows: Pick three states $A \in \mathcal{H}_i$, $B \in \mathcal{H}_j$, $C \in \mathcal{H}_k$ and write down the matrix element

$$\langle A| \, \mathrm{V}^i_{jk}(B,z) \, |C\rangle \ . \tag{4.14}$$

Define

$$\mathrm{V}^i_{jk}(L_{-1}B,z) = \frac{d}{dz}\mathrm{V}^i_{jk}(B,z) \tag{4.15}$$

and if α,β,γ are Virasoro highest weight states fix, for some number[2] $|\mathrm{V}^\alpha_{\beta\gamma}|$:

$$\langle\alpha|\mathrm{V}^i_{jk}(\beta,z)|\gamma\rangle = |\mathrm{V}^\alpha_{\beta\gamma}| z^{h_\alpha - h_\beta - h_\gamma} \tag{4.16}$$

Then one uses contour deformations as in fig. 4.1 to manipulate the general matrix element (4.14) and to relate it to a few basic ones.

Let us now derive some consequences of this definition. First of all note that a general matrix element (4.14) can be reduced to one involving only Vir–primary fields. That can be seen by combining the commutation relations (4.7), (4.8) for $J(z)$=$T(z)$ and (4.15), (4.16). This determines the operator V^i_{jk} up to the constants $|\mathrm{V}^\alpha_{\beta\gamma}|$ in (4.16). If $\mathcal{A} = \mathsf{Vir}$ then there is only one of these and we have defined V^i_{jk} by its matrix elements up to an overall normalisation. So, in the Virasoro case the space of possible $\mathrm{V} : \mathcal{H}_j \otimes \mathcal{H}_k \to \mathcal{H}_i$ has either dimension 0 or 1. The case of dimension zero can occur if a relation forces the normalisation constant to vanish. The easiest example involves the vacuum representation with h=0. It has a null

[2] The notation $|\mathrm{V}^\alpha_{\beta\gamma}|$ does not imply that it has to be a positive number. In this thesis it will however always be real.

vector at level 1, $L_{-1}|0\rangle = 0$ and we find

$$0 = \langle i|\mathrm{V}^i_{j0}(z)L_{-1}|0\rangle = -\frac{d}{dz}\langle i|\mathrm{V}^i_{j0}(z)|0\rangle = |\mathrm{V}^i_{j0}|\ (h_j - h_i)\ z^{h_i-h_j-1}\ . \tag{4.17}$$

So $|\mathrm{V}^i_{j0}| = 0$ unless $h_i{=}h_j$.

However for $\mathcal{A} \supset \mathsf{Vir}$ an $\mathcal{A}$-irrep can contain infinitely many Vir-irreps. In fact this is the interesting case because otherwise the space of states $\mathcal{H}$, even though it is now organised in irreps of a larger algebra $\mathcal{A}$, would still only consist of finitely many Vir-irreps and thus be a minimal model. This latter possibility is the case for the D– and the E–series minimal models.

So up to now our CVO V^i_{jk} is only determined up to infinitely many constants. To see how this can possibly be reduced to finite number let us look at the case with one extra chiral field $J(z)$. For a general matrix element $\langle A|\mathrm{V}^i_{jk}(B,z)|C\rangle$ we can use contour deformations as in fig. 4.1b to make B $\mathcal{A}$-primary. Next employ (4.8) to move all J_n's with $n{>}0$ from the left side of V to the right and then use (4.8) once more to express all J_{-n}'s with $n \geq h_J$ through $J_0, \ldots, J_{1-h_J}$. We are left with a matrix element of the form

$$\langle i|\ (J_0)^m\ \mathrm{V}^i_{jk}(|j\rangle, z)\ (J_{1-h_J})^{n_{h_J-1}} \ldots (J_{-1})^{n_1}(J_0)^{n_0}\ |k\rangle\ . \tag{4.18}$$

To get any further we assume the presence of null vectors in the modules $\mathcal{H}_i$, $\mathcal{H}_k$ which allow us to reduce the strings of J_n in (4.18) to a finite number of basic three point functions. Example: In the case of WZW models the zero modes of the currents $J^a(z)$ form a Lie algebra. In an integrable highest weight irrep $\mathcal{H}_k$, the action of the zero modes on the highest weight vector $|k\rangle$ generates a finite dimensional representation of this Lie algebra. Let J_0 be the raising operator. Then $\langle i|(J_0)^M = 0 = (J_0)^N|k\rangle$ for M, N large enough. Since for a current we have $h_J{=}1$, (4.18) implies that any matrix element can be reduced to the form $\langle i|(J_0)^m\mathrm{V}^i_{jk}(|j\rangle, z)(J_0)^n|k\rangle$. There can thus at most be $M \cdot N$ independent CVOs of type V^i_{jk}.

In general we denote the dimension of the space of possible CVOs by $\mathrm{N}_{jk}{}^i$, i.e.

$$\mathrm{N}_{jk}{}^i = \dim\left(\mathrm{V}^i_{jk} : \mathcal{H}_j \otimes \mathcal{H}_k \to \mathcal{H}_i\right)\ . \tag{4.19}$$

We assume that all CVOs which can occur in the theory we are looking at are determined in terms of finitely many constants, i.e. $\mathrm{N}_{ij}{}^k{<}\infty$. This will be part of (A3) in the end of this section.

If $\mathrm{N}_{jk}{}^i > 1$ we can pick a basis in the space of CVOs $V : \mathcal{H}_j{\otimes}\mathcal{H}_k{\to}\mathcal{H}_i$ and introduce an extra index to distinguish the different linearly independent CVOs:

$\mathrm{V}^i_{jk,a}(\cdot,z)$. It will be convenient to make a particular choice of basis: Let $N = \mathrm{N}_{ij}{}^k$. We can find N basic triples of states on which the normalisation of the CVO can be chosen arbitrarily

$$(\alpha_1,\beta_1,\gamma_1),(\alpha_2,\beta_2,\gamma_2),\ldots,(\alpha_N,\beta_N,\gamma_N) \tag{4.20}$$

with $\alpha_m \in \mathcal{H}_{i^\vee}$, $\beta_m \in \mathcal{H}_j$, $\gamma_m \in \mathcal{H}_k$. It is always possible to make all α,β,γ Vir–primary, so we demand that they are. We choose a basis $\mathrm{V}^i_{jk,a}(\cdot,z)$ of CVOs with the property

$$\langle\alpha_n|\mathrm{V}^i_{jk,a}(\beta_n,z)|\gamma_n\rangle = \delta_{a,n}\cdot V^a\cdot z^{h_\alpha-h_\beta-h_\gamma} \qquad ;\ V^a\in\mathbb{R}\ . \tag{4.21}$$

Furthermore we will link the choice of basis between the following six types of CVOs:

$$\mathrm{V}^i_{jk}(\cdot,z)\,,\ \ \mathrm{V}^i_{kj}(\cdot,z)\,,\ \ \mathrm{V}^{j^\vee}_{ki^\vee}(\cdot,z)\,,\ \ \mathrm{V}^{j^\vee}_{i^\vee k}(\cdot,z)\,,\ \ \mathrm{V}^{k^\vee}_{i^\vee j}(\cdot,z)\,,\ \ \mathrm{V}^{k^\vee}_{ji^\vee}(\cdot,z)\ . \tag{4.22}$$

To do so note that the normalisation of $\mathrm{V}^{k^\vee}_{ji^\vee}$, say, can be chosen freely[3] on the triples $(\gamma_n^\vee,\beta_n,\alpha_n^\vee)$, where the individual α,β,γ are the same as in (4.20). Similar results hold for all other types of CVOs in the list (4.22).

This leads us to a natural identification of bases and normalisations in the spaces of CVOs of the types listed in (4.22). E.g. for a basic triple $(\alpha_n,\beta_n,\gamma_n)$ chosen from

[3] This can be seen as follows: Let V_{ABC} denote the matrix element $\langle A|\mathrm{V}^i_{jk}(B,z)|C\rangle$. Relations between the matrix elements obtained by commuting modes take the form $\sum_{ABC} M_n^{ABC}V_{ABC}=0$. For $z\in\mathbb{R}^+$ and in our normalisation of chiral fields M_n^{ABC} is a real matrix. Furthermore in each row n there are only finitely many entries ABC s.t. $M_n^{ABC}\neq 0$. Let M be the matrix of all such relations. We need not keep all rows in M. It is sufficient to keep rows with only Vir–primary fields α,β,γ. To see this let V_{ABC} be a CVO involving Vir–descendents. Then there is a linear differential operator D_{ABC} s.t. $V_{ABC}=D_{ABC}V_{\alpha\beta\gamma}$ where α,β,γ are the primary states associated to A,B,C. The functional form of $V_{\alpha\beta\gamma}$ is given in (4.16) and thus D_{ABC} can be replaced by a number d_{ABC}. For a relation we get the equalities $0=\sum M^{ABC}V_{ABC}=\sum M^{ABC}d_{ABC}V_{\alpha\beta\gamma}=\sum\tilde{M}^{\alpha\beta\gamma}V_{\alpha\beta\gamma}$. Since we demanded M to be complete, the relation $\sum\tilde{M}^{\alpha\beta\gamma}V_{\alpha\beta\gamma}=0$ is already present in M. From hereon we restrict M to involve only Vir–primary states. A choice $V^a_{\alpha\beta\gamma}$ of matrix elements for the CVO V^i_{jk} is consistent with the commutation relations of $\mathcal{A}$ if and only if $M.V^a=0$. It follows that $\mathrm{N}_{ij}{}^k=\dim\ker M$. Consider the conjugate of each relation: $0=(\sum M_n^{\alpha\beta\gamma}V_{\alpha\beta\gamma})^*=\sum M_n^{\alpha\beta\gamma}\bar{z}^{-2h_\beta}\langle\gamma^\vee|\mathrm{V}^{k^\vee}_{ji^\vee}(\beta,\bar{z}^{-1})|\alpha^\vee\rangle$. This defines a set of relations for the matrix elements of $\mathrm{V}^{k^\vee}_{ji^\vee}$: $\sum M_n^{\alpha\beta\gamma}z^{2h_\beta}V_{\gamma^\vee\beta\alpha^\vee}=0$. In fact repeating the above argument for $V_{\gamma^\vee\beta\alpha^\vee}$, the completeness of M implies that these are all relations for $\mathrm{V}^{k^\vee}_{ji^\vee}$. Let D be the diagonal matrix $D^{\alpha\beta\gamma}_{\alpha\beta\gamma}=z^{2h_\beta}$. Then $\mathrm{N}_{k^\vee j}{}^{i^\vee}=\dim\ker(M.D)=\dim\ker M=\mathrm{N}_{ij}{}^k$. Furthermore, since it is consistent to choose a set of CVOs $\mathrm{V}^i_{jk,a}$ with property (4.21), the above argument tells us that it is equally consistent to choose a basis $\mathrm{V}^{k^\vee}_{ji^\vee,a}$ with $\langle\gamma_n^\vee|\mathrm{V}^i_{jk}(\beta_n,z)|\alpha_n^\vee\rangle=\delta_{a,n}\cdot V^a\cdot z^{h_\gamma-h_\beta-h_\alpha}$. To obtain a similar result for exchanging the bottom two indices $\mathrm{V}^i_{jk}\leftrightarrow\mathrm{V}^i_{kj}$ one can for example consider the translation/rotation $\zeta\to z-\zeta$.

(4.20) we get:

$$\begin{aligned}\langle\alpha_n|\mathrm{V}^i_{kj,a}(\gamma_n,z)|\beta_n\rangle &= \delta_{n,a}\cdot V^a\cdot z^{h_\alpha-h_\gamma-h_\alpha}\\ \langle\beta^\vee_n|\mathrm{V}^{j^\vee}_{i^\vee k,a}(\alpha^\vee_n,z)|\gamma_n\rangle &= \delta_{n,a}\cdot V^a\cdot z^{h_\beta-h_\alpha-h_\gamma}\\ &\dots\end{aligned} \tag{4.23}$$

An immediate consequence[3] of the identification of bases for the CVOs in (4.22) is that

$$\mathrm{N}_{jk}{}^{i} = \mathrm{N}_{kj}{}^{i} = \mathrm{N}_{ki}{}^{j^\vee} = \mathrm{N}_{i^\vee k}{}^{j^\vee} = \mathrm{N}_{i^\vee j}{}^{k^\vee} = \mathrm{N}_{ji^\vee}{}^{k^\vee} \tag{4.24}$$

If one of the three representations is the identity $\mathcal{H}_0$, the space of CVOs V^j_{i0} is zero dimensional if $i{\neq}j$ and one dimensional if $i{=}j$. We choose the normalisation to be

$$\langle i|\mathrm{V}^i_{i0}(|i\rangle,z)|0\rangle = 1\ . \tag{4.25}$$

Since the CVOs in the list (4.22) are all normalised in the same way, this implies in particular that $\langle i|\mathrm{V}^i_{0i}(|0\rangle,z)|i\rangle{=}\langle i|i\rangle{=}1$. In the convention used in this thesis the normalisation of the highest weight vector in the $\mathcal{A}$–irrep $\mathcal{H}_i$ and the $\mathcal{A}{\otimes}\mathcal{A}$–highest weight vectors in the space of states $\mathcal{H}$ are not linked. I.e. for $a\in\mathcal{H}$, $a{=}i{\otimes}\bar{\imath}$ we can have $\langle a|a\rangle\neq 1$.

The CVOs can also be interpreted as intertwiners between the representations $\mathcal{H}_j\otimes\mathcal{H}_k$ and $\mathcal{H}_i$. If ρ_i,ρ_j are representations of $\mathcal{A}$ on $\mathcal{H}_i,\mathcal{H}_j$ the linear map $\Phi:\mathcal{H}_i\to\mathcal{H}_j$ is an intertwiner if for any $A\in\mathcal{A}$, $\rho_i(A)\Phi=\Phi\rho_j(A)$.

To turn $\mathcal{H}_j{\otimes}\mathcal{H}_k$ into a representation of $\mathcal{A}$ we need a co-product $\Delta:\mathcal{A}\to\mathcal{A}{\otimes}\mathcal{A}$. This can be constructed from the contour deformation in fig. 4.1a. Writing this in terms of modes, we see that an operator J_m, with a large contour encircling z and 0 acts on the individual field insertions at z and 0 as

$$\Delta(J_m) = \mathbb{1}{\otimes}J_m + \sum_{k=0}^{\infty}\binom{h_J{+}m{-}1}{k}\, z^{h_J+m-k-1}\cdot J_{k+1-h_J}{\otimes}\mathbb{1} \tag{4.26}$$

where in $a\otimes b$ the left component a is understood to act on the field at z in the sense of eqn. (4.5) and the right component b acts on the field at 0. By evaluating $\Delta([L_2,L_{-2}])$ on the state $|j\rangle{\otimes}|k\rangle$ we see that the co-product (4.26) preserves the central charge. The usual definition $\Delta(X)=X{\otimes}\mathbb{1}+\mathbb{1}{\otimes}X$ does not have this property.

The CVOs introduced above are now intertwiners by construction: $\rho\mathrm{V}=\mathrm{V}\rho$. In more detail (and a slight struggle with notation), for all states $\alpha\in\mathcal{H}_i$, $\beta\in\mathcal{H}_j$,

$\gamma \in \mathcal{H}_k$ and all modes $J_m \in \mathcal{A}$ we have

$$\langle\alpha|\, \rho_i(J_m)\, \mathrm{V}^i_{jk}(\cdot,z)\, |\beta\rangle\otimes|\gamma\rangle = \langle\alpha|\, \mathrm{V}^i_{jk}(\cdot,z)\, (\rho_j\otimes\rho_k)(\Delta(J_m))\, |\beta\rangle\otimes|\gamma\rangle\ . \tag{4.27}$$

If for a triple of irreducible highest weight representations $\mathcal{H}_i, \mathcal{H}_j, \mathcal{H}_k$ we can find a nonzero intertwiner $\mathrm{V}^i_{jk} \neq 0$ we say $\mathcal{H}_j$ and $\mathcal{H}_k$ *can fuse to* $\mathcal{H}_i$.

In the next section we will need to sum over all representations occurring in the fusion of two given ones. To make these sums finite, as well as the individual dimensions $\mathrm{N}_{ij}{}^k$, we make the assumption

(A3) Let $\mathcal{R}_0$ be the set of all $\mathcal{A}$-irreps that occur in the space of states $\mathcal{H}$. Denote by $\mathcal{R}$ its closure under fusion, i.e. when $\mathcal{H}_j, \mathcal{H}_k \in \mathcal{R}$ then all $\mathcal{H}_i$ that $\mathcal{H}_j$ and $\mathcal{H}_k$ can fuse to are also contained in $\mathcal{R}$. We assume that the total number of such $\mathcal{H}_i$ is finite, i.e.

$$\text{for all } i,j \in \mathcal{R}\ :\ \sum_{p\in\mathcal{R}} \mathrm{N}_{ij}{}^p < \infty \tag{4.28}$$

We furthermore assume that $\mathcal{R}$ is closed under conjugation, i.e. $i\in\mathcal{R} \Rightarrow i^\vee\in\mathcal{R}$.

Let us summarise what we have done so far. We demanded that the correlators of the CFT in question have an operator description (assumption (A1)). Then we defined a symmetry algebra $\mathcal{A}$ generated by (some of) the chiral fields of the CFT. By assumption (A2) our CFT is rational in the sense that the action of $\mathcal{A}\otimes\mathcal{A}$ decomposes the space of states $\mathcal{H}$ into a direct sum of finitely many irreps $\mathcal{H}_p\otimes\mathcal{H}_q$. For a given field in the CFT we constructed CVOs to have the same commutation relations with the modes in $\mathcal{A}$ and then interpreted them as intertwiners of $\mathcal{A}$-representations. By assumption (A3) a CVO is completely determined by only finitely many numbers.

4.3 Braiding and Fusion

We will now proceed to build conformal blocks from CVOs. Consider a chain of CVOs

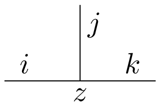
$$\langle\alpha_i|\mathrm{V}^i_{j_1p_1}(\beta_1,z_1)\mathrm{V}^{p_1}_{j_2p_2}(\beta_2,z_2)\dots\mathrm{V}^{p_{n-1}}_{j_nk}(\beta_n,z_n)|\gamma_k\rangle \tag{4.29}$$

where $\alpha_i \in \mathcal{H}_i$, $\beta_m \in \mathcal{H}_{j_m}$, $\gamma_k \in \mathcal{H}_k$. As in (A1) we assume that this function converges for $|z_1| > |z_2| > \dots > |z_n| > 0$. We will also assume that all chains with two or more CVOs are linearly independent[4] as functions of $z_1,\dots,z_n$. Since the

[4] One would suspect that the linear independence of chains of two or more CVOs as functions in $z_1,\dots,z_n$ is linked to the linear independence of the CVOs as operators $\mathcal{H}_j\otimes\mathcal{H}_k\to\mathcal{H}_i$ for fixed

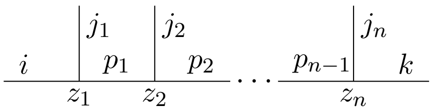
CVOs in general involve fractional powers of z, (4.29) may be multivalued. We define (4.29) initially to be the power series evaluated for $z_m \in \mathbb{R}^+$ and $z_1 > z_2 > \cdots > z_n$, so that no ambiguities arise from the fractional powers. Values outside this domain have to be obtained by analytic continuation.

For calculations it is convenient to introduce a graphical notation for the conformal blocks. Denote

$\mathrm{V}^i_{jk}(\cdot, z) =$ 

$: \mathcal{H}_j \otimes \mathcal{H}_k \longrightarrow \mathcal{H}_i$ (4.30)

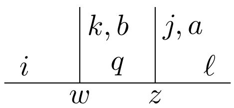
then the chain of operators in (4.29) becomes



(4.31)

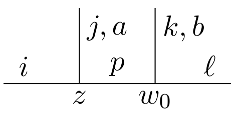
Evaluating this with the states $\beta_1, \ldots, \beta_n, \gamma_k$ and taking the inner product with α_i then gives exactly the expression (4.29).

Consider the following chiral block, with $w, z \in \mathbb{R}^+$ and $w > z$:



$: \; \mathcal{H}_k \otimes \mathcal{H}_j \otimes \mathcal{H}_\ell \longrightarrow \mathcal{H}_i$ (4.32)

This operator is an intertwiner $\mathcal{H}_k \otimes \mathcal{H}_j \otimes \mathcal{H}_\ell \to \mathcal{H}_i$, again by construction of the CVOs. We can construct another intertwiner $\Phi : \mathcal{H}_k \otimes \mathcal{H}_j \otimes \mathcal{H}_\ell \to \mathcal{H}_i$ by analytic continuation. Start with the intertwiner



$: \; \mathcal{H}_j \otimes \mathcal{H}_k \otimes \mathcal{H}_\ell \longrightarrow \mathcal{H}_i \; .$ (4.33)

which is initially defined for $z > w_0 > 0$. Then analytically continue[5] w_0 anti–clockwise around z up to the point w in (4.32). The resulting operator, which we will denote by Φ, is an intertwiner $\mathcal{H}_k \otimes \mathcal{H}_j \otimes \mathcal{H}_\ell \to \mathcal{H}_i$. This can again be shown by contour deformation.

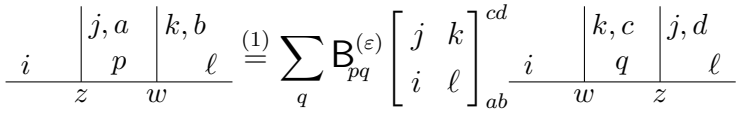
We now assume[6] that the set of intertwiners in eqn. (4.32) is complete in the sense

z. But no general theorem is known to the author.

[5] That is, analytically continue each matrix element as a function of w_0

[6] In the case of $\mathcal{A} = \mathsf{Vir}$ this can be shown to follow from (A1) and the fact that a conformal block involving descendent states can always be expressed as a linear differential operator acting on the block with only primary states, as described in [MSb89a]. No general proof is known to the author.

that the newly constructed intertwiner Φ can be expressed as a linear combination thereof. This defines the *braiding matrix* B and we get the operator identity



(4.34)

where ε denotes the direction of analytic continuation, as is detailed below. The initial domains of definition of both sides of (4.34) do not intersect and we have to be careful what precisely we mean by "=". In the rest of this chapter (but not necessarily elsewhere in the text), whenever "=" means "equal upon analytic continuation" it is marked with a superscript to describe the move. The move will always be starting from the lhs and ending at the rhs. In particular in (4.34) what is meant is:

(1) Choose four vectors $\alpha \in \mathcal{H}_i$, $\beta \in \mathcal{H}_j$, $\gamma \in \mathcal{H}_j$, $\delta \in \mathcal{H}_k$. Fix three values $z, w_0, w_1 \in \mathbb{R}^+$ with $w_1 > z_0 > w_0$. Evaluate the operator on the rhs for $\alpha, \gamma, \beta, \delta$ at w_1, z. Next fix a path P running from w_0 to w_1 anti–clockwise (if $\varepsilon = +$) or clockwise (if $\varepsilon = -$) around z_0. Starting at w_0 continue the $\alpha, \beta, \gamma, \delta$-matrix element on the lhs as a function of w along P to w_1. The result is the same as on the rhs.

This will be written in a clearer but less precise way as

(1) continue w anti–clockwise (for $\varepsilon = +$) or clockwise (for $\varepsilon = -$) around z.

One property of the braiding matrix that can be deduced immediately is its invertibility: if we continue w around z and then follow back the same path, we did nothing and get back the original function. Expressed in terms of B this means

$$\sum_r \sum_{s,t} \mathsf{B}^{(\varepsilon)}_{pr} \begin{bmatrix} j & k \\ i & \ell \end{bmatrix}^{st}_{ab} \mathsf{B}^{(-\varepsilon)}_{rq} \begin{bmatrix} k & j \\ i & \ell \end{bmatrix}^{cd}_{st} = \delta_{p,q}\delta_{ab,cd} \,. \tag{4.35}$$

We can also (almost) work out the braiding matrices involving the identity representation. To do so recall that we required the defining triples of states (α, β, γ) in (4.20) to be Vir–primary. Now a four point block involving an identity representation reduces to a three point function and SL(2)-invariance fixes its functional form.

E.g. we get, for the particular choice of basis introduced in the previous section:

$$\begin{array}{ccccc} & & j,a & & k \\ i & & k & & 0 \\ & z & & w & \end{array}(\alpha_a,\beta_a,\gamma_a,|0\rangle) = V^a\cdot(z-w)^{h_i-h_j-h_k}(z-w)^{n(\alpha_a)-n(\beta_a)-n(\gamma_a)}$$

$$\begin{array}{ccccc} & & k,a & & j \\ i & & j & & 0 \\ & w & & z & \end{array}(\alpha_a,\beta_a,\gamma_a,|0\rangle) = V^a\cdot(w-z)^{h_i-h_j-h_k}(w-z)^{n(\alpha_a)-n(\beta_a)-n(\gamma_a)} \tag{4.36}$$

Here $n(\alpha_a)\in\mathbb{Z}$ denotes the level of α_a in $\mathcal{H}_i$. In terms of the braiding matrix we have

$$\begin{array}{ccccc} & & j,a & & k \\ i & & k & & 0 \\ & z & & w & \end{array} \overset{(1)}{=} \mathsf{B}^{(\varepsilon)}_{kj}\begin{bmatrix} j & k \\ i & 0 \end{bmatrix}^{a0}_{a0} \begin{array}{ccccc} & & k,a & & j \\ i & & j & & 0 \\ & w & & z & \end{array}, \tag{4.37}$$

where (1): continue w in direction ε around z. Carrying out the analytic continuation in (4.36) explicitly gives the result

$$\mathsf{B}^{(\varepsilon)}_{kj}\begin{bmatrix} j & k \\ i & 0 \end{bmatrix}^{a0}_{a0} = e^{i\pi\varepsilon(h_i-h_j-h_k)}\cdot\xi^a_{ijk}\ , \tag{4.38}$$

where we defined the sign ξ^a_{ijk} as $\xi^a_{ijk}=(-1)^{n(\alpha_a)+n(\beta_a)+n(\gamma_a)}$.

The three-point B's will appear several times, so it is convenient to introduce an abbreviation

$$\mathsf{B}^{(\varepsilon)}_{kj}\begin{bmatrix} j & k \\ i & 0 \end{bmatrix}^{a0}_{a0} =: \Omega^i_{jk,a}(\varepsilon) = \xi^a_{ijk}\cdot e^{i\pi\varepsilon(h_i-h_j-h_k)}\ . \tag{4.39}$$

Similarly, in the normalisation of CVOs we chose, from

$$\begin{array}{ccccc} & & i^\vee & & j,a \\ 0 & & i & & k \\ & z & & w & \end{array}(|0\rangle,\alpha_a,\beta_a,\gamma_a)$$
$$= V^a\cdot z^{h_j-h_i-h_k}w^{h_i-h_j-h_k}(z-w)^{h_k-h_i-h_j}$$
$$\cdot\, z^{n(\beta_a)-n(\alpha_a)-n(\gamma_a)}w^{n(\alpha_a)-n(\beta_a)-n(\gamma_a)}(z-w)^{n(\gamma_a)-n(\alpha_a)-n(\beta_a)} \tag{4.40}$$

we get

$$\mathsf{B}^{(\varepsilon)}_{ij^\vee}\begin{bmatrix} i^\vee & j \\ 0 & k \end{bmatrix}^{0a}_{0a} = \Omega^k_{ij,a}(\varepsilon)\ . \tag{4.41}$$

It is worth noting two facts about the Ω's and ξ's just introduced. First, neither sees the difference between a representation i and its conjugate $i^\vee$, since both $h_i = h_{i^\vee}$ and $n_\alpha = n_{\alpha^\vee}$. Second, (and all this heavily depends on the normalisation of CVOs we chose) any sign ξ involving an identity representation $\mathcal{H}_0$ is $+1$.

The two remaining braiding matrix elements can be obtained by the method outlined above as:

$$\begin{array}{ccccc} & & j,a & & 0 \\ i & & k & & k \\ \hline & z & & w & \end{array} \quad : \quad \mathsf{B}^{(\varepsilon)}_{ki} \begin{bmatrix} j & 0 \\ i & k \end{bmatrix}^{0a}_{a0} = 1$$

$$\begin{array}{ccccc} & & 0 & & j,a \\ i & & i & & k \\ \hline & z & & w & \end{array} \quad : \quad \mathsf{B}^{(\varepsilon)}_{ik} \begin{bmatrix} 0 & j \\ i & k \end{bmatrix}^{a0}_{0a} = 1 \tag{4.42}$$

Now we turn to another transformation of conformal block: fusion. Using $\mathrm{V}^k_{k0}(\beta,0)|0\rangle = |\beta\rangle \in \mathcal{H}_k$, one can verify

$$\mathrm{V}^i_{jk,a}(\alpha,z)\mathrm{V}^k_{k0}(\beta,w)|0\rangle = \mathrm{V}^i_{i0}(\mathrm{V}^i_{jk,a}(\alpha,z-w)|\beta\rangle,w)|0\rangle \tag{4.43}$$

by applying the generator of translations $\exp(-wL_{-1})$ (which is invertible) to both sides. To see that (4.43) is true for any state in $\mathcal{H}_0$, consider the following series of transformations:

$$\begin{aligned} \mathrm{V}^i_{jk,a}(\alpha,z)\mathrm{V}^k_{k0}(\beta,w)\mathrm{V}^0_{00}(\chi,\zeta)|0\rangle &\overset{(*)}{=} \mathrm{V}^i_{jk,a}(\alpha,z)\mathrm{V}^k_{0k}(\chi,\zeta)\mathrm{V}^k_{k0}(\beta,w)|0\rangle \\ &\overset{(*)}{=} \mathrm{V}^i_{0i}(\chi,\zeta)\mathrm{V}^i_{jk,a}(\alpha,z)\mathrm{V}^k_{k0}(\beta,w)|0\rangle = \mathrm{V}^i_{0i}(\chi,\zeta)\mathrm{V}^i_{i0}(\mathrm{V}^i_{jk,a}(\alpha,z-w)|\beta\rangle,w)|0\rangle \\ &\overset{(*)}{=} \mathrm{V}^i_{i0}(\mathrm{V}^i_{jk,a}(\alpha,z-w)|\beta\rangle,w)\mathrm{V}^0_{00}(\chi,\zeta)|0\rangle \end{aligned} \tag{4.44}$$

where (*): The direction of analytic continuation does not matter as all braiding matrix elements have been computed to be equal to one just before.

Setting $\zeta = 0$ in the first and last equation we obtain, for an arbitrary state $|\chi\rangle \in \mathcal{H}_0$:

$$\mathrm{V}^i_{jk,a}(\alpha,z)\mathrm{V}^k_{k0}(\beta,w)|\chi\rangle = \mathrm{V}^i_{i0}(\mathrm{V}^i_{jk,a}(\alpha,z-w)|\beta\rangle,w)|\chi\rangle \tag{4.45}$$

Note that in (4.45) we can just write "=" because the power series on both sides are defined for $z,w \in \mathbb{R}^+$ and $z > w$. Furthermore (4.44) overall did not introduce any monodromy, so that no analytic continuation is involved and the rhs of (4.45) is just a resummation of the lhs. Eqn. (4.43) is thus not only valid on the vacuum, but on any state $|\chi\rangle \in \mathcal{H}_0$, i.e. the following operator identity holds:

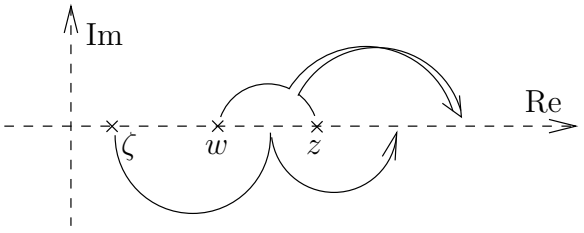
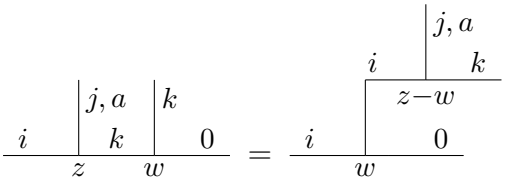
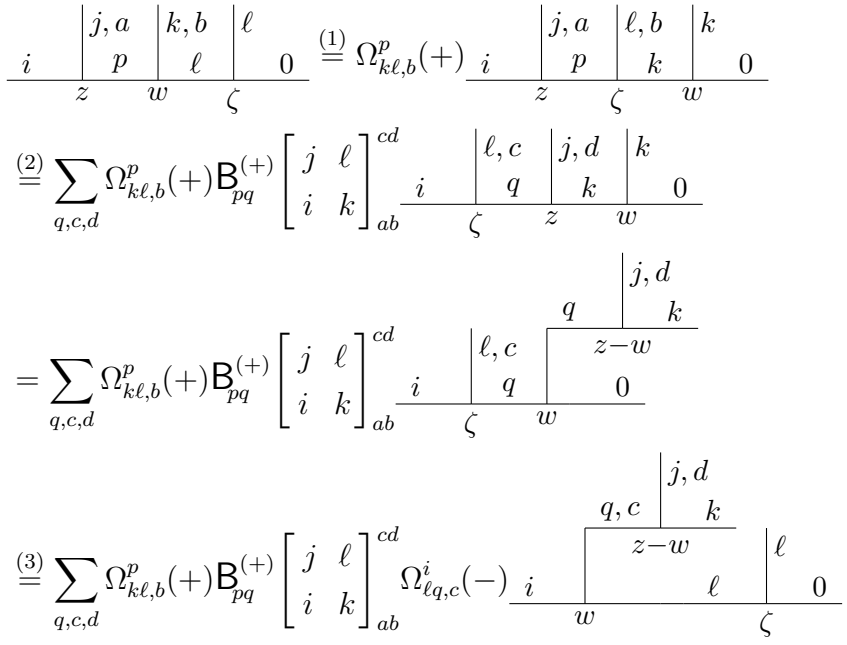


Figure 4.2: Analytic continuations to obtain the fusion matrix



(4.46)

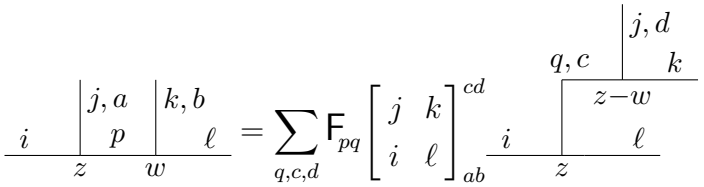
We have gathered all the ingredients for the general fusion matrix. The following is a repetition of the calculation (4.44) in graphical notation, this time involving nontrivial braiding matrices:



(4.47)

where the analytic continuations are (fig. 4.2): (1) ζ anti–clockwise around w, (2) ζ anti–clockwise around z, (3) w clockwise around ζ, keeping $z-w$ fixed.

Setting ζ=0 in the first and last expression gives the fusion procedure:



(4.48)

with

$$\mathsf{F}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}^{cd}_{ab} = \Omega^{p}_{k\ell,b}(+)\mathsf{B}^{(+)}_{pq}\begin{bmatrix} j & \ell \\ i & k \end{bmatrix}^{cd}_{ab}\Omega^{i}_{\ell q,c}(-) \tag{4.49}$$

Note that for the same reason as in (4.45), no analytic continuation is involved, and the rhs of (4.48) is just a resummation of the lhs.

When a conformal block has enough points one can imagine analytic continuation along different, but equivalent[7] paths. Each path corresponds to an expression involving the B and F matrices. Since the final function does not depend on which of the equivalent paths has been chosen, this leads to a large number of nontrivial identities on the braiding and fusion matrices. A detailed study of these relations, called polynomial equations, together with a proof that the a priori infinite set of resulting equations are generated by a finite number, can be found in the papers by Moore and Seiberg, e.g. [MSb89b].

In the following we will mainly be working with the fusion matrix, so rather than elaborating on the polynomial equations we will restrict ourself to presenting some symmetry properties and a recursion relation for the F's. The derivation has been shifted to Appendix A.3.

The indices inside the F-matrix can be permuted according to the following rules: Let ε denote the sign $\varepsilon = \xi^{a}_{ijp}\,\xi^{b}_{klp}\,\xi^{c}_{ilq}\,\xi^{d}_{jkq}$, then

$$\mathsf{F}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}^{cd}_{ab} = \varepsilon\cdot\mathsf{F}_{pq^\vee}\begin{bmatrix} i^\vee & \ell \\ j^\vee & k \end{bmatrix}^{dc}_{ab} = \varepsilon\cdot\mathsf{F}_{p^\vee q}\begin{bmatrix} k & j \\ \ell^\vee & i^\vee \end{bmatrix}^{cd}_{ba} = \mathsf{F}_{p^\vee q^\vee}\begin{bmatrix} \ell & i^\vee \\ k^\vee & j \end{bmatrix}^{dc}_{ba} \tag{4.50}$$

This, together with (4.35) and (4.49) allows us to invert the fusion matrix:

$$\sum_{rst}\mathsf{F}_{pr}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}^{st}_{ab}\mathsf{F}_{r^\vee q}\begin{bmatrix} k & \ell \\ j^\vee & i^\vee \end{bmatrix}^{cd}_{ts} = \delta_{pq}\delta_{ac}\delta_{bd}\;; \qquad \Big(\mathsf{F}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}^{-1}\Big)^{cd}_{rq,st} = \mathsf{F}_{r^\vee q}\begin{bmatrix} k & \ell \\ j^\vee & i^\vee \end{bmatrix}^{cd}_{ts} \tag{4.51}$$

For the recursion relation we suppose the theory contains some representations Δ, which are sufficiently simple to work out all fusion matrices involving at least one Δ. If the Δ's generate all representations in the theory by mutual fusion, this allows us to obtain all fusion matrices by a recursive procedure. Let $k{+}\Delta$ be any representation that appears in the fusion of k and Δ. We then have the relation

[7] In the sense that the two sets of continuation paths can be deformed into each other without intersecting any CVO insertion points.

$$\mathsf{F}_{pq}\left[\begin{array}{cc} j & k{+}\Delta \\ i & \ell \end{array}\right]^{cd}_{ab} = \sum_{r,s;efghm} \mathsf{F}_{(k+\Delta)^\vee,r}\left[\begin{array}{cc} \Delta & \ell \\ k^\vee & p^\vee \end{array}\right]^{ef}_{xb} \mathsf{F}_{ps}\left[\begin{array}{cc} j & k \\ i & r \end{array}\right]^{gh}_{ae} \mathsf{F}_{rq}\left[\begin{array}{cc} s & \Delta \\ i & \ell \end{array}\right]^{cm}_{gf} \mathsf{F}_{s^\vee,k+\Delta}\left[\begin{array}{cc} k & \Delta \\ j^\vee & q^\vee \end{array}\right]^{dx}_{hm} \tag{4.52}$$

The recursion runs on the index k. Suppose for a fixed k we know the matrices $\mathsf{F}[i\ell; jk]$ for arbitrary (allowed) i, j, ℓ. Then (4.52) gives us all matrices $\mathsf{F}[i\ell; j(k{+}\Delta)]$. Knowing these we can proceed to $(k{+}\Delta){+}\Delta$ etc.

4.4 The Virasoro case

For the remainder of the thesis we will restrict ourselves to the Virasoro algebra $\mathcal{A}{=}\mathsf{Vir}$ at central charge $c{=}1{-}6(t{+}t^{-1}{-}2)$ where $t{=}p/q$ and p, q are coprime integers ≥ 2. As mentioned in section 2.2, for these values of c one can find a finite set of irreps that close under fusion. They are labelled by Kac-labels (r, s). Let $i{=}(r_i, s_i)$, etc, and define a function $\mathcal{N}^c_{ab}(r)$ as

$$\mathcal{N}^c_{ab}(r) = \begin{cases} 1: & |a{-}b| < c < \min(a{+}b,\, 2r{-}a{-}b)\ ,\ a{+}b{+}c \text{ odd} \\ 0: & \text{otherwise} \end{cases}$$

The Verlinde fusion numbers are

$$\mathrm{N}_{ij}{}^k = \mathcal{N}^{r_k}_{r_i r_j}(p) \cdot \mathcal{N}^{s_k}_{s_i s_j}(q) + \mathcal{N}^{p-r_k}_{r_i r_j}(p) \cdot \mathcal{N}^{q-s_k}_{s_i s_j}(q) \in \{0, 1\}\ . \tag{4.53}$$

There are (at least) two routes to obtain numbers for the F-matrix elements. The first method implements the recursion relation and is aimed mainly at numerical studies. The second method uses the fact that the B-matrices form representations of braid groups, which may also be realised by a quantum group.

For the first route consider the chiral fields $\phi_{1,2}$ and $\phi_{2,1}$. All representations of the minimal theory can be obtained through repeated fusion of these fields. Furthermore a four-point correlator involving one of the above fields satisfies a second order differential equation. This can be solved and the F-matrix can be read off explicitly.

Consider the minimal model $M(p, q)$. The highest weight representations have weights $h_{r,s}{=}\frac{1}{4t}({d_{rs}}^2{-}{d_{11}}^2)$ where $t{=}p/q$ and $d_{rs}{=}r{-}st$. The Kac labels run over $r{=}1...p{-}1$, $s{=}1...q{-}1$ and h has the symmetry $h_{r,s}{=}h_{p-r,q-s}$.

Consider the case $\phi_{1,2}$. For the representation $x = (r, s)$ let x^+ denote the

representation $x^+=(r,s+1)$, with weight h_k^+ and similar $x^-=(r,s-1)$. Then we find

$$\begin{aligned}\langle i|\mathrm{V}^i_{jk^-}(1)\mathrm{V}^{k^-}_{k(12)}(x)|1,2\rangle &= x^{h_k^- - h_k - h_{12}}\cdot(1-x)^{h_i^+ - h_j - h_k}\\ &\quad\times {}_2F_1\big(\tfrac{1}{2}(1-d_i+d_j+d_k),\ \tfrac{1}{2}(1-d_i-d_j+d_k);\ 1+d_k;\ x\big)\ ,\\ \langle i|\mathrm{V}^i_{jk^+}(1)\mathrm{V}^{k^+}_{k(12)}(x)|1,2\rangle &= x^{h_k^+ - h_k - h_{12}}\cdot(1-x)^{h_i^+ - h_j - h_k}\\ &\quad\times {}_2F_1\big(\tfrac{1}{2}(1-d_i+d_j-d_k),\ \tfrac{1}{2}(1-d_i-d_j-d_k);\ 1-d_k;\ x\big)\ .\end{aligned}\tag{4.54}$$

The corresponding result for $\phi_{2,1}$ can be obtained by using the fact that $M(p,q)$ and $M(q,p)$ are identical when swapping the Kac-labels of all fields $(r,s)\leftrightarrow(s,r)$. In the formulas this amounts to replacing $t\to t^{-1}$ and $d\to -t^{-1}d$.

The transformation properties of the hypergeometric functions can now be used to obtain the F–matrices. In the following formula Δ stands for either $(2,1)$ or $(1,2)$. If j stands for (r,s) then $j\pm\Delta$ is $(r\pm1,s)$ and $(r,s\pm1)$ respectively. Altogether one gets:

$$\mathsf{F}\begin{bmatrix} j & \Delta\\ i & \ell\end{bmatrix} = \begin{pmatrix}\mathsf{F}_{\ell-\Delta,j-\Delta} & \mathsf{F}_{\ell-\Delta,j+\Delta}\\ \mathsf{F}_{\ell+\Delta,j-\Delta} & \mathsf{F}_{\ell+\Delta,j+\Delta}\end{pmatrix} = \cdots \tag{4.55}$$

for $\Delta=(2,1)$:

$$= \begin{pmatrix} \frac{\Gamma(t^{-1}d_j)\,\Gamma(1-t^{-1}d_\ell)}{\Gamma\left(\frac{1}{2t}(t-d_i+d_j-d_\ell)\right)\Gamma\left(\frac{1}{2t}(t+d_i+d_j-d_\ell)\right)} & \frac{\Gamma(-t^{-1}d_j)\,\Gamma(1-t^{-1}d_\ell)}{\Gamma\left(\frac{1}{2t}(t-d_i-d_j-d_\ell)\right)\Gamma\left(\frac{1}{2t}(t+d_i-d_j-d_\ell)\right)}\\ \frac{\Gamma(t^{-1}d_j)\,\Gamma(1+t^{-1}d_\ell)}{\Gamma\left(\frac{1}{2t}(t-d_i+d_j+d_\ell)\right)\Gamma\left(\frac{1}{2t}(t+d_i+d_j+d_\ell)\right)} & \frac{\Gamma(-t^{-1}d_j)\,\Gamma(1+t^{-1}d_\ell)}{\Gamma\left(\frac{1}{2t}(t-d_i-d_j+d_\ell)\right)\Gamma\left(\frac{1}{2t}(t+d_i-d_j+d_\ell)\right)}\end{pmatrix}$$

for $\Delta=(1,2)$:

$$= \begin{pmatrix} \frac{\Gamma(-d_j)\,\Gamma(1+d_\ell)}{\Gamma\left(\frac{1}{2}(1+d_i-d_j+d_\ell)\right)\Gamma\left(\frac{1}{2}(1-d_i-d_j+d_\ell)\right)} & \frac{\Gamma(d_j)\,\Gamma(1+d_\ell)}{\Gamma\left(\frac{1}{2}(1+d_i+d_j+d_\ell)\right)\Gamma\left(\frac{1}{2}(1-d_i+d_j+d_\ell)\right)}\\ \frac{\Gamma(-d_j)\,\Gamma(1-d_\ell)}{\Gamma\left(\frac{1}{2}(1+d_i-d_j-d_\ell)\right)\Gamma\left(\frac{1}{2}(1-d_i-d_j-d_\ell)\right)} & \frac{\Gamma(d_j)\,\Gamma(1-d_\ell)}{\Gamma\left(\frac{1}{2}(1+d_i+d_j-d_\ell)\right)\Gamma\left(\frac{1}{2}(1-d_i+d_j-d_\ell)\right)}\end{pmatrix}$$

Depending on the fusion rules between i,j,Δ,ℓ none or only one of the above matrix elements may be allowed.

The recursion formula (4.52) now reads:

$$\mathsf{F}_{pq}\begin{bmatrix} j & k+\Delta\\ i & \ell\end{bmatrix} = \sum_{r,s}\mathsf{F}_{k+\Delta,r}\begin{bmatrix}\ell & \Delta\\ p & k\end{bmatrix}\cdot\mathsf{F}_{ps}\begin{bmatrix} j & k\\ i & r\end{bmatrix}\cdot\mathsf{F}_{rq}\begin{bmatrix} s & \Delta\\ i & \ell\end{bmatrix}\cdot\mathsf{F}_{s,k+\Delta}\begin{bmatrix} k & \Delta\\ j & q\end{bmatrix} \tag{4.56}$$

The index range of the sum in (4.56) is determined through the requirement that the eight diagrams associated to the four F-matrices exist. The independent conditions

on r, s are: $\mathrm{N}_{pk}{}^{r} \cdot \mathrm{N}_{l\Delta}{}^{r} \neq 0$, $\mathrm{N}_{jk}{}^{s} \cdot \mathrm{N}_{q\Delta}{}^{s} \neq 0$ and $\mathrm{N}_{rs}{}^{i} \neq 0$. In particular the maximal range of the summation indices is $r = \ell \pm \Delta$ and $s = q \pm \Delta$. Depending on i, j, k, p it may however be smaller than that.

Concerning the second route to obtain F, it can be found in the literature that the quantum 6j–symbols of $sl(2)_q$ are related to the minimal model F-matrices [AGS89b, FFK89, FGP90]. There is a catch, however, since the quantum group formulation gives the matrix structure of F, but not the associated normalisation of the CVOs. This has to be obtained by some other means, for example by explicit calculation of Coulomb gas integrals, as done by Dotsenko and Fateev in [DFa84]. The result of combining the two formulas and adapting to the notation used in this thesis (just) fits on one page and can be found in appendix A.4.

One can now check, at least numerically, that both methods indeed lead to the same result for the F's. Note that in the present normalisation the F–matrices are all real.

Identities satisfied by the fusion matrices will be useful in the calculation of the structure constants and some are listed in appendix A.5. These and many more can be found in the lecture notes by Moore and Seiberg [MSb89a]. Here we just state the pentagon identity:

$$\sum_s \mathsf{F}_{qs}\begin{bmatrix} j & k \\ p & b \end{bmatrix} \mathsf{F}_{p\ell}\begin{bmatrix} i & s \\ a & b \end{bmatrix} \mathsf{F}_{sr}\begin{bmatrix} i & j \\ \ell & k \end{bmatrix} = \mathsf{F}_{pr}\begin{bmatrix} i & j \\ a & q \end{bmatrix} \mathsf{F}_{q\ell}\begin{bmatrix} r & k \\ a & b \end{bmatrix} \tag{4.57}$$

Chapter 5.

Sewing Constraints

In the previous chapter we have constructed all functions which, for a given collection of fields, are consistent with the commutation relations of the symmetry algebra $\mathcal{A}$. Conformal blocks constitute a basis in the space of these functions. In this chapter we will investigate how the conformal blocks have to be combined to form physical correlators. This will lead to a set of relations for the structure constants, called sewing constraints.

For example a correlator of four bulk fields can be written as

$$\begin{aligned}&\langle\phi_i(z_1,z_1^*)\phi_j(z_2,z_2^*)\phi_k(z_3,z_3^*)\phi_\ell(z_4,z_4^*)\rangle\\&=\sum_{p,q}C_{pq}\;\underset{z_1^*\;\;z_2^*\;\;z_3^*\;\;z_4^*}{\underline{\begin{array}{ccccccccc}0&&i&&j&&k&&l\\&i^\vee&&p&&l&&0&\end{array}\;\;\begin{array}{ccccccccc}0&&\bar\imath&&\bar\jmath&&\bar k&&\bar\ell\\&\bar\imath^\vee&&q&&\bar\ell&&0&\end{array}}}\end{aligned}\tag{5.1}$$

where the sums run over all p for which ${\mathrm{N}_{jp}}^{i^\vee}\mathrm{N}_{k\ell}{}^p\neq 0$ and over all q s.t. ${\mathrm{N}_{\bar\jmath q}}^{\bar\imath^\vee}\mathrm{N}_{\bar k\bar\ell}{}^q\neq 0$. The first aim of this chapter is to see how the constants appearing in the (bi–)linear combinations of conformal blocks are related to the structure constants in the OPE of primary fields. Secondly, the basis of conformal blocks chosen in (5.1) is not the only one available. Each basis will result in a particular combination of OPE structure constants for the linear coefficients. The transformation matrices derived in the previous chapter describe the change of basis between the different sets of blocks. This will lead to polynomial equations on the structure constants, called sewing– or duality constraints.

5.1 Structure constants

The OPE allows us to express the product of two nearby fields $\chi_a(x)$, $\chi_b(y)$ through a sum of fields of the form

$$\chi_a(x)\chi_b(y) = \sum_c C_{ab}^c(x-y)\cdot\chi_c(y) \tag{5.2}$$

The chiral algebra $\mathcal{A}$ will fix the functional form of the C's and give a large number of relations between the different OPE coefficients. Loosely speaking the structure constants are the input in the OPE which is not fixed by requiring consistency with the action of $\mathcal{A}$ on both sides of (5.2).

More precisely there are (at least) three different ways of thinking about structure constants. In listing them we ignore all notational complications, such as those arising from non-diagonal fields and multiplicities.

(A) Structure constants are the coefficients of three point functions of primary fields:

$$\begin{aligned}&\langle\phi_i(z,\bar z)\phi_j(w,\bar w)\phi_k(0,0)\rangle\\ &\quad= C_{ijk}|z-w|^{2(h_k-h_i-h_j)}|w|^{2(h_i-h_j-h_k)}|z|^{2(h_j-h_i-h_k)}\end{aligned} \tag{5.3}$$

(B) Structure constants determine the leading behaviour of the OPE:

$$\begin{aligned}&\phi_i(z,\bar z)\phi_j(w,\bar w)\\ &\quad=\sum_k C_{ij}{}^k\,|z-w|^{2(h_k-h_i-h_j)}\phi_k(w,\bar w)+\ \text{(higher terms)}\end{aligned} \tag{5.4}$$

(C) Structure constants give the linear combination of CVOs that make up the operator description of a primary field:

$$\phi_i(z,\bar z) = \sum_{j,k} C_{ij}{}^k\,\mathrm{V}_{ij}^k(|i\rangle,z)\otimes\mathrm{V}_{ij}^k(|i\rangle,\bar z) \tag{5.5}$$

We will be working with (B), as it is the most convenient and intuitive when deriving the sewing constrains. (B) does have to be treated with care though – we will come back to that in a moment.

First it should be mentioned that it is (C) that extends most easily beyond the case $\mathcal{A}=\mathsf{Vir}$. Moore and Seiberg use this definition of structure constants in their proof of the naturality theorem [MSb89c]. The main reason why (C) is superior to (A) and (B) is the following: It might happen that two $\mathcal{A}$–primary fields ϕ_i and ϕ_j couple to a family $[\phi_k]$, in such a way that the first term in the OPE is not the

primary field ϕ_k, but a descendent. In this case the coefficient of $(z-w)^{h_k-h_i-h_j}$ would be zero and cannot be used to define the structure constant. In other words the three point function of three $\mathcal{A}$–primaries might vanish, but might be nonzero for some descendents.

In a fully fledged notation for (C) we would have to give a name α to each distinct pair of representations $\mathcal{H}_{r_\alpha}\otimes\mathcal{H}_{\bar{r}_\alpha}$ in the space of states $\mathcal{H}$ and accommodate two types of multiplicities: of representations in the space of states and of CVOs for a given triple of representations. One possible notation would be:

$$\phi_\alpha(z,\bar{z}) = \sum_{\beta,\gamma,a,b} d^{\gamma}_{\alpha\beta,ab}\cdot \mathrm{V}^{r_\gamma}_{r_\alpha r_\beta,a}(|r_\alpha\rangle,z)\otimes \mathrm{V}^{\bar{r}_\gamma}_{\bar{r}_\alpha \bar{r}_\beta,b}(|\bar{r}_\alpha\rangle,\bar{z}) \tag{5.6}$$

where the $V\otimes V$–terms act on $\mathcal{H}_\alpha\otimes\mathcal{H}_\beta\to\mathcal{H}_\gamma$. Also we have used d instead of C to denote the structure constants. This is because the relation to (A) and (B) is no longer immediate (for a start d has two indices to many).

For the rest of the text we will restrict ourselves to $\mathcal{A}=\mathsf{Vir}$. In this case one can show that the three numbers defined in (A)–(C) have the same meaning. In (A) one can compute the leading behaviour $w,\bar{w}\to 0$ directly, or by inserting the OPE (B). Comparing the two yields $C_{ijk} = C_{ij}{}^{k}\ C_{kk}{}^{1}$. Alternatively one can replace the three primary fields in (A) by the sums of CVOs given in (C). This gives a product of three structure constants $C_{kk}{}^{1}\ C_{ij}{}^{k}\ C_{i1}{}^{i}$. The last one has to be equal to one as it describes multiplication by the identity field. We again end up with the previous relation.

Let us now focus on the structure constants as defined by (B). First the notation has to be refined to handle pairs of representations in $\mathcal{H}$ which occur with multiplicities. Denote with $\mathcal{R}$ the set of all representations of Vir entering the model under consideration (in the sense of (A4) in section 4.3), i.e. all representations in the Kac-table. Let $\mathcal{F}$ be the set of all pairs of representation $\mathcal{H}_i\otimes\mathcal{H}_j$ in $\mathcal{H}$, *without* multiplicities. I.e. an element $r\in\mathcal{F}$ corresponds to a pair of representations $(i,\bar{\imath})$. To keep notation at bay we will often use $i\in\mathcal{F}$ to denote the pair $(i,\bar{\imath})$. The two representations have conformal weights $h_i, h_{\bar{\imath}}$ or, for better readability, $h_i,\bar{h}_i$.

To specify a primary field uniquely, an element $i\in\mathcal{F}$ gets an additional index i_α to account for possible multiplicities. That is, ϕ_{i_α}, ϕ_{i_β}, ... are distinct primary fields that transform in the same representation $\mathcal{H}_i\otimes\mathcal{H}_{\bar{\imath}}$. As Vir–highest weight representations are self-dual, we can always pick a basis of primary fields s.t. the two point function satisfies the following orthogonality condition:

$$\langle\phi_{i_\alpha}(x)\phi_{j_\beta}(y)\rangle = \delta_{i,j}\delta_{\alpha,\beta}f(x-y) \tag{5.7}$$

Now we can point out the first subtlety in definition (B). Suppose we have some correlator $f(z,w)=\langle\cdots\phi_{i_\alpha}(z,\bar{z})\phi_{j_\beta}(w,\bar{w})\cdots\rangle$. Next suppose that $f(z,w)$ presents us with a leading behaviour $A\cdot(z-w)^a(\bar{z}-\bar{w})^b$ in the limit $z\to w$. Then from $a=h_k-h_i-h_j$, $b=\bar{h}_k-\bar{h}_i-\bar{h}_j$ we can identify which element $k\in\mathcal{F}$ this corresponds to, but not necessarily which primary field ϕ_{k_γ}. So all we learn is

$$A=\sum_\gamma C_{i_\alpha j_\beta}{}^{k_\gamma} \quad . \tag{5.8}$$

Thus in a way the duality constraints will be less restrictive if $\mathcal{H}$ contains fields with multiplicities.

The next subtlety is that we do not only want the leading behaviour of $f(z,w)$ but we want to identify all primary fields which contribute in the limit $z\to w$. That is we want to be able to write

$$\begin{aligned} f(x,y)=\sum_{k\in\mathcal{F}} A_k\cdot(z-w)^{h_k-h_i-h_j}(1+a_1\cdot(z-w)+\cdots) \\ \cdot(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}(1+b_1\cdot(\bar{z}-\bar{w})+\cdots) \end{aligned} \tag{5.9}$$

The constants A_k are precisely the coefficients of the conformal blocks and it is essential for the duality constraints that they can be extracted unambiguously. This will work fine unless there are two weights $h_{k_1}<h_{k_2}$ which differ by an integer. Then the contribution of the primary field h_{k_2} will occur with the same power, $(z-w)^a$ say, as a descendent in the h_{k_1}–family. The coefficient A_{k_2} can however still be extracted. Since we know the constant A_{k_1} we can just subtract the contribution of the descendent from the number in front of $(z-w)^a$ to obtain A_{k_2}. In deriving the constraints we do not actually have to do this calculation, but we will use the fact that the A_k are well–defined.

Let us fix some notation used in the following. The vacuum expectation value of the UHP with boundary condition a imposed on the real line will be denoted with $\langle 1\rangle^a_{\text{UHP}}$. The vacuum expectation value of the full complex plane with no boundaries present will be denoted with $\langle 1\rangle$ or $\langle 0|0\rangle$.

The leading terms in the bulk–bulk, bulk–boundary and boundary–boundary operator product expansions of primary fields are, in this order:

$$\phi_{i_\alpha}(z)\phi_{j_\beta}(w)=\sum_{k,\gamma} C_{i_\alpha j_\beta}{}^{k_\gamma}\,(z-w)^{h_k-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}\left(\phi_{k_\gamma}(w)+\cdots\right) \quad |z|>|w| \tag{5.10}$$

$$\phi_{i_\alpha}(x+iy) = \sum_{k,\gamma} {}^aB_{i_\alpha}{}^{k_\gamma} \cdot (2y)^{h_k-h_i-\bar{h}_i} \cdot \Big(\psi_{k_\gamma}^{(aa)}(x) + \cdots \Big) \qquad y>0 \tag{5.11}$$

$$\psi_{i_\alpha}^{(ab)}(x)\, \psi_{j_\beta}^{(bc)}(y) = \sum_{k,\gamma} C_{i_\alpha j_\beta}^{(abc)k_\gamma} \cdot (x-y)^{h_k-h_i-h_j} \cdot \Big(\psi_{k_\gamma}^{(ac)}(y) + \cdots \Big) \qquad x>y \tag{5.12}$$

The omissions stand for an infinite sum of descendents of the primary field in question. A method to work out the coefficients of the descendent fields is described in appendix A.1. Eqns. (5.10)–(5.12) define the three sets of structure constants which are necessary to compute the correlation functions of minimal models: the bulk structure constants $C_{i_\alpha j_\beta}{}^{k_\gamma}$ with three bulk fields i_α, j_β, k_γ, the bulk-boundary couplings ${}^aB_{i_\alpha}{}^{k_\gamma}$ with boundary condition a, bulk field i_α and boundary field k_γ and the boundary structure constants $C_{i_\alpha j_\beta}^{(abc)k_\gamma}$ with boundary conditions a, b, c and boundary fields i_α, j_β, k_γ.

5.2 Duality constraints

It has already been pointed out in section 2.3 that one has to distinguish between pure and mixed boundary conditions. On a pure boundary condition the identity is the unique boundary field with weight h=0. In the derivation of the constraints for the structure constants below, we restrict ourselves to pure boundary conditions.

To illustrate the method used to obtain the sewing constraints, the constraint arising from two bulk fields in the presence of a pure boundary is derived in detail [CLe91]. For simplicity suppose there are only diagonal primary fields without multiplicities. Say the bulk fields have distance d from each other and distance y from the boundary. There are two limits we can compare. First we can take the two bulk fields to approach each other. The bulk OPE (5.10) then gives the asymptotics for $d \to 0$. The resulting sum of primary bulk field can be expanded in terms of boundary fields using (5.11). As there is no other boundary field to couple to, only the identity has a non-vanishing contribution. We obtain the asymptotics

$$\langle \phi_i(d+iy)\phi_j(iy)\rangle^a_{\text{UHP}} \underset{d\to 0}{\sim} \sum_m C_{ij}{}^m\, {}^aB_m{}^1\, \langle 1\rangle^a_{\text{UHP}} \cdot d^{2h_m-2h_i-2h_j}\, (2y)^{-2h_m} \,. \tag{5.13}$$

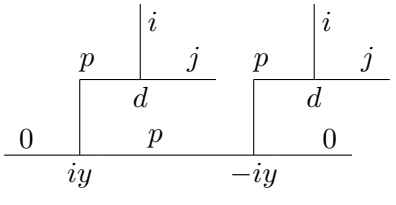
On the other hand we could expand the two bulk fields in terms of boundary fields, resulting in the $y \to 0$ asymptotics. We are left with a sum of boundary two point functions, of which the primary boundary fields will appear in the leading term for

$y \to 0$. We get:

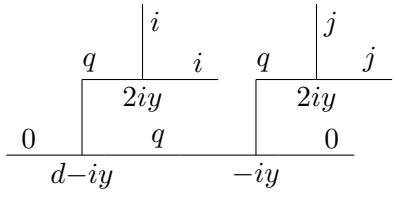
$$\langle \phi_i(d{+}iy)\phi_j(iy)\rangle^a_{\text{UHP}} \underset{y\to 0}{\sim} \sum_\ell {}^aB_i{}^\ell \; {}^aB_j{}^\ell \; C_{\ell\ell}^{(aaa)1} \, \langle 1\rangle^a_{\text{UHP}} \cdot (2y)^{2h_\ell - 2h_i - 2h_j} \, d^{-2h_\ell} \; . \quad (5.14)$$

We can express the complete functional dependence of the two-point correlator through the appropriate conformal blocks. Recall that, due to Cardy's doubling trick mentioned in section 2.3, a bulk field at iy on the UHP corresponds to two insertions of CVOs, one at iy and one at the reflected point $-iy$. Here we write the correlator as a linear combination of two different bases of blocks, each adapted to one of the asymptotic behaviours (5.13), (5.14):

$$\langle \phi_i(d{+}iy)\phi_j(iy)\rangle^a_{\text{UHP}} = \sum_p c_p \cdot$$



$$\langle \phi_i(d{+}iy)\phi_j(iy)\rangle^a_{\text{UHP}} = \sum_q d_q \cdot$$

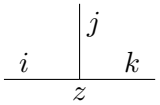


(5.15)

where are functional arguments of the blocks (i.e. the external legs) are set to the respective highest weight vectors.

There is another subtlety at this point. We have defined products of CVOs as a convergent power series for positive, real and ordered insertion points. In (5.15) we have complex arguments, so we have to specify an analytic continuation. E.g. for the first line we start with some $a > b > 0$ and continue a to iy and b to $-iy$.

The CVOs have been normalised in (4.25) to have the asymptotics

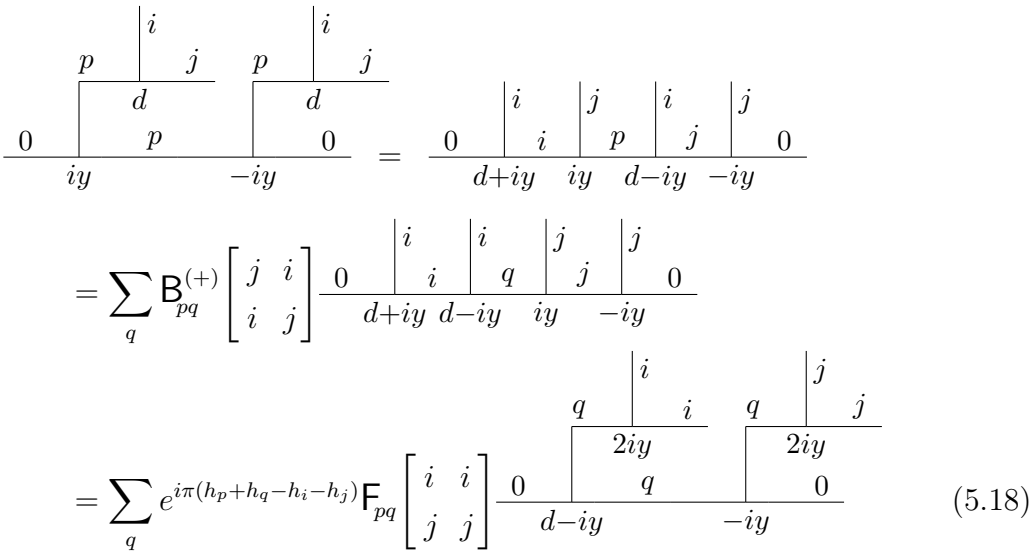


$$|j\rangle \otimes |k\rangle = z^{h_k - h_i - h_j}\big(|i\rangle + a_1 \, z \cdot L_{-1}|i\rangle + \dots\big) \quad , \; \langle i|i\rangle = 1 \; . \quad (5.16)$$

Using this we can determine the asymptotics of the two-point correlator given in terms of conformal blocks (5.15) as

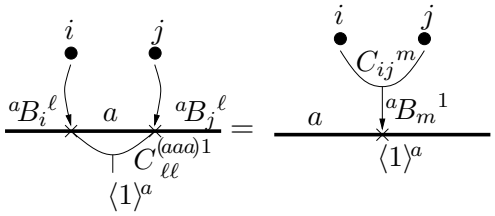
$$\begin{aligned} \langle \phi_i(d{+}iy)\phi_j(iy)\rangle^a_{\text{UHP}} &\underset{d\to 0}{\sim} \sum_p c_p \cdot e^{-i\pi h_p} \cdot d^{2h_p - 2h_i - 2h_j} \, (2y)^{-2h_p} \\ \langle \phi_i(d{+}iy)\phi_j(iy)\rangle^a_{\text{UHP}} &\underset{y\to 0}{\sim} \sum_q d_q \cdot e^{i\pi(h_q - h_i - h_j)} \cdot d^{-2h_q} \, (2y)^{2h_q - 2h_i - 2h_j} \end{aligned} \quad (5.17)$$

Comparing this to (5.13), (5.14) fixes the values c_p and d_q in terms of structure constants. We can also perform the basis transformation explicitly in (5.15) in terms of F-matrices:



(5.18)

Reinserting this relation into the expression for the correlator (5.15) gives a linear relation between c_p and d_q, which translates in a (nonlinear) constraint on the structure constants. Below the constraint is listed, together with a picture detailing the different limits that have been taken on each side



$$ {}^aB_i{}^\ell \; {}^aB_j{}^\ell \; C^{(aaa)1}_{\ell\ell} = \sum_m C_{ij}{}^m \; {}^aB_m{}^1 \cdot \mathsf{F}_{m\ell} \begin{bmatrix} i & i \\ j & j \end{bmatrix} \tag{5.19} $$

There are certainly infinitely many different ways to take limits in correlators resulting in infinitely many constraints for the structure constants. It is the work of Sonoda [Son88b] (for correlators in Riemann surfaces without boundaries) and Lewellen [Lew92] (who extends the treatment to surfaces with boundaries) to show that of all constraints that can be obtained by taking different limits, only finitely many are independent. Once these are satisfied all others hold automatically.

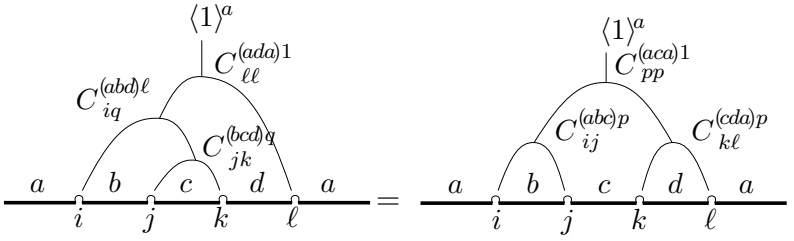
The method used by Sonoda and later Lewellen to obtain this result, was to take a given Riemann surface with field insertions and cut it apart into a few basic building blocks (this corresponds to taking a certain limit in the correlator). If all

different cuttings give the same answer this is equivalent to all different limits being consistent. They showed that all different cuttings can be related to each other by a finite number of elementary moves. Each of these moves corresponds to a fundamental sewing constraint.

For orientable surfaces without boundaries there are only two fundamental constraints. The four point function on the sphere at genus zero and the one-point function on the torus at genus one. In the presence of boundaries we get four more constraints. At genus zero the new fundamental constraints arise from considering four boundary fields, two boundary/one bulk field and two bulk/one boundary field. At genus one the additional conditions arises from the one point function on the cylinder. In this thesis we only consider orientable surfaces. The additional constraints for non-orientable surfaces are given in [FPS93, PSS95a, PSS95b].

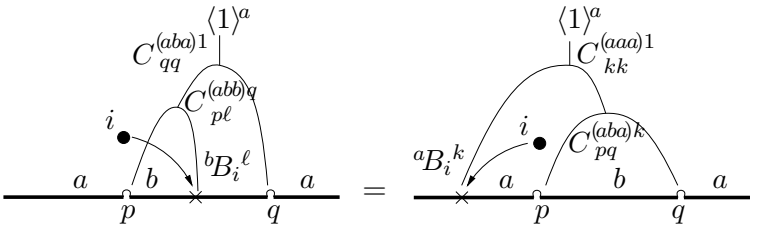
Below is a list of all fundamental sewing constraints for genus zero, as given by Lewellen [Lew92]. First we give the constraints in "light" (i.e. multiplicity free and diagonal fields) notation, together with pictures detailing the limits.

For four boundary fields i, j, k, ℓ:



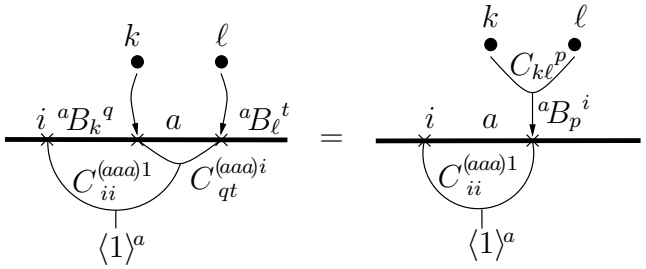
$$C^{(bcd)q}_{jk}\; C^{(abd)\ell}_{iq}\; C^{(ada)1}_{\ell\ell}\; \langle 1\rangle^a_{\mathrm{UHP}} = \sum_p C^{(abc)p}_{ij}\; C^{(cda)p}_{k\ell}\; C^{(aca)1}_{pp}\; \langle 1\rangle^a_{\mathrm{UHP}} \cdot \mathsf{F}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} \tag{5.20}$$

For one bulk field i and two boundary fields p, q:



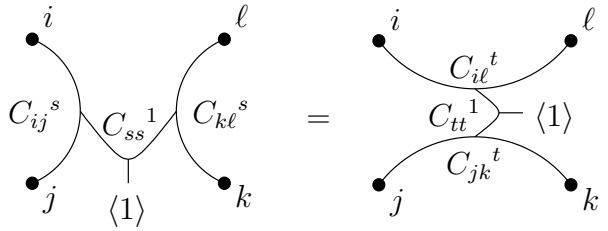
$$ {}^bB_i{}^\ell \; C_{p\ell}^{(abb)q} \; C_{qq}^{(aba)1} \; \langle 1\rangle^a_{\mathrm{UHP}} = \sum_{k,m} {}^aB_i{}^k \; C_{pq}^{(aba)k} \; C_{kk}^{(aaa)1} \; \langle 1\rangle^a_{\mathrm{UHP}} \cdot e^{i\pi(2h_m+\frac{1}{2}h_k-h_p-h_q-2h_i+\frac{1}{2}h_\ell)} \cdot \mathsf{F}_{km}\begin{bmatrix} q & i \\ p & i \end{bmatrix} \mathsf{F}_{m\ell}\begin{bmatrix} i & i \\ p & q \end{bmatrix} \tag{5.21} $$

For two bulk fields and one boundary field:



$$ {}^aB_k{}^q \; {}^aB_\ell{}^t \; C_{qt}^{(aaa)i} \; C_{ii}^{(aaa)1} \; \langle 1\rangle^a_{\mathrm{UHP}} = e^{i\frac{\pi}{2}(h_t-h_i-h_q-2h_\ell)} \sum_{p,r} C_{k\ell}{}^p \; {}^aB_p{}^i \; C_{ii}^{(aaa)1} \; \langle 1\rangle^a_{\mathrm{UHP}} e^{i\pi h_r} \cdot \mathsf{F}_{pr}\begin{bmatrix} i & \ell \\ p & k \end{bmatrix} \mathsf{F}_{pq}\begin{bmatrix} \ell & r \\ k & k \end{bmatrix} \mathsf{F}_{rt}\begin{bmatrix} \ell & \ell \\ q & i \end{bmatrix} \tag{5.22} $$

For four bulk fields:



$$ C_{ij}{}^s \; C_{k\ell}{}^s \; C_{ss}{}^1 \; \langle 1\rangle \cdot \mathsf{F}_{st}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} = C_{jk}{}^t \; C_{\ell i}{}^t \; C_{tt}{}^1 \; \langle 1\rangle \cdot \mathsf{F}_{ts}\begin{bmatrix} \ell & k \\ i & j \end{bmatrix} \tag{5.23} $$

To compare the above formulas to the corresponding expressions in Lewellen's paper two remarks are in order: The form of the constraints given here is slightly different from [Lew92]. Firstly, the definition of the boundary OPE (5.12) differs slightly

and the labels have to be permuted correspondingly. Second, different limits of the correlators were chosen in (5.20)–(5.23) with the result that the expectation value of the identity field cancels from all expressions. Otherwise (5.20)–(5.23) are equivalent to the corresponding equations in [Lew92].

In "heavy" notation, including multiplicity labels and non-diagonal fields the corresponding constraints read:

For four boundary fields $i_\alpha, j_\beta, k_\gamma, \ell_\delta$ and boundary conditions a, b, c, d:

$$\sum_\varepsilon C^{(bcd)q_\varepsilon}_{j_\beta k_\gamma}\; C^{(abd)\ell_\delta}_{i_\alpha q_\varepsilon}\; C^{(ada)1}_{\ell_\delta \ell_\delta}\; \langle 1\rangle^a_{\mathrm{UHP}} = \sum_p \left(\sum_\nu C^{(abc)p_\nu}_{i_\alpha j_\beta}\; C^{(cda)p_\nu}_{k_\gamma \ell_\delta}\; C^{(aca)1}_{p_\nu p_\nu}\; \langle 1\rangle^a_{\mathrm{UHP}} \right) \mathsf{F}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} \tag{5.24}$$

For two boundary fields p_ν, q_ε and one bulk field i_α:

$$\begin{aligned} &\sum_\delta {}^bB_{i_\alpha}{}^{\ell_\delta}\; C^{(abb)q_\varepsilon}_{p_\nu \ell_\delta}\; C^{(aba)1}_{q_\varepsilon q_\varepsilon}\; \langle 1\rangle^a_{\mathrm{UHP}} \\ &\quad = \sum_k \left(\sum_\gamma {}^aB_{i_\alpha}{}^{k_\gamma}\; C^{(aba)k_\gamma}_{p_\nu q_\varepsilon}\; C^{(aaa)1}_{k_\gamma k_\gamma}\; \langle 1\rangle^a_{\mathrm{UHP}} \right) \\ &\qquad \cdot \sum_m e^{i\pi(2h_m - 2h_i - h_p - h_q + \frac{1}{2}(h_k + h_\ell))} \cdot \mathsf{F}_{km}\begin{bmatrix} \bar\imath & q \\ i & p \end{bmatrix} \mathsf{F}_{m\ell}\begin{bmatrix} i & \bar\imath \\ p & q \end{bmatrix} \end{aligned} \tag{5.25}$$

For two bulk fields i_α, j_β and one boundary field k_γ:

$$\begin{aligned} &\sum_\rho C_{i_\alpha j_\beta}{}^{m_\rho}\; {}^aB_{m_\rho}{}^{k_\gamma}\; C^{(aaa)1}_{k_\gamma k_\gamma}\; \langle 1\rangle^a_{\mathrm{UHP}} \\ &\quad = \sum_{p,q} \left(\sum_{\nu,\varepsilon} {}^aB_{i_\alpha}{}^{p_\nu}\; {}^aB_{j_\beta}{}^{q_\varepsilon}\; C^{(aaa)k_\gamma}_{p_\nu q_\varepsilon}\; C^{(aaa)1}_{k_\gamma k_\gamma}\; \langle 1\rangle^a_{\mathrm{UHP}} \right) \\ &\qquad \cdot \sum_r e^{i\frac{\pi}{2}(h_k + h_p - h_q - 2h_r + h_m - \bar h_m - h_i + \bar h_i + h_j + \bar h_j)} \cdot \mathsf{F}_{qr}\begin{bmatrix} k & \bar\jmath \\ p & j \end{bmatrix} \mathsf{F}_{pm}\begin{bmatrix} \bar\imath & r \\ i & j \end{bmatrix} \mathsf{F}_{r\bar m}\begin{bmatrix} \bar\imath & \bar\jmath \\ m & k \end{bmatrix} \end{aligned} \tag{5.26}$$

For four bulk fields $i_\alpha, j_\beta, k_\gamma, \ell_\delta$:

$$\begin{aligned} &\sum_\varepsilon C_{i_\alpha k_\gamma}{}^{q_\varepsilon}\; C_{j_\beta \ell_\delta}{}^{q_\varepsilon}\; C_{q_\varepsilon q_\varepsilon}{}^{1}\; \langle 1\rangle \\ &\quad = \sum_{p,\bar p} \left(\sum_\nu C_{i_\alpha j_\beta}{}^{p_\nu}\; C_{k_\gamma \ell_\delta}{}^{p_\nu}\; C_{p_\nu p_\nu}{}^{1}\; \langle 1\rangle \right) \\ &\qquad \cdot e^{i\pi(h_i - \bar h_i + h_\ell - \bar h_\ell - h_p + \bar h_p - h_q + \bar h_q)} \mathsf{F}_{pq}\begin{bmatrix} j & \ell \\ i & k \end{bmatrix} \mathsf{F}_{\bar p \bar q}\begin{bmatrix} \bar\jmath & \bar\ell \\ \bar\imath & \bar k \end{bmatrix} \end{aligned} \tag{5.27}$$

The constraints have been derived using notation and techniques for calculating with conformal blocks presented in the previous chapter. To illustrate the method, the calculation that leads to constraint (5.26) is given in appendix A.6.

Next consider the two genus one constraints in the case of no field insertions, i.e. the torus and cylinder partition functions. For the torus (circumferences R and L) we get the condition that the partition function of the bulk theory has to be modular invariant:

$$tr_{\mathcal{H}} e^{-\frac{2\pi R}{L}(L_0+\bar{L}_0-\frac{c}{12})} = tr_{\mathcal{H}} e^{-\frac{2\pi L}{R}(L_0+\bar{L}_0-\frac{c}{12})} \tag{5.28}$$

For a cylinder of length R and circumference L we get the condition [Car89b]:

$$\langle a| e^{-\frac{2\pi R}{L}(L_0+\bar{L}_0-\frac{c}{12})} |b\rangle = tr_{\mathcal{H}_{ab}} e^{-\frac{\pi L}{R}(L_0-\frac{c}{24})} \tag{5.29}$$

where $\mathcal{H}_{ab}$ is the space of states of a strip with boundary conditions a and b.

Let $\mathcal{E}$ be the set of all diagonal bulk fields $\mathcal{E} = \{\phi_{i_\alpha} | i{=}\bar{\imath}\}$. Recall that a conformal boundary state $\langle a|$ can be written as a linear combination of Ishibashi states (see eqn. (2.57))

$$\langle a| = \sum_{i_\alpha \in \mathcal{E}} g_a^{i_\alpha} \langle\!\langle i_\alpha| \tag{5.30}$$

where the sum runs over all diagonal bulk fields i_α. If we also write $\mathcal{H}_{ab} = \sum_j n_{ja}{}^b \mathcal{H}_j$ and make use of the modular transformation properties of the characters, we obtain the formula [Car89b]

$$\sum_{i_\alpha \in \mathcal{E}} g_a^{i_\alpha} (g_b^{i_\alpha})^* \langle i_\alpha | i_\alpha \rangle S_i{}^j = n_{ja}{}^b \ . \tag{5.31}$$

To make the connection between $g_a^{i_\alpha}$ and the structure constants we have to link the normalisations in the bulk and boundary CFT: We demand that the one-point function in unit disk geometry is given by the inner product with a boundary state:

$${}^a B_{i_\alpha}{}^1 \langle 1 \rangle^a_{\text{disc}} = \langle \phi_{i_\alpha}(0,0) \rangle^a_{\text{disc}} = \langle a | i_\alpha \rangle = g_a^{i_\alpha} \langle i_\alpha | i_\alpha \rangle \tag{5.32}$$

Recall from (2.31) and (2.37) the relations $\phi_{i_\alpha}(0,0)|0\rangle = |i_\alpha\rangle$ and $\langle i_\alpha | i_\alpha \rangle = C_{i_\alpha i_\alpha}{}^1 \langle 0|0\rangle$. Thus the $g_a^{i_\alpha}$ are not independent quantities, but can be ex-

pressed as

$$g_a^{i_\alpha} = \frac{{}^aB_{i_\alpha}{}^1\,\langle 1\rangle^a_{\rm disc}}{C_{i_\alpha i_\alpha}{}^1\,\langle 0|0\rangle}\ . \tag{5.33}$$

We have now introduced all the objects we aim to determine by solving the sewing constraints:

$$C_{i_\alpha j_\beta}{}^{k_\gamma}\ ,\ {}^aB_{i_\alpha}{}^{p_\beta}\ ,\ C^{(abc)r_\gamma}_{p_\alpha q_\beta}\ ,\ \langle 1\rangle^a_{\rm disc}\,,\ \langle 0|0\rangle \tag{5.34}$$

Of course the sewing constraints cannot determine the quantities (5.34) uniquely, but at most up to rescalings[1] of the fields. The effect of rescaling on the structure constants can be summarized as follows: suppose we change (suppressing the multiplicity indices)

$$\phi_i \to \alpha_i\phi_i\,,\ \ \psi_p^{(ab)} \to \beta_p^{ab}\,\psi_p^{(ab)}\,,\ \langle 0|0\rangle \to \mu^2\langle 0|0\rangle\ . \tag{5.35}$$

Then the structure constants change according to

$$\begin{aligned} &C_{ij}{}^k \to \frac{\alpha_i\alpha_j}{\alpha_k}\,C_{ij}{}^k\ ,\ {}^aB_i{}^p \to \frac{\alpha_i}{\beta_p^{aa}}\,{}^aB_i{}^p\ ,\ C^{(abc)r}_{pq} \to \frac{\beta_p^{ab}\beta_q^{bc}}{\beta_r^{ac}}\,C^{(abc)r}_{pq}\\ &g_a^{i_\alpha} \to (\alpha_i\mu)^{-1}g_a^{i_\alpha}\,,\ \langle 1\rangle^a_{\rm disc} \to \mu\langle 1\rangle^a_{\rm disc}\ . \end{aligned} \tag{5.36}$$

The behaviour of $g_a^{i_\alpha}$ follows from (5.31), where the rhs is normalisation independent.

The rescalings are a symmetry of the sewing constraints in the sense that, if we have a solution for all quantities in (5.34), then the rescaled versions (5.36) form a solution as well[2].

5.3 Cylinder partition function

As mentioned in the previous section, we have to distinguish two kinds of boundary conditions: *pure*– or Cardy–boundary conditions and *mixed* boundary conditions. Pure boundary conditions are characterised by the property that they have a unique

[1] In fact in the presence of multiplicities not only rescalings are possible, but also linear combinations of equivalent fields. This fact will be used when choosing a suitable basis of primary fields to solve the D–series sewing constraints.

[2] There are two subtleties hidden in (5.36), both connected to choosing imaginary scale factors. If α_i is imaginary we have to reintroduce the sign $\sigma(i)$ from eqn. (2.37), which was set to one in section 2.1. This is necessary to insure that $\langle i|i\rangle \to |\alpha_i|^2\mu^2\langle i|i\rangle$ even though $C_{ii}{}^1 \to \alpha_i{}^2\,C_{ii}{}^1$. If μ is imaginary, say $\mu = i$ for simplicity, the inner product itself changes sign, i.e. $\langle a|b\rangle \to -\langle a|b\rangle$. These two subtleties have to be taken into account to verify that (5.31) is invariant under the rescalings (5.35).

($sl(2)$–invariant) vacuum. In other words, there is only one weight zero field on a pure boundary a: ${n_{0a}}^a{=}1$.

Once we know all pure boundary conditions, we can generate all mixed ones as superpositions thereof. If Z_{ab} denotes the partition function of a strip with pure boundary conditions a, b, the partition functions of mixed boundary conditions are of the form

$$Z_{\text{mixed}} = \sum_{a,b \text{ pure}} k_a \cdot k_b \cdot Z_{ab} \tag{5.37}$$

for $k_a, k_b \in \mathbb{Z}_0^+$. In particular on a mixed boundary x we have ${n_{0x}}^x{>}1$: We get one weight zero field from each identity field on the pure boundaries that make up x.

In more physical terms, for pure boundary conditions the correlators obey the cluster decomposition principle, whereas for mixed boundary conditions they do in general not.

In this section we will be concerned with finding all pure conformal boundary conditions. Equation (5.31) places strong constraints on the field content and the possible boundary conditions of the theory on the strip. This was first exploited by Cardy [Car89b] and later extended to more general situations in [BPZ98, BPPZ98, BPPZ99]. The result is that the matrices ${n_{ia}}^b$ form an integer valued representation of the fusion algebra. Here we will follow closely the arguments concerning this point from [BPPZ98], specialised to the case of minimal models.

It will be helpful to fix a specific scaling of the *diagonal* bulk fields $\phi_{i_\alpha} {\in} \mathcal{E}$ and the bulk vacuum. We choose

$$\langle 0|0\rangle = {S_1}^1 \,, \quad {C_{i_\alpha i_\alpha}}^1 = \frac{{S_i}^1}{{S_1}^1} \text{ for all } i_\alpha \in \mathcal{E} \,. \tag{5.38}$$

Then we can define quantities $\psi_a^{i_\alpha}$ which are rescaled versions of $g_a^{i_\alpha}$:

$$\psi_a^{i_\alpha} = {}^aB_{i_\alpha}{}^1 \langle 1\rangle_{\text{disc}}^a \,, \qquad {}^aB_{i_\alpha}{}^1 = \frac{\psi_a^{i_\alpha}}{\psi_a^1} \tag{5.39}$$

The second equation follows from the first, since for $i_\alpha{=}\mathbb{1}$ it reads $\psi_a^1{=}\langle 1\rangle_{\text{disc}}^a$. In terms of $\psi_a^{i_\alpha}$ the cylinder constraint (5.31) takes the more convenient form

$$\sum_{i_\alpha \in \mathcal{E}} \frac{{S_i}^j}{{S_i}^1} \psi_a^{i_\alpha} (\psi_b^{i_\alpha})^* = {n_{ja}}^b \,. \tag{5.40}$$

In section 2.3 we argued that changing the boundary conditions along the real line gives rise to a nontrivial field insertion. Consequently ${n_{1a}}^b = \delta_{a,b}$. Substituting this

into (5.40) shows that the $\psi_a^{i_\alpha}$ are orthonormal:

$$\sum_{i_\alpha \in \mathcal{E}} \psi_a^{i_\alpha} (\psi_b^{i_\alpha})^* = \delta_{a,b} \tag{5.41}$$

We now make the assumption that the $\psi_a^{i_\alpha}$ are also *complete*, i.e. form an ON-basis:

$$\sum_{a \in \mathcal{B}} \psi_a^{i_\alpha} (\psi_a^{j_\alpha})^* = \delta_{i_\alpha, j_\beta} \tag{5.42}$$

where the sum runs over the set of all pure conformal boundary conditions $\mathcal{B}$. In particular there are as many diagonal bulk fields as pure conformal boundary conditions $|\mathcal{B}|=|\mathcal{E}|$.

At this point (5.42) is a technical assumption to make the rest of the calculation work. To motivate why with this assumption we will not neglect a large class of conformal boundary conditions, let us deviate slightly from the course of arguments in [BPPZ98] and elaborate a bit on (5.42).

Let, for the following discussion, a "theory" denote a solution to the list of quantities (5.34), which solves the sewing constraints (5.24)–(5.29).

Since the vectors ψ_a are orthonormal, if there are $|\mathcal{E}|$ of them, they automatically form an ON-basis. The only problems with assumption (5.42) can thus come form theories which have *less* than $|\mathcal{E}|$ conformal boundary conditions. On the other hand, if we have a theory with $|\mathcal{E}|$ boundary conditions, we can construct a new solution to all sewing constraints by restricting the theory to a subset of these boundary conditions. So the assumption entering formula (5.42) can be phrased more precisely as "In every theory the set of $\psi_a^{i_\alpha}$ can be extended to an ON-basis s.t. (5.40) is satisfied".

In an attempt to prove or disprove this statement one can look at the remaining sewing constraints. To start with, rewrite (5.26) with k_γ=1 and the remaining F–matrix multiplied to the other side. By the fusion rules ${}^aB_{i_\alpha}{}^1$ can be nonzero only if $i=\bar{\imath}$, so all phases cancel. We get

$${}^aB_{i_\alpha}{}^1 \, {}^aB_{j_\beta}{}^1 = \sum_{k_\gamma \in \mathcal{E}} \left(C_{i_\alpha j_\beta}{}^{k_\gamma} \, \mathsf{F}_{k1} \begin{bmatrix} i & i \\ j & j \end{bmatrix} \right) \cdot {}^aB_{k_\gamma}{}^1 \ , \tag{5.43}$$

which looks as though the B's want to form an algebra, separately for each boundary condition a. Now *suppose* (5.43) does indeed define an algebra. Then this algebra is associative, commutative, finite dimensional (with dimension $|\mathcal{E}|$) and has a *-

structure[3]. As a general result in representation theory[4], there are then exactly $|\mathcal{E}|$ one-dimensional representations, i.e. (5.43) has exactly $|\mathcal{E}|$ sets of solutions for the ${}^aB_{i_\alpha}{}^1$. Furthermore[5] we can always find an orthogonal $|\mathcal{E}|\times|\mathcal{E}|$–matrix $\psi_a^{i_\alpha}$ s.t. ${}^aB_{i_\alpha}{}^1 = \psi_a^{i_\alpha}/\psi_a^1$. On the other hand we will show below that the converse holds as well: If the $\psi_a^{i_\alpha}$ form an ON-basis then (5.43) defines an algebra.

The assumption is thus equivalent to the statement "In every theory (5.43) defines an algebra". This is interesting, because the validity of (5.42) can now be verified entirely in terms of bulk–data and does not involve the theory on the UHP at all.

To prove this statement it would be enough to demonstrate associativity of (5.43). Written out completely, the associativity condition $(B_iB_j)B_k = B_i(B_jB_k)$ evaluates to

$$\begin{aligned}\sum_{t_\tau\in\mathcal{E}} C_{i_\alpha j_\beta}{}^{t_\tau}\; C_{t_\tau k_\gamma}{}^{m_\mu}\, \mathsf{F}_{t1}\begin{bmatrix} j & j\\ i & i\end{bmatrix}\mathsf{F}_{m1}\begin{bmatrix} k & k\\ t & t\end{bmatrix}\\ = \sum_{s_\sigma\in\mathcal{E}} C_{j_\beta k_\gamma}{}^{s_\sigma}\; C_{s_\sigma i_\alpha}{}^{m_\mu}\, \mathsf{F}_{s1}\begin{bmatrix} j & j\\ k & k\end{bmatrix}\mathsf{F}_{m1}\begin{bmatrix} l & l\\ i & i\end{bmatrix}\,,\end{aligned} \tag{5.44}$$

which is quite similar to the bulk duality relation (5.27). In fact, for the diagonal modular invariant bulk theory (the A–series) one can show that (5.27) implies (5.44). It follows that for the A-series the conformal boundary conditions found by Cardy in [Car89b] are indeed the only possible solution to the sewing constraints.

For the D-series the explicit solution (by construction, see chapter 6) shows that (5.43) defines an algebra. However for non-diagonal theories, the author is not aware of a way to deduce the associativity of (5.43) from (5.27).

A possible solution might come from Moore and Seiberg's naturality theorem [MSb89c]. Let $M_{k\ell}$ be the number of bulk fields transforming in $\mathcal{H}_k\otimes\mathcal{H}_\ell$. The theorem states that either the field content is of the form $M_{k\ell} = \delta_{k,\pi(\ell)}$ for some permutation $\pi(\ell)$, or there are additional chiral fields in the theory so that the

[3] The *-operation is an anti–linear anti–homomorphism. It has the property that $A^*A{=}0 \Rightarrow A{=}0$. Its existence can be seen as follows: Let $\hat{g}_{i_\alpha}$ be the generators of an algebra X with structure constants $a_{i_\alpha j_\beta}^{k_\gamma}$ defined by (5.43). Define the *-operation on generators as $(\hat{g}_{i_\alpha})^* = \hat{g}_{i_\alpha}$. Using (5.38) and (A.49) we see that $a_{i_\alpha j_\beta}^{\mathbf{1}}{=}\delta_{i_\alpha,j_\beta}$. Let $A = \sum b_{i_\alpha}\hat{g}_{i_\alpha}$ be a general element in X. Then $A^*A = \sum (b_{i_\alpha})^* b_{j_\beta}\cdot a_{i_\alpha j_\beta}^{k_\gamma}\cdot\hat{g}_{k_\gamma}$. The coefficient of $\hat{g}_{\mathbf{1}}$ is given by $\sum|b_{i_\alpha}|^2$. Thus, $A{\neq}0 \Leftrightarrow A^*A{\neq}0$.

[4] To be more specific we use that an n–dimensional commutative *-algebra can always be written in terms of n orthogonal projectors $P_aP_b = \delta_{ab}P_a$.

[5] Let $a_{i_\alpha j_\beta}^{k_\gamma}$ denote the structure constants of the algebra (5.43) as in footnote 3. Associativity and commutativity imply the matrix equation $a_{i_\alpha}a_{j_\beta} = a_{j_\beta}a_{i_\alpha}$. The matrices a_{i_α} can thus be simultaneously diagonalised, which is done by $\psi_a^{i_\alpha}$. That the eigenvalues of a_{i_α} are of the form $\psi_a^{i_\alpha}/\psi_a^1$ follows from $a_{i_\alpha 1}^{k_\gamma} = \delta_{i_\alpha,k_\gamma}$.

chiral algebra can be extended. The argument presented above should apply for the maximally extended algebra, and one could try to draw conclusions for the situation with restricted symmetry from this point of view. It would be interesting to pursue this line of thought in future research.

After this short digression we return to the line of argument in [BPPZ98]. Using completeness of the ψ_a's one can verify that the matrices $(n_i)_a{}^b$ form a representation of the Verlinde fusion algebra

$$n_i \cdot n_j = \sum_{k\in\mathcal{R}} \mathrm{N}_{ij}{}^k n_k \ . \tag{5.45}$$

Furthermore, from (5.40) we see that $n_i{=}{n_i}^T$ (since n_i is real).

It follows that an orthonormal and complete solution to (5.40) is equivalent to finding $\mathbb{N}_0$–valued matrices n_i with $n_1{=}\mathbb{1}$, ${n_i}^T = n_i$ that form a representation of the fusion algebra.

In the next chapter we state the solution to this problem for the A– and D–series minimal models without going into details of the complete classification.

Now it is also possible to verify that, if the ψ_a form an ON-basis, (5.43) does indeed define an algebra. To do so, note that the ratios $\psi_a^{i_\alpha}/\psi_a^1$ also represent an algebra. In fact

$$\frac{\psi_a^{i_\alpha}}{\psi_a^1}\cdot\frac{\psi_a^{j_\beta}}{\psi_a^1} = \sum_{k_\gamma\in\mathcal{E}}\Big(\sum_{b\in\mathcal{B}}\frac{\psi_b^{i_\alpha}\psi_b^{j_\beta}(\psi_b^{k_\gamma})^*}{\psi_b^1}\Big)\cdot\frac{\psi_a^{k_\alpha}}{\psi_a^1} \tag{5.46}$$

i.e. in terms of bulk-boundary couplings B we have

$${}^aB_{i_\alpha}{}^1\ {}^aB_{j_\beta}{}^1 = \sum_{k_\gamma\in\mathcal{E}} M_{i_\alpha j_\beta}{}^{k_\gamma}\cdot{}^aB_{k_\gamma}{}^1 \ , \qquad M_{i_\alpha j_\beta}{}^{k_\gamma} = \sum_{b\in\mathcal{B}}\frac{\psi_b^{i_\alpha}\psi_b^{j_\beta}(\psi_b^{k_\gamma})^*}{\psi_b^1} \ . \tag{5.47}$$

The structure constants M define the so-called Pasquier, graph fusion or classifying algebra [Pas87, PSS96, FSc97a, BPPZ99]. In particular it follows that (5.43) defines an algebra[6].

[6] Suppose we have a set of numbers $\tilde{M}_{i_\alpha j_\beta}^{k_\gamma}$, about which we only know that ${}^aB_{i_\alpha}{}^1\ {}^aB_{j_\beta}{}^1 = \sum\tilde{M}_{i_\alpha j_\beta}^{k_\gamma}\cdot{}^aB_{k_\gamma}{}^1$ for all a. Then from (5.39), the fact that ψ is an ON-matrix and inverting (5.46) it follows that $\tilde{M}=M$.

Chapter 6.

Minimal Model Structure Constants

In this chapter we will calculate the bulk– and boundary structure constants for the A– and D–series minimal models. The rough procedure will be as follows: We use the set of allowed boundary conditions, the boundary field content and the bulk field content as an input in the calculation. The first and most difficult step is to find the boundary structure constants. Once these are known, the sewing constraints can be inverted and the bulk–boundary couplings and bulk structure constants can be read off directly. It is worth pointing out that the sewing constraints are over determined and the construction given below is not a proof that the final expressions for the structure constants actually solve all the constraints. The statement is rather: If a solution to the sewing constrains exist it can be brought to the given form by appropriately redefining the fields. In particular, for the A– and D–series at least, the solution to the sewing constraints is unique in this sense. Extensive numerical tests have been performed with the expressions for the structure constants presented here, and no contradiction with the full set[1] of sewing constraints (5.24)–(5.27) was found.

In each of the following steps we will treat the A–series first, as it will have all essential features, and then turn to the D-series.

As already pointed out in section 4.1, the D_{even} minimal models have a larger symmetry algebra (a W–algebra, see [EFHHNV92, BSc92] and references therein) that contains the Virasoro algebra as subalgebra. In principle one could understand these CFTs in terms of the W-algebra, with W-algebra chiral blocks and F–matrices. To obtain all conformally invariant boundary conditions in this case one has to consider boundary conditions that break part of the extended symmetry. An appropriate formalism has been developed by Fuchs et al. in [FSc99a, FSc99b, FSc00]. But since the Hilbert space of the D-series minimal models is finitely reducible with respect to

[1] It should be mentioned that the one point genus one constraints are not included in (5.24)–(5.27). The solution has not been tested against these constraints, but it seems unlikely that it would fail, since the consequence would be that minimal models do not exist.

the Virasoro algebra alone, it is possible and indeed simple to use only the Virasoro symmetry.

6.1 Boundary conditions and boundary fields

In this section we exploit the sewing constraint arising from the cylinder partition function (5.29). More precisely we summarise the solution to the condition (5.40) obtained in [BPZ98], which will give us the list of conformal boundary conditions and their respective field content, in short the matrix $n_{ia}{}^{b}$.

General construction

First choose an odd integer $p \geq 2$ and an integer $q \geq 2$ coprime to p. We will construct the cylinder partition function of a minimal model with the following central charge:

$$c = 1 - 6\frac{(p-q)^2}{pq} \tag{6.1}$$

Let A be the adjacency matrix of the Dynkin diagram associated to the Lie-algebra A_{p-1} and G be the adjacency matrix for a Lie-algebra with Coxeter number q. For $X{=}A$ or $X{=}G$ define the matrix valued functions $V_n(X)$ recursively via

$$V_n(X) = V_2(X)V_{n-1}(X) - V_{n-2}(X) \qquad ; \; V_1(X) = \mathrm{id} \text{ and } V_2(X) = X \tag{6.2}$$

The V_n are called fused adjacency matrices.

Let α be an odd node of A and β be any node of G. Then $a{=}(\alpha,\beta)$ labels the possible boundary conditions. Let $a{=}(\alpha,\beta)$ and $b{=}(\bar{\alpha},\bar{\beta})$. Then the partition function of a cylinder of circumference L and length R with (pure) boundary conditions a and b is given by:

$$Z_{a|b} = \sum_{\substack{r=1..p-1 \\ s=1..q-1}} V_r(A)_{\alpha}{}^{\bar{\alpha}} V_s(G)_{\beta}{}^{\bar{\beta}} \cdot \chi_{r,s}(q) \qquad ; q = \exp(-\pi L/R) \tag{6.3}$$

When we take into account the identity $\chi_{r,s} = \chi_{p-r,q-s}$ for Virasoro characters we can rewrite (6.3) in the following unique way:

$$Z_{a|b} = \sum_{i=(r \text{ odd},s)} n_{ia}{}^{b} \cdot \chi_i(q) \tag{6.4}$$

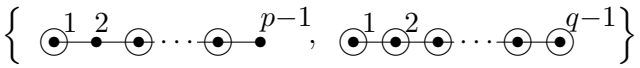
In [BPZ98] the genus one sewing constraint for a cylinder with no field insertions

was analysed with the result that G has to be the adjacency matrix of an A–D–E type Dynkin diagram or of a tadpole diagram.

The numbers ${n_{ia}}^b$ can be interpreted as the number of times the representation i occurs in between the boundary conditions a and b and they thus describe the field content on the boundary (in this case the real line).

A-series

For the pair of Lie algebras (A_{p-1}, A_{q-1}) one finds ${n_{ia}}^b = {\mathrm{N}_{ia}}^b$, i.e. the field content is just given by the Verlinde fusion numbers [Car89b]. Distinct boundary conditions are given by pairings of an odd node in the first A-diagram with any node in the second (recall that p is odd):



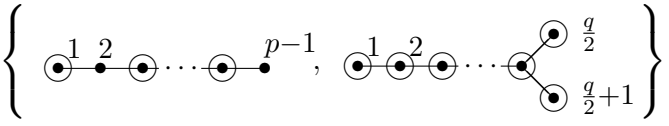
(6.5)

There is a distinguished boundary condition, which we will call 1–boundary, that corresponds to the first node in each A-diagram, i.e. the $(1,1)$-node.

There are no fields with multiplicity and there is a unique field $i{=}a$ between the a– and 1–boundary: $\underset{\frown}{a\ \ i\ \ 1}\ \Rightarrow i{=}a$. The formula ${n_{ia}}^b = {\mathrm{N}_{ia}}^b$ can be understood as saying that the representations i that can live between the a– and b–boundary are exactly those occurring in the fusion of the representations a and b: $\underset{\frown\ \frown\ \ \ \frown}{a\ \ a\ \ 1\ \ b\ \ b}\ \rightarrow\ \underset{\frown}{a\ \ i\ \ b}$.

D-series

Here q has to be an even number and the boundary conditions are given by pairings of an odd node in the A-diagram with any node in the D-diagram:

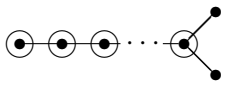


(6.6)

The total number of boundary conditions is $\frac{1}{4}(p-1)(q+2)$. Again the boundary condition associated with the first node of each diagram, i.e. the $(1,1)$-node, will get a special name. It will be called ω so that it is not confused with the 1–boundary in the A-series.

When looking at the D-series boundary field content one observes the following: The boundary conditions can be organised in two categories. In the first case, which we will denote as 'i-type' boundaries, a boundary condition $x{=}(\alpha,\beta)$ is associated with an odd node α of the A-diagram and any node β of the D-diagram except for

the two at the split end. The second case we will call 'n-type'. An n-type boundary a is associated with any odd A-node and one of the two end nodes in the D-diagram.

i-type:  n-type:  (6.7)

The names i-type and n-type stand for 'invariant' and 'non-invariant' and are related to the $\mathbb{Z}_2$–symmetry of the D-diagram.

Carrying through the procedure outlined in the beginning of this section, one finds that an n-type boundary $a{=}(\alpha,\beta)$ has exactly one field living on $\underline{\quad a \quad \omega \quad}$, denoted by a_u ('unique'). It has representation labels $(\alpha, \frac{q}{2})$.

An i-type boundary $x{=}(\alpha,\beta)$ has two fields living on $\underline{\quad x \quad \omega \quad}$. Their Kac-labels are (α,β) and $(\alpha, q-\beta)$ and they will be denoted by x_e and x_o (x 'even' and x 'odd'), respectively. The labelling is arbitrary, but we choose to always give the identity field on the ω–boundary an 'even' label.

For definiteness we will fix a specific even/odd labelling for boundary fields. It will turn out later that the structure constants involving only i-type boundaries have an explicit $\mathbb{Z}_2$–symmetry that sends even fields to themselves and odd fields to minus themselves. This symmetry is not assumed, but a consequence of the constraints and the gauge we choose. Following the arguments in [Rue99] we would like the ground state, i.e. the state of lowest conformal weight, for each pair of boundary conditions to be invariant under that symmetry. This can be achieved for all pairs $\underline{\quad x \quad \omega \quad}$ with $x{=}(\alpha,\beta)$ an i-type boundary by assigning e/o-labels to the two representations living between these boundary conditions in the following way:

$$x_e = \begin{cases} (\alpha,\beta) & : \ \alpha < \frac{p}{2} \\ (\alpha, q-\beta) & : \ \alpha > \frac{p}{2} \end{cases} \qquad x_o = \begin{cases} (\alpha, q-\beta) & : \ \alpha < \frac{p}{2} \\ (\alpha,\beta) & : \ \alpha > \frac{p}{2} \end{cases} \tag{6.8}$$

As discussed in [Rue99] the physical motivation for this choice comes from relating the $\mathbb{Z}_2$–action to the effect of a disorder line stretching from boundary to boundary on a cylinder and trying to interpret the resulting amplitude as a partition function. It is also found there that in general it is not possible to fix a labelling s.t. the ground state is invariant for all possible pairs of boundary conditions.

The conformal weight of a highest weight representation with Kac-labels (r,s) is given by $h_{r,s} = \frac{1}{4pq}((qr-ps)^2-(p-q)^2)$. Thus with definition (6.8) we find:

$$h(x_o) - h(x_e) = \left|(\alpha - \tfrac{p}{2})(\beta - \tfrac{q}{2})\right| > 0 \tag{6.9}$$

boundary types	ℓ–labels	field content
i–i $\underline{x \overset{\ell}{\frown} y}$	e,o	$n_{\ell x}{}^{y} = \mathrm{N}_{\ell x_e}{}^{y_e} + \mathrm{N}_{\ell x_e}{}^{y_o} = \mathrm{N}_{\ell x_o}{}^{y_o} + \mathrm{N}_{\ell x_o}{}^{y_e}$
n–i $\underline{a \overset{\ell}{\frown} y}$	u	$n_{\ell a}{}^{y} = \mathrm{N}_{\ell a_u}{}^{y_e} = \mathrm{N}_{\ell a_u}{}^{y_o}$
n–n $\underline{a \overset{\ell}{\frown} b}$	u	$\beta_a \neq \beta_b : n_{\ell a}{}^{b} = \mathrm{N}_{\ell a_u}{}^{b_u}$ if $s_\ell \equiv 3 \mod 4$ and $n_{\ell a}{}^{b} = 0$ otherwise $\beta_a = \beta_b : n_{\ell a}{}^{a} = \mathrm{N}_{\ell a_u}{}^{a_u}$ if $s_\ell \equiv 1 \mod 4$ and $n_{\ell a}{}^{b} = 0$ otherwise

Table 6.1: Field content and labels between the two types of boundary conditions.

In particular the ω–boundary itself is of i-type and it has two fields living on it, the identity ω_e=1 and the field ω_o with $h(\omega_o) = (\frac{p}{2}-1)(\frac{q}{2}-1)$. We see that $h(\omega_o)$ is integer for $\frac{q}{2}+1$ even (the "D_{even}" case) and half-integer for $\frac{q}{2}+1$ odd (the "D_{odd}" case).

We will now proceed to assign e/o/u-labels to all boundary fields, not just the ones adjacent to the ω–boundary, in the following way: Any field adjacent to an n-type boundary will get the label u. In this case no multiplicities occur. In the case $\underline{x \overset{\ell_\delta}{\frown} y}$ where both x and y are of i-type, the possible representations ℓ are those that occur in the fusion of x_e, y_e (which is the same set as in the fusion of x_o, y_o) or x_e, y_o (which is the same as x_o, y_e). If ℓ occurs in the x_e, y_e–fusion it gets an e-label, i.e. δ=e and if it occurs in the x_e, y_o–fusion an o-label δ=o. If ℓ occurs in both fusions, this representation has multiplicity two and the two corresponding fields have labels ℓ_e and ℓ_o.

The field content between different boundary conditions is summed up in table 6.1 listing the numbers $n_{ia}{}^{b}$ from (6.4) for the different cases (here the representation ℓ has Kac-labels $\ell = (r_\ell, s_\ell)$ and the boundary conditions a, b have labels (α_a, β_a) and (α_b, β_b)).

Notice the double role of the $e/o/u$–indices: When x is a boundary condition, then x_e, x_o or x_u denote the even/odd/unique field living between the x– and ω–boundary. In particular x_e and x_o denote different representations. If ℓ is a representation, then ℓ_e, ℓ_o or ℓ_u all denote the same representation and distinguish fields with multiplicities by labelling one as even and the other as odd.

Ordering of boundary conditions

The constraint equations (5.24)–(5.27) allow for a large amount of gauge freedom of the structure constants. Here we find a particular set of structure constants and show that any solution of the constraints can be transformed into this set by a regauging.

For the construction of the structure constants presented here an ordering of the boundary conditions has to be introduced. Let $x{=}(\alpha,\beta)$ and $y{=}(\bar\alpha,\bar\beta)$. Then we define:

$$x<y \quad\Leftrightarrow\quad (\beta<\bar\beta) \text{ or } (\beta=\bar\beta \text{ and } \alpha<\bar\alpha) \tag{6.10}$$

6.2 Boundary structure constants

It turns out that for both, the A– and D–series it is possible to find the boundary structure constants by only considering the sewing constraint arising from four boundary fields (5.24).

The procedure to solve the constraints is similar in the A– and D–series. First the freedom to rescale the boundary fields is used to fix some structure constants to convenient values. Next a subset of the full set of equations arising from the four boundary fields sewing constraint (5.24) is considered and shown to determine all remaining structure constants.

As a starting point we use

(a) the field content obtained in section 6.1

(b) the assumption that all two-point functions $\langle\,\psi_{i_\alpha}^{(ab)}(x)\,\psi_{i_\alpha}^{(ba)}(y)\rangle$ are nonzero, i.e. $C_{i_\alpha i_\alpha}^{(aba)1}\neq 0$ for all a,b,i_α that are allowed by (a).

An interpretation of assumption (b) is that a zero two-point function implies that the field in question can be removed from the theory and hence we effectively have a field content different from what we demanded in (a). This can be seen as follows: Suppose $C_{i_\alpha i_\alpha}^{(aba)1}=0$ for some choice of a,b,i_α. Consider a correlator which contains the field $\psi_{i_\alpha}^{(ab)}$. Any correlator can be expressed as a sum of conformal blocks with coefficients given as products of structure constants. It is always possible to take a limit of this correlator where all bulk fields are taken to the boundary and then all boundary fields except for $\psi_{i_\alpha}^{(ab)}$ are taken together. In the end we are left with the two-point function $\langle\,\psi_{i_\alpha}^{(ab)}(x)\,\psi_{i_\alpha}^{(ba)}(y)\rangle$. Thus in this limit the coefficient in front of each conformal block contains the factor $C_{i_\alpha i_\alpha}^{(aba)1}$, which is zero. Therefore any correlator involving $\psi_{i_\alpha}^{(ab)}$ vanishes and this field can as well be removed from the theory.

A–series

In this case one can almost guess the solution. The sewing constraint involving four boundary fields can be rewritten (see (A.71) for details) so that the two point

structure constants cancel. The resulting equation

$$C_{jk}^{(bcd)q}\, C_{iq}^{(abd)\ell} = \sum_p C_{ij}^{(abc)p}\, C_{pk}^{(acd)\ell}\, \mathsf{F}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} \tag{6.11}$$

bears a curious resemblance to the pentagon identity[2] obeyed by the F-matricies,

$$\mathsf{F}_{cq}\begin{bmatrix} b & d \\ j & k \end{bmatrix}\mathsf{F}_{b\ell}\begin{bmatrix} a & d \\ i & q \end{bmatrix} = \sum_p \mathsf{F}_{bp}\begin{bmatrix} a & c \\ i & j \end{bmatrix}\mathsf{F}_{c\ell}\begin{bmatrix} a & d \\ p & k \end{bmatrix}\mathsf{F}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} \tag{6.12}$$

Indeed, setting

$$C_{ij}^{(abc)k} = \mathsf{F}_{bk}\begin{bmatrix} a & c \\ i & j \end{bmatrix} \qquad \forall a, b, c, i, j, k \tag{6.13}$$

turns (6.11) into (6.12) and is thus a solution of the sewing constraint.

It is more work to show that it is the *only* solution (up to rescaling) of (6.11). This calculation is presented in appendix A.7.

In the form (6.13) the boundary structure constants are symmetric under reflection, i.e. $C_{ij}^{(abc)k} = C_{ji}^{(cba)k}$. This follows from the F-matrix idenities collected in appendix (A.5).

Note that all parameters in the F-matrices refer to irreducible highest weight representations. The identification (6.13) was only possible because in the A–series there is a one-to-one correspondance between boundary fields resp. boundary conditions and highest weight representations. This is not the case for the D-series and the solution there takes a much more complicated form.

D-series

Again we only present the results for the boundary structure constants, the calculation has been shifted to appendix A.7. In the gauge we chose there, two conditions have to be satisfied before the boundary–boundary coupling $\psi_{i_\alpha}^{(xy)}\,\psi_{j_\beta}^{(yz)} \to \psi_{k_\gamma}^{(xz)}$ can exist. First the corresponding Verlinde fusion number $\mathrm{N}_{ij}{}^{k}$ has to be nonzero and second the $e/u/o$-labels $\{\alpha, \beta, \gamma\}$ have to be one of the sets $\{u, *, *\}$, $\{e, e, e\}$ or $\{e, o, o\}$ (where $*$ stands for any label). From this even/odd–coupling rule we see immediately that the boundary structure constants involving only i-type boundaries have a $\mathbb{Z}_2$–symmetry of the form $\psi_{\ell_e}^{(xy)} \to \psi_{\ell_e}^{(xy)}$ and $\psi_{\ell_o}^{(xy)} \to -\psi_{\ell_o}^{(xy)}$.

We can also give the behaviour of the boundary structure constants under re-

[2] (6.12) has been obtained from (4.57) by relabelling indices and using (4.50).

flection. Define the sign $\varepsilon(x,y,i_\alpha)$ to be

$$\varepsilon(x,x,i_o) = \mathsf{F}_{x_o x_o}\begin{bmatrix} x_e & \omega_o \\ i & x_e \end{bmatrix} = \pm 1 \quad \text{and} \quad \varepsilon(x,y,i_\alpha) = 1 \text{ in all other cases.} \tag{6.14}$$

Then a boundary structure constant is related to its reflected counterpart by

$$C^{(xyz)k_t}_{i_r j_s} = \frac{\varepsilon(x,y,i_r)\varepsilon(y,z,j_s)}{\varepsilon(x,z,k_t)} \cdot C^{(zyx)k_t}_{j_s i_r} \ . \tag{6.15}$$

In the following ω stands for the boundary condition associated to the $(1,1)$ pair of nodes in the diagram (6.6) and μ for $(1,\frac{q}{2})$. Consequently $\mu_u = (1,\frac{q}{2})$ and $\omega_o = (1,q{-}1)$. Let the constants A, B_a, C_x be given by

$$\begin{aligned} &A = \mathsf{F}_{\mu_u 1}\begin{bmatrix} \mu_u & \mu_u \\ \omega_o & \omega_o \end{bmatrix}, \quad B_a = \mathsf{F}_{\mu_u a_u}\begin{bmatrix} \omega_o & a_u \\ \mu_u & \ell \end{bmatrix} \quad \text{where } \underline{\mu \ \ell_u \ a} \ , \\ &C_x = \mathsf{F}_{x_e 1}\begin{bmatrix} \omega_o & \omega_o \\ x_o & x_o \end{bmatrix} \text{ for } x \text{ of i-type}, \quad C_x = \mathsf{F}_{x_u 1}\begin{bmatrix} \omega_o & \omega_o \\ x_u & x_u \end{bmatrix} \text{ for } x \text{ of n-type}. \end{aligned} \tag{6.16}$$

Here ℓ_u can be any field living on $\underline{\mu \ \ a}$, B_a turns out to be independent of the specific choice made. With these definitions, the structure constants involving the boundary condition ω take the following form:

- Two-point functions with ω

$$C^{(\omega x\omega)1}_{x_u x_u} = 1 \qquad C^{(\omega x\omega)1}_{x_e x_e} = 1 \qquad C^{(\omega x\omega)1}_{x_o x_o} = A \cdot C_x \tag{6.17}$$

- For x any type and y of n-type:

$$C^{(\omega xy)y_u}_{x_u \ell_u} = 1 \qquad C^{(\omega xy)y_u}_{x_e \ell_u} = 1 \qquad C^{(\omega xy)y_u}_{x_o \ell_u} = \frac{A}{B_y}\mathsf{F}_{x_e y_u}\begin{bmatrix} \omega_o & y_u \\ x_o & \ell \end{bmatrix} \tag{6.18}$$

- For x,y of i-type and $x{\le}y$:

$$\begin{aligned} C^{(\omega xy)y_e}_{x_e \ell_e} &= 1 & C^{(\omega xy)y_o}_{x_e \ell_o} &= 1 \\ C^{(\omega xy)y_o}_{x_o \ell_e} &= \mathsf{F}_{x_e y_o}\begin{bmatrix} \omega_o & y_e \\ x_o & \ell \end{bmatrix} & C^{(\omega xy)y_e}_{x_o \ell_o} &= A \cdot C_y \cdot \mathsf{F}_{x_e y_e}\begin{bmatrix} \omega_o & y_o \\ x_o & \ell \end{bmatrix} \end{aligned} \tag{6.19}$$

All structure constants involving the ω–boundary, which are not covered in this list

can be obtained by

$$C^{(\omega xy)y_\gamma}_{x_\alpha \ell_\beta} = \varepsilon(x,y,\ell_\beta)\cdot \frac{C^{(\omega x\omega)1}_{x_\alpha x_\alpha}}{C^{(\omega y\omega)1}_{y_\gamma y_\gamma}} C^{(\omega yx)x_\alpha}_{y_\gamma \ell_\beta} \quad . \tag{6.20}$$

For the general boundary structure constants we distinguish two cases. First, for x, z of n-type and y of i-type we get:

$$C^{(xyz)k_u}_{i_u j_u} = \mathsf{F}_{y_e k}\begin{bmatrix} x_u & z_u \\ i & j \end{bmatrix} + \frac{B_x}{B_z}\cdot \mathsf{F}_{x_u y_o}\begin{bmatrix} y_e & \omega_o \\ i & x_u \end{bmatrix}\mathsf{F}_{y_e z_u}\begin{bmatrix} z_u & \omega_o \\ j & y_o \end{bmatrix}\mathsf{F}_{y_o k}\begin{bmatrix} x_u & z_u \\ i & j \end{bmatrix} \tag{6.21}$$

In all other cases we have

$$C^{(xyz)k_t}_{i_r j_s} = \frac{C^{(\omega xy)y_\beta}_{x_\alpha i_r}\, C^{(\omega yz)z_\gamma}_{y_\beta j_s}}{C^{(\omega xz)z_\gamma}_{x_\alpha k_t}} \mathsf{F}_{y_\beta k}\begin{bmatrix} x_\alpha & z_\gamma \\ i & j \end{bmatrix} . \tag{6.22}$$

The rhs does not depend on the specific choice of α, β, γ as long as the combinations $\{\alpha, r, \beta\}$, $\{\beta, s, \gamma\}$ and $\{\alpha, t, \gamma\}$ are allowed by the even/odd coupling rule. The structure constant in the denominator is then automatically nonzero.

Note in particular that if x, y, z are all of n-type or if the fields i_r, j_s, k_t are even, the solution takes the same form as in the A-series (see eqn. (6.13)):

$$C^{(xyz)k_e}_{i_e j_e} = \mathsf{F}_{y_e k}\begin{bmatrix} x_e & z_e \\ i & j \end{bmatrix} \quad \text{and} \quad C^{(xyz)k_u}_{i_u j_u} = \mathsf{F}_{y_u k}\begin{bmatrix} x_u & z_u \\ i & j \end{bmatrix} \tag{6.23}$$

In fact this form also holds for all mixed cases with only e/u field labels, except when the boundaries x, z are of n-type and y is of i-type, when we obtained (6.21).

6.3 Bulk field content

At this point we have completely determined the boundary theory. The next step is to find which bulk theory, if any[3], it matches to. The classification of boundary conditions and their field content establishes a one–to–one correspondence between boundary conditions and diagonal bulk fields (see section 5.3). In this approach specifying the possible boundary conditions is equivalent to giving the diagonal part of the bulk partition function, and from the A–D–E–classification in [CIZ87] one knows which off-diagonal parts have to be added to make it modular invariant.

In this section we make the observation that the modular invariant bulk field

[3] Just looking at the cylinder partition function, [BPZ98] gave an A–D–E–T classification of the boundary conditions for minimal models. T stands for a tadpole graph, which has to be discarded, since there is no modular invariant bulk theory with a matching diagonal field content.

content is, at least for the A– and D–series, also the maximal consistent one, in a sense made precise below.

The sewing constraint (5.25) can be rewritten in the form:

$$\sum_{\delta} {}^{b}B_{i_\alpha}{}^{\ell_\delta}\; C^{(abb)q_\varepsilon}_{p_\nu \ell_\delta} = \sum_{k,\gamma} {}^{a}B_{i_\alpha}{}^{k_\gamma}\; C^{(aab)q_\varepsilon}_{k_\gamma p_\nu} \cdot \sum_{m} e^{i\pi(2h_m-2h_i-h_p-h_q+\frac{1}{2}(h_k+h_\ell))} \cdot \mathsf{F}_{km}\begin{bmatrix} \bar{\imath} & q \\ i & p \end{bmatrix} \mathsf{F}_{m\ell}\begin{bmatrix} i & \bar{\imath} \\ p & q \end{bmatrix} \tag{6.24}$$

Suppose now there is a boundary condition a and a bulk field i_α s.t. i_α does not couple to any boundary field on the a–boundary, i.e. ${}^{a}B_{i_\alpha}{}^{k_\gamma}=0$ for all k_γ. In the detailed treatment of the A– and D–series below it will be argued that (6.24) implies that i_α does not couple to any boundary: ${}^{x}B_{i_\alpha}{}^{k_\gamma}=0$ for all x and k_γ.

With an argument similar to that justifying assumption (b) in the previous section, it follows that the bulk field i_α can be removed from the theory: Any n-point function has a limit in which we take all bulk fields to the boundary. In this limit the factors in front of the conformal blocks contain the product of the B's of all bulk fields, and in particular ${}^{a}B_{i_\alpha}{}^{k_\gamma}$, which is zero. Thus any n-point function involving the field i_α is identically zero and i_α can be removed from the theory.

Eqn. (6.24) also limits the maximal multiplicity of a given representation $i\otimes\bar{\imath}$ in the bulk spectrum. Let a be the boundary with the least number of boundary fields living on it. Suppose by the fusion rules $i\otimes\bar{\imath}$ can couple to N boundary fields on a, i.e. $N=\sum_k \mathrm{N}_{i\bar{\imath}}{}^{k}\cdot n_{ka}{}^{a}$. Denote these boundary fields by $k_1,\ldots,k_N$. Next suppose that there are M bulk fields $i_1,\ldots,i_M$ transforming in the representation $i\otimes\bar{\imath}$. We can change the basis in the M–dimensional space of these fields by an orthogonal transformation (w.r.t. to the metric $g_{\alpha\beta}=C_{i_\alpha i_\beta}{}^{1}$). This will leave the two-point functions invariant. By the algorithm leading to the QR-decomposition of a matrix one can show that with an appropriate change of basis the bulk field i_2 does not couple to the boundary field k_1, the bulk field i_3 does not couple to k_1,k_2 and so on. If $M>N$, then i_M does not couple to any boundary field on the a–boundary and can, by the above argument, be removed from the theory.

Using this idea we can now determine the maximal bulk field content consistent with (6.24) in the case of the A– and D–series, and find that it is precisely the modular invariant one found in [CIZ87].

A-Series

For the A–series the boundary condition with the fewest fields living on it is the 1–boundary, with only the identity field. In this case (6.24) reads (we use (6.13) to

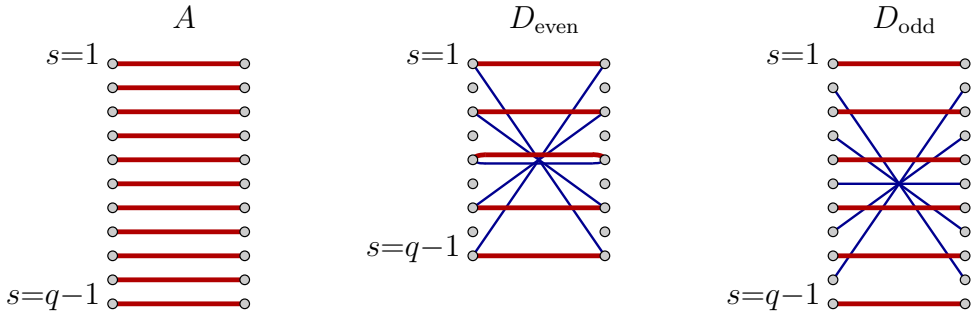


Figure 6.1: Modular invariant bulk field content in the minimal model $M(p,q)$ with p odd, a) for the A–series b) for D_{even}, i.e. $\frac{q}{2}$+1 even and c) for D_{odd}, i.e. $\frac{q}{2}$+1 odd. The picture indicates the left–right pairing of representations $(r,s_l)\otimes(r,s_r)$. r takes only odd values and the pairing is the same for all r.

replace the boundary structure constants)

$$ {}^{b}B_{i_\alpha}{}^{\ell} \;=\; {}^{1}B_{i_\alpha}{}^{1} \sum_m e^{i\pi(2h_m-2h_i-2h_b+\frac{1}{2}h_\ell)} \cdot \mathsf{F}_{1m}\begin{bmatrix} \bar{\imath} & b \\ i & b \end{bmatrix} \mathsf{F}_{m\ell}\begin{bmatrix} i & \bar{\imath} \\ b & b \end{bmatrix} . \tag{6.25} $$

The first conclusion is that every bulk field i_α has to couple to the 1–boundary: ${}^{1}B_{i_\alpha}{}^{1}\neq 0$. Otherwise the lhs is zero for any boundary condition b and the field i_α is not present in the theory. Say i_α transforms in $i\otimes\bar{\imath}$. For i and $\bar{\imath}$ to fuse to the identity we need $i=\bar{\imath}$ and thus the A–series cannot contain any bulk fields with spin. Next we note that, since the 1–boundary contains exactly one field, every bulk field has to occur with multiplicity one.

These are all the restrictions that arise from (6.25) and thus the maximal consistent field content is one diagonal bulk field $i\otimes i$ for every representation i in the Kac-table (see fig. 6.1). This is precisely the A–series modular invariant bulk field content.

D-Series

In the D–series, the boundary with the least number of boundary fields is the ω–boundary, with two fields, the identity $\omega_e=\mathbb{1}$ and the field ω_o.

For an arbitrary boundary x consider (6.24) with $a=\omega$, $b=x$, $p_\nu=x_\nu$, $q_\varepsilon=x_\varepsilon$. The ℓ_δ sum reduces to one element and we get:

$$ \begin{aligned} {}^{x}B_{i_\alpha}{}^{\ell_\delta}\; C^{(\omega x x)x_\varepsilon}_{x_\nu \ell_\delta} \;&=\; {}^{\omega}B_{i_\alpha}{}^{1}\; C^{(\omega\omega x)x_\varepsilon}_{1x_\nu} \cdot \big(\text{ F's }\big)\Big|_{k=1} \\ &\quad + {}^{\omega}B_{i_\alpha}{}^{\omega_o}\; C^{(\omega\omega x)x_\varepsilon}_{\omega_o x_\nu} \cdot \big(\text{ F's }\big)\Big|_{k=\omega_o} \end{aligned} \tag{6.26} $$

In appendix A.7 it is argued that that all boundary structure constants appearing

in (6.26) are nonzero if they are allowed by fusion and the even/odd coupling rule.

To avoid ${}^{x}B_{i_\alpha}{}^{\ell_\delta} = 0$ for all x, ℓ_δ the bulk field i_α thus has to couple to either $\mathbb{1}$ or ω_o on the ω–boundary. Say i_α transforms in $i\otimes\bar{\imath}$. Suppose i_α is a field with spin, i.e. $i\neq\bar{\imath}$. Then i and $\bar{\imath}$ cannot fuse to the identity and hence it follows that ${}^{\omega}B_{i_\alpha}{}^{1}=0$. So for the field i_α to exist we need ${}^{\omega}B_{i_\alpha}{}^{\omega_o}\neq 0$. This is only possible if the fusion $i,\bar{\imath}\to\omega_o$ exists, i.e. only for pairs with Kac-labels $i=(r,s)$ and $\bar{\imath}=(r,q-s)$. From (6.26) with $x=\omega$, $\ell_\delta=x_\nu=\omega_o$, $x_\varepsilon=1$ we obtain (compare also with (6.9)):

$$ {}^{\omega}B_{i_\alpha}{}^{\omega_o} = {}^{\omega}B_{i_\alpha}{}^{\omega_o}\cdot e^{2\pi i(\bar{h}_i-h_i)} = {}^{\omega}B_{i_\alpha}{}^{\omega_o}\cdot e^{2\pi i(\frac{p}{2}-r)(\frac{q}{2}-s)} \tag{6.27}$$

For the exponential to be $+1$ we need $\frac{q}{2}-s$ to be even (recall that for the D-series p is always odd and q always even).

Suppose next that i_α is diagonal and that it cannot fuse to ω_o: $\mathrm{N}_{ii}{}^{\omega_o}=0$. Define the operation "*" on Kac-labels $i=(r,s)$ as $i^*=(r,q-s)$. Then (6.26) with $x=\omega$, $\ell_\delta=\mathbb{1}$, $x_\nu=x_\varepsilon=\omega_o$, together with (A.46) gives

$$ {}^{\omega}B_{i_\alpha}{}^{1} = {}^{\omega}B_{i_\alpha}{}^{1}\cdot \mathsf{F}_{1i^*}\begin{bmatrix} i & \omega_o \\ i & \omega_o\end{bmatrix}\mathsf{F}_{i^*1}\begin{bmatrix} i & i \\ \omega_o & \omega_o\end{bmatrix}\cdot e^{2\pi i(h_{i^*}-h_i-h_{\omega_o})} = {}^{\omega}B_{i_\alpha}{}^{1}\cdot e^{i\pi(s+1)} \tag{6.28}$$

Consequently diagonal fields $i\otimes i$ with ${}^{\omega}B_{i_\alpha}{}^{\omega_o}=0$ can only occur for $i=(r,s)$ with s odd.

The only bulk-representations that allow coupling to both fields on the ω–boundary are the pairs $i\otimes i$ where i has Kac-labels $(r,\frac{q}{2})$ and only in this case can multiplicity two occur. All other bulk fields have multiplicity one.

This is precisely the D–series modular invariant bulk field content. Note that the only case in which some bulk fields have multiplicity greater than one is for diagonal fields in the representation $(r,\frac{q}{2})$ with $\frac{q}{2}+1$ even, i.e. in D_{even} models. Figure 6.1 gives a graphical representation of the field content.

6.4 Bulk boundary couplings

In the previous section we argued, that for our boundary condition of choice, the one with minimal field content, ${}^{a}B_{i_\alpha}{}^{k_\gamma}$, as a matrix in α and k_γ, can be brought to triangular form. That it then automatically has to be diagonal can be seen from (5.26) with $m=\mathbb{1}$, $k_\gamma=\mathbb{1}$, $i=j$, i.e.

$$ C_{i_\alpha i_\beta}{}^{\mathbf{1}} = \sum_{p_\nu} {}^{a}B_{i_\alpha}{}^{p_\nu}\cdot {}^{a}B_{i_\beta}{}^{p_\nu}\cdot C^{(aaa)\mathbf{1}}_{p_\nu p_\nu}\mathsf{F}_{p1}\begin{bmatrix}\bar{\imath} & \bar{\imath}\\ i & i\end{bmatrix}. \tag{6.29}$$

We normalised the bulk fields s.t. $C_{i_\alpha i_\beta}{}^{\mathbf{1}} = \delta_{\alpha,\beta}\, C_{i_\alpha i_\alpha}{}^{\mathbf{1}}$. If the F-matrix elements in (6.29) are nonzero, this imples that ${}^aB_{i_\alpha}{}^{p_\nu}$ is diagonal as a matrix in α and p_ν.

With this choice of basis the bulk–boundary couplings can be read off from (6.24).

A-series

The A–series bulk–boundary couplings were first worked out (up to signs, in certain cases) in [CLe91]. In the approach presented here, no sign ambiguities arise. For the A–series there are no bulk fields with multiplicities and thus we do not need the above argument; for a=1 (6.24) reads

$$ {}^bB_i{}^\ell \;=\; {}^1B_i{}^1 \sum_m e^{i\pi(2(h_m-h_b-h_i)+\frac{1}{2}h_\ell)} \mathsf{F}_{1m}\begin{bmatrix} b & i \\ b & i \end{bmatrix} \mathsf{F}_{m\ell}\begin{bmatrix} i & i \\ b & b \end{bmatrix} , \tag{6.30} $$

since from (6.13) we see that the boundary structure constants appearing in (6.24) are equal to one. To fix the normalisation of the bulk fields, we require

$$ {}^1B_i{}^1 \;=\; \frac{S_1{}^i}{S_1{}^1} \,. \tag{6.31} $$

Petkova has observed [prP] that up to a normalisation the ${}^aB_i{}^k$ are just the S-matrix elements $S_a{}^i(k)$ for the torus with one operator insertion[4]. This S-Matrix is given by equation (A.53) in the appendix. For k=1 it reduces to the S-matrix implementing the modular transformation of characters: $S_i{}^j(1) = S_i{}^j$. Combining (6.30) and (A.53), the relation between ${}^aB_i{}^k$ and $S_a{}^i(k)$ becomes:

$$ {}^aB_i{}^k \;= e^{i\frac{\pi}{2}h_k} \left(\mathsf{F}_{a1}\begin{bmatrix} a & a \\ k & k \end{bmatrix} \right)^{-1} \cdot \frac{S_a{}^i(k)}{S_a{}^1} \qquad\qquad {}^aB_i{}^1 \;=\; \frac{S_a{}^i}{S_a{}^1} \,. \tag{6.32} $$

The phase on the rhs cancels with the phases in $S_a{}^i(k)$ and ${}^aB_i{}^k$ is real.

D-series

As a first step to determine the bulk-boundary couplings we will choose a basis of primary bulk fields s.t. each field couples either to 1 or to ω_o on the ω–boundary, but not to both simultaneously.

As we have seen when determining the D–series bulk-field content, this demand is automatically enforced by the fusion rules except for diagonal bulk fields in the

[4] see also [BPPZ99] for a general discussion of the relation between the sets of duality relations in [Lew92] and [MSb89a]

representation $i\otimes i$ where i has Kac-labels $(r,\frac{q}{2})$ and we are looking at a D_{even} model. In this case the corresponding bulk fields have multiplicity two. Denote these two fields by i_α, i_β. By change of basis we can achive ${}^{\omega}B_{i_1}{}^{\omega_o}=0$. For α=1, β=2 equation (6.29) then reads

$$0 = {}^{a}B_{i_1}{}^{\mathbf{1}}\,{}^{a}B_{i_2}{}^{\mathbf{1}}\cdot \mathsf{F}_{11}\begin{bmatrix} i & i \\ i & i \end{bmatrix}. \tag{6.33}$$

The F-matrix element is nonzero by appendix A.5 and thus ${}^{a}B_{i_2}{}^{\mathbf{1}}=0$, as required. We will denote bulk fields that couple to 1 on the ω–boundary as 'even' and bulk fields that couple to the boundary field ω_o as 'odd'. The new bulk–boundary couplings then fulfil ${}^{\omega}B_{i_e}{}^{\omega_o}=0$ for the even bulk fields i_e and ${}^{\omega}B_{i_o}{}^{1}=0$ for the odd fields i_o.

We are still free to rescale all the bulk fields i_α. We use this freedom to set[5]:

$$\begin{aligned}
{}^{\omega}B_{i_o}{}^{\omega_o} &= \frac{B_i}{A}\cdot\frac{S_1{}^{i}}{S_1{}^{1}} && \text{for diagonal odd fields } i_o \text{ (i.e. } i{=}\bar{\imath}) \\
{}^{\omega}B_{i_e}{}^{1} &= {}^{\omega}B_{i_o}{}^{\omega_o} = \frac{S_1{}^{i}}{S_1{}^{1}}\cdot e^{i\frac{\pi}{2}(h_i-\bar{h}_i)} && \text{for all other cases} \\
{}^{\omega}B_{i_o}{}^{1} &= {}^{\omega}B_{i_e}{}^{\omega_o} = 0 && \text{from the coupling rule}
\end{aligned} \tag{6.34}$$

With this redefinition of the fields the sum on the rhs of (6.26) reduces to one term. Using (6.26) with $\varepsilon{=}\delta$ and $k_\gamma{=}\omega_\alpha$ the general bulk–boundary coupling takes the form:

$$\begin{aligned}
{}^{x}B_{i_\alpha}{}^{\ell_\delta} \;=\; {}^{\omega}B_{i_\alpha}{}^{\omega_\alpha}\cdot\sum_m \exp\left(i\pi(2h_m-2h_i-h_{x_\nu}-h_{x_\delta}+\tfrac{1}{2}(h_{\omega_\alpha}+h_\ell))\right) \\
\cdot\,\mathsf{F}_{\omega_\alpha m}\begin{bmatrix} \bar{\imath} & x_\delta \\ i & x_\nu \end{bmatrix}\mathsf{F}_{m\ell}\begin{bmatrix} i & \bar{\imath} \\ x_\nu & x_\delta \end{bmatrix}\cdot\begin{cases} 1 & : \alpha{=}e \text{ and } \delta{=}e \text{ or } u \\ C_x{}^{-1} & : \alpha{=}o \text{ and } \delta{=}o \\ B_x & : \alpha{=}o \text{ and } \delta{=}u \\ 0 & : \text{otherwise} \end{cases}
\end{aligned} \tag{6.35}$$

where $\nu{=}e$ if x is an i-type boundary and $\nu{=}u$ if x is an n-type boundary.

Note in particular that any bulk field can couple to a u–field on an n-type boundary, but for an i-type boundary an even bulk field can only couple to an even boundary field and an odd bulk field to an odd boundary field. This indicates that the

[5] the odd diagonal fields are normalised differently from the odd fields with spin, so that the ${}^{x}B_{i_\alpha}{}^{1}$ represent the Pasquier algebra as seen in section 6.6

$\mathbb{Z}_2$–symmetry of the boundary structure constants for i-type boundaries carries over to the bulk.

6.5 Bulk structure constants

The bulk structure constants for the A– and D–series minimal models have been found in e.g. [DFa84, FKl89, Pet89, PZu94]. The difference in the approach taken here is that we obtain the bulk structure constants directly from the known boundary data, specifically from the sewing constraint (5.26).

A-series

Consider (5.19) in the form:

$$\sum_\ell {}^aB_i{}^\ell \; {}^aB_j{}^\ell \; C_{\ell\ell}^{(aaa)1} \, \mathsf{F}_{\ell n}\begin{bmatrix} j & i \\ j & i \end{bmatrix} = C_{ij}{}^n \; {}^aB_n{}^1 \tag{6.36}$$

Since by assumption only $\psi_1^{(11)}$ can exist on the 1-boundary, for a=1 the sum reduces to one term ℓ=1. Using F–matrix identities in appendix A.5 we obtain:

$$C_{ij}{}^k = \frac{{}^1B_i{}^1 \; {}^1B_j{}^1}{{}^1B_k{}^1} \mathsf{F}_{1k}\begin{bmatrix} i & j \\ i & j \end{bmatrix} = \left(\mathsf{F}_{k1}\begin{bmatrix} i & i \\ j & j \end{bmatrix}\right)^{-1} . \tag{6.37}$$

To compare the present normalisation of bulk fields to alternative choices it is helpful to know the two-point function. For k=1 the above formula reduces to

$$C_{ii}{}^1 = \frac{S_1{}^i}{S_1{}^1} . \tag{6.38}$$

D-series

Take (5.26) with a=ω. The sum on the lhs then reduces to the term where ${}^\omega B_{m_\rho}{}^{k_\gamma} \neq 0$, i.e. k_γ=ω_ρ. On the rhs the sum over p_ν and q_ε reduces to p_ν=ω_α, q_ε=ω_β. The r–sum is also reduced to one element. We are left with:

$$\begin{aligned} C_{i_\alpha j_\beta}{}^{m_\gamma} &= \exp\left(i\tfrac{\pi}{2}(h_{\omega_\gamma}+h_{\omega_\alpha}-h_{\omega_\beta}+2(h_j-h_r)+h_m-\bar h_m-h_i+\bar h_i-h_j+\bar h_j)\right) \\ &\cdot \frac{{}^\omega B_{i_\alpha}{}^{\omega_\alpha} \; {}^\omega B_{j_\beta}{}^{\omega_\beta}}{{}^\omega B_{m_\gamma}{}^{\omega_\gamma}} \cdot \mathsf{F}_{\mu_u\omega_\gamma}\begin{bmatrix} \mu_u & \mu_u \\ \omega_\alpha & \omega_\beta \end{bmatrix} \mathsf{F}_{\omega_\beta r}\begin{bmatrix} \omega_\gamma & \bar\jmath \\ \omega_\alpha & j \end{bmatrix} \mathsf{F}_{\omega_\alpha m}\begin{bmatrix} \bar\imath & r \\ i & j \end{bmatrix} \mathsf{F}_{r\bar m}\begin{bmatrix} \bar\imath & \bar\jmath \\ m & \omega_\gamma \end{bmatrix} \end{aligned} \tag{6.39}$$

where r=j if α=e and r=j^* if α=o. Recall that for $i = (r,s)$ the *-operation was defined as $i^* = (r, q-s)$. The first F-matrix element in (6.39) implements an

even/odd coupling rule for bulk fields, i.e. the only combinations for $\{\alpha, \beta, \gamma\}$ which can be nonzero are $\{e, e, e\}$ and $\{e, o, o\}$.

This implies that apart from other symmetries the bulk structure constants may have there is a manifest $\mathbb{Z}_2$–symmetry which sends even bulk fields to themselves and odd ones to minus themselves. Together with section 6.2 and 6.4 we can conclude that any correlator with no or only i-type boundaries is manifestly invariant under the $\mathbb{Z}_2$–symmetry $e \to e$ and $o \to -o$ applied to bulk and boundary simultaneously.

The bulk two-point function follows from (6.39) to be:

$$\begin{aligned} C_{i_\alpha i_\alpha}{}^{1} &= \frac{S_1{}^{i}}{S_1{}^{1}} && \text{for diagonal fields } i_\alpha \text{ (i.e. } i{=}\bar{\imath}) \\ C_{i_o i_o}{}^{1} &= A \cdot \mathsf{F}_{i1}\begin{bmatrix} \omega_o & \omega_o \\ \bar{\imath} & \bar{\imath} \end{bmatrix} \cdot \frac{S_1{}^{i}}{S_1{}^{1}} \cdot (-1)^{s(i_o)} && \text{for nondiagonal fields } i_o \end{aligned} \tag{6.40}$$

where $s(i_\alpha) = h_i - \bar{h}_i \in \mathbb{Z}$ is the spin of the bulk field ϕ_{i_α}.

In [PZu94] it was shown that the minimal model bulk structure constants in the A– and D–series are related by rational numbers, called relative structure constants. Taking the explicit expression (6.39) together with the normalisation (6.40) we find (numerical) agreement with these results. Also, for unitary models the signs of the two-point structure constants in (6.40) are the same as in [PZu94], where they were shown to lead to real bulk structure constants (in the unitary case).

Note that if all bulk fields are even the solution (6.39) takes the same form as in the A-series:

$$C_{i_e j_e}{}^{m_e} = \left(\mathsf{F}_{m1}\begin{bmatrix} i & i \\ j & j \end{bmatrix}\right)^{-1} \tag{6.41}$$

Together with equations (6.23) for the boundary structure constants and (6.35) for the bulk-boundary couplings this illustrates another interesting point. If one considers the even fields alone, that is all bulk and boundary fields that are invariant under the $\mathbb{Z}_2$–symmetry mentioned above, they form a subalgebra (as the coupling $e, e \to o$ is not allowed) and the structure constants are identical to the A-series in the following way:

Consider the A-series boundary theory associated to the pair of diagrams (A_{p-1}, A_{q-1}). In this theory consider only the boundary conditions (α, β) with $\alpha < \frac{p}{2}$ and $\beta < \frac{q}{2}$ or $\alpha > \frac{p}{2}$ and $\beta > \frac{q}{2}$ (compare to (6.8)) and the bulk fields $i \otimes i$ where i has Kac-labels (r, s) with both r and s odd. Then this is a closed subset of fields of the A-series theory and in the normalisation chosen its structure constants coincide with those of the invariant fields of the D-series theory $(A_{p-1}, D_{\frac{q}{2}+1})$. This

correspondence seems natural from the point of view that the invariant part of the D-diagram is an A-diagram.

Real structure constants

In the normalisation chosen in this thesis, the F-matrices are all real. This implies first of all that all structure constants are real for the A–series, both for unitary and non-unitary minimal models. In the D–series all bulk–bulk and boundary–boundary structure constants are real, but not all bulk–boundary couplings. For the boundary structure constants this follows from (6.17)–(6.22). For the bulk structure constants we see from (6.41) that they are clearly real as long as only even fields are involved. If odd fields are present consider (6.39) in the case $C_{i_o j_o}{}^{m_e}$. The phase factors cancel and we are left with a real expression. All other cases are real as well, as can for example be seen by rotating indices using the three-point function $C_{i_\alpha j_\beta}{}^{k_\gamma}\ C_{k_\gamma k_\gamma}{}^{1} = C_{j_\beta k_\gamma}{}^{i_\alpha}\ C_{i_\alpha i_\alpha}{}^{1}$.

For the D–series bulk-boundary couplings in (6.35) we can apply the identity (A.52) to obtain the result:

$$\left({}^{x}B_{(i\otimes\bar{\imath})_\alpha}{}^{\ell_\delta}\right)^* = {}^{x}B_{(\bar{\imath}\otimes i)_\alpha}{}^{\ell_\delta} \tag{6.42}$$

This means that complex conjugation relates the bulk-boundary coupling for a bulk field of spin s to that of the field of spin $-s$. In particular all bulk-boundary couplings for diagonal bulk fields are real.

It is in general not possible to choose a basis of primary fields s.t. all structure constants are real. To see this we construct a gauge invariant expression from the sewing constraints which cannot be fulfiled by real structure constants. Consider for example two diagonal bulk fields i_α and m_ρ and a field j_β with spin one. Take (5.26) with k=1 and a=ω. This forces p_ν=q_ε=ω_o and $r = \bar{\jmath}$. We get:

$$\sum_\rho C_{i_\alpha j_\beta}{}^{m_\rho}\ {}^{\omega}B_{m_\rho}{}^{1} = e^{i\frac{\pi}{2}} \cdot \mathsf{F}_{\omega_o m}\begin{bmatrix} i & \bar{\jmath} \\ i & j \end{bmatrix} \cdot {}^{\omega}B_{i_\alpha}{}^{\omega_o}\ {}^{\omega}B_{j_\beta}{}^{\omega_o}\ C^{(\omega\omega\omega)1}_{\omega_o\omega_o} \tag{6.43}$$

The F-matrix entry is real and will in general be nonzero, provided that all fusions are allowed. As discussed in section 6.3 the spin one field j_β has to couple to ω_o (i.e. ${}^{\omega}B_{j_\beta}{}^{\omega_o} \neq 0$) and we can find a diagonal field i_α that couples to ω_o. The boundary two-point function has to be nonzero by assumption (b) in section 6.2. Hence the rhs is nonzero and some of the D-series structure constants will have nonzero imaginary part.

We can however consider the following non-primary basis of bulk fields to obtain

real structure constants: Diagonal (even or odd) bulk fields stay as they are and for a pair ϕ_s, ϕ_{-s} of nondiagonal bulk fields with spins $\pm s$ we define a new set of fields as follows:

$$\phi^r = \tfrac{1}{2}\left(\phi_s + \phi_{-s}\right) \qquad\qquad \phi^i = \tfrac{1}{2i}\left(\phi_s - \phi_{-s}\right) \tag{6.44}$$

The new fields ϕ^r and ϕ^i are no longer primary, but one finds that the coefficients describing their behaviour under arbitrary conformal mappings are all real. Since the OPE is linked to the transformation behaviour of the fields one expects the coefficients appearing in the OPE to be real as well.

One can see explicitly from (6.42) that all bulk-boundary couplings ${}^{x}B_{\phi^r}{}^{\psi}$ and ${}^{x}B_{\phi^i}{}^{\psi}$ are real and one can verify numerically that all bulk structure constants are real in the new basis as well. This holds for both unitary and non-unitary models.

6.6 g-functions

The g–functions, or generalised ground state degeneracies, will be discussed in more detail in section 7.1. They are defined by taking the $R \to \infty$ limit in the cylinder partition function (5.29),

$$Z_{(ab)} \underset{R\to\infty}{\sim} (g_a \cdot g_b) \cdot e^{-RE_0(L)} \tag{6.45}$$

where $E_0(L)$ is the ground state energy and g_a resp. g_b is the factor of the ground state degeneracy coming from the boundary a resp. b. Let ϕ_Ω be the field of lowest conformal weight. Then the S-matrix (for Virasoro minimal models) satisfies $S_\Omega{}^a > 0\ \forall a$. From (5.29) and (5.30) we get the relation $g_a \cdot g_b = g_{a\Omega} \cdot g_{b\Omega} \cdot \langle\Omega|\Omega\rangle$. Thus

$$g_a = g_{a\Omega}\sqrt{\langle\Omega|\Omega\rangle} = \frac{{}^{a}B_{\Omega}{}^{1} \cdot \langle 1\rangle^a_{\text{disc}}}{\sqrt{C_{\Omega\Omega}{}^{1}\,\langle 0|0\rangle}}\,. \tag{6.46}$$

Note that by (5.36) this expression is invariant under both rescaling of the fields and rescaling of the identity-1-pt-functions.

The values of the unit-disc one point functions, or unit disc partition functions, follow from (5.39):

$$\langle 1\rangle^a_{\text{disc}} = \psi^1_a \tag{6.47}$$

Recall that the ψ^1_a were defined via the Pasquier algebra (5.47). It is worth pointing out that (6.47) is valid only for the unit-disc. The dependence of the partition

function on the geometry is discussed in [CPe88]. For the radius dependence of (6.47) we would get $\langle 1\rangle^x_{\rm disc}\cdot r^{c/6}$, where c is the central charge.

Substituting the explicit values of the structure constants (5.39) and (6.38) (which in the case of spinless bulk fields holds for A– and D–series alike), the g-functions become

$$g_a = \psi_a^\Omega/(S_1{}^\Omega)^{1/2}\ . \tag{6.48}$$

In section 3.1 we already derived a constraint on the unit-disc partition functions. It originated from requiring continuity of the two-point functions and read

$$C^{(xyx)1}_{i_\alpha i_\alpha}\,\langle 1\rangle^x_{\rm disc} = C^{(yxy)1}_{i_\alpha i_\alpha}\,\langle 1\rangle^y_{\rm disc}\ . \tag{6.49}$$

Note that this expression relates partition functions with different boundary conditions in the same geometry. Since both, the boundary structure constants and the unit-disc partition functions have already been determined, this condition provides a consistency check of the formalism.

A-series

For the A–series, the Pasquier algebra is just the Verlinde fusion algebra $M_{ij}{}^k{=}\mathrm{N}_{ij}{}^k$. Consequently the quantities ψ_a^i are equal to the S-matrix:

$$\psi_a^i = S_a{}^i \tag{6.50}$$

In agreement with the original result by Affleck and Ludwig [ALu91], the g-functions in this case are given by

$$g_a = S_a{}^\Omega/(S_1{}^\Omega)^{1/2}\ . \tag{6.51}$$

The constraint (6.49) is equally satisfied, since from (6.13) we see $C^{(xyx)1}_{ii}\,/\,C^{(yxy)1}_{ii}{=}S_1{}^y/S_1{}^x$.

D-series

From the relations (5.39), (6.35) together with the constraint that the vectors ψ_x form an orthonormal basis we get the following expression:

$$\psi_x^{i_\alpha} = {}^xB_{i_\alpha}{}^1\,\psi_x^1\,, \qquad \psi_x^1 = \begin{cases} \sqrt{2}\cdot {S_1}^{x_e} & \text{if } x \text{ is of i-type} \\ \frac{1}{\sqrt{2}}\cdot {S_1}^{x_u} & \text{if } x \text{ is of n-type} \end{cases} \tag{6.52}$$

Since the bulk-boundary couplings are known from (6.35) one can now check that the ψ's so defined are all real and verify numerically that they fulfil (5.40), with ${n_{ix}}^y$ on the lhs taken from section 6.1.

To verify the constraint (6.49) we use the structure constants in section 6.2. Let x, y be i-type boundaries and a, b be n-type boundaries. In the different cases (6.49) reads:

$$\frac{\langle 1\rangle^x_{\text{disc}}}{{S_1}^{x_e}} = \frac{\langle 1\rangle^y_{\text{disc}}}{{S_1}^{y_e}}\,, \qquad \frac{\langle 1\rangle^x_{\text{disc}}}{{S_1}^{x_e}} = 2\cdot\frac{\langle 1\rangle^a_{\text{disc}}}{{S_1}^{a_u}}\,, \qquad \frac{\langle 1\rangle^a_{\text{disc}}}{{S_1}^{a_u}} = \frac{\langle 1\rangle^b_{\text{disc}}}{{S_1}^{b_u}}\,. \tag{6.53}$$

One checks that this is consistent with (6.52).

Chapter 7.

Away from the Critical Point

As an application of the results obtained in the previous chapter we will consider the conformal field theory perturbed away from the critical point. If the perturbation is integrable, i.e. still allows for an infinite number of conserved charges, the results can be compared to other methods in integrable systems. We will carry out a comparison to thermodynamic Bethe ansatz results for the Lee-Yang model.

From the CFT side we use two methods to study the perturbed theory. The first one, the truncated conformal space approach, is purely numerical. It has the advantage that it is relatively straightforward and can give accurate information still quite far away from the fixed point. It can, however, never give exact results. The second method is conformal perturbation theory in the traditional sense, i.e. an expansion of the perturbed action in terms of its coupling. We will see that the integrals involved in the first few terms can be done explicitly.

7.1 g-functions in CFT

Of particular interest will be how the conformal boundary condition is affected by the perturbation. A good indicator (in the sense that it is accessible in all three methods), apart from the energy spectrum, is the so-called g-function. It was introduced by Affleck and Ludwig [ALu91] as a measure of the ground state degeneracy of a conformal boundary condition.

Consider the partition function of a classical statistical-mechanical system defined on a cylinder of length R and circumference L. Among the characteristics of the model might be a bulk mass scale M and boundary scales depending on the boundary conditions α and β imposed at the two ends of the cylinder; we will highlight the role of these quantities by denoting the partition function $Z_{\alpha\beta}(M, R, L)$. If R is taken to infinity with all other variables held fixed, then

$$Z_{\alpha\beta}(M, R, L) \sim A_{\alpha\beta}(M, L)\, e^{-R\, E_0^{\mathrm{circ}}(M,L)}\ , \tag{7.1}$$

where $E_0^{\rm circ}(M,L)$ is the ground state energy of the model on a circle of circumference L. To derive this asymptotic R-dependence, it is sufficient to treat the boundary conditions as boundary states $|\alpha\rangle$ in a formalism where time runs along the length of the cylinder, and states are propagated by a bulk Hamiltonian $H_{\rm circ}(M,L)$:

$$Z_{\alpha\beta}(M,R,L) = \langle\alpha|\, \exp(-RH_{\rm circ}(M,L))\, |\beta\rangle \ . \tag{7.2}$$

At large R the contribution of the ground state $|\Omega\rangle$ dominates, establishing (7.1) and also giving

$$A_{\alpha\beta}(M,L) = \frac{\langle\alpha|\Omega\rangle\, \langle\Omega|\beta\rangle}{\langle\Omega|\Omega\rangle} \ . \tag{7.3}$$

The inner products appearing in (7.3) should in general contain a term corresponding to a free-energy per unit length, i.e.

$$\log(\, \frac{\langle\Omega|\alpha\rangle}{\langle\Omega|\Omega\rangle^{1/2}}\,) = -Lf_\alpha + \log(g_\alpha(M,L)) \ . \tag{7.4}$$

This linear term can in principle be extracted unambiguously from the large L behaviour of $\log(\, \langle\Omega|\alpha\rangle\,)$. The question is then whether the functions $\log(g_\alpha(M,L))$ now contain universal information.

In the case M=0, i.e. for critical bulk, Affleck and Ludwig [ALu91] pointed out that the UV and IR limits of these functions, $\log(g_\alpha(0,0))$ and $\log(g_\alpha(0,\infty))$, play the role of a generalised ground state degeneracy for the UV and IR conformal boundary conditions respectively. These can be easily calculated for many conformal field theories, and can enable one to identify the boundary conditions uniquely.

The universality of the g-functions is however a somewhat delicate issue when the model has a mass scale, either in the bulk or at the boundary. For example, one could imagine that in some calculational schemes the boundaries acquire a finite thickness, and so the effective cylinder length would decrease by some finite amount δ to $R-\delta$, in which case $\log(g_\alpha)$ would be altered

$$\log(g_\alpha(M,L)) \to \log(g_\alpha(M,L)) + \frac{\pi\delta}{12L} c(ML) \ , \tag{7.5}$$

where $c(ML)$ is related to the ground state energy on the circle by

$$E_0^{\rm circ}(M,L) = f_{\rm bulk}L - \frac{\pi}{6L} c(ML) \ . \tag{7.6}$$

So in search of potential universal information contained in the g–functions it seems

that differences of $\log g$, either between the UV/IR values of a given flow or of the g–functions for different boundary condition along the flow, stand a better chance of being independent of the scheme used to calculate them.

7.2 Truncated conformal space approach

The truncated conformal space approach (TCSA) was originally divised by Yurov and Zamolodchikov in [YZa90]. The method was later adapted by Dorey et al. to situations with boundaries [DPTW97]. This section has two parts. First a quick review of the theoretical workings of TCSA is given. As already said, the mathematics of TCSA is straightforward enough, and the only difficult bit (apart from having the idea in the first place, of course) is to not to get confused when writing the program. So the second part of this section is devoted to a more detailed description of how one could set up a TCSA program.

Some theory

The idea behind TCSA is to diagonalise the perturbed Hamiltonian numerically. Consider an infinite strip of width R. At the fixed point the right/left boundary conditions are a, b and the unperturbed Hamitonian is denoted as H^{CFT}. We will consider a perturbation by a relevant bulk field $\lambda \cdot \phi(x, y)$ and a relevant boundary field on the right boundary $\mu \cdot \psi(0, y)$. The extension to perturbing both boundaries is straightforward, but makes notation more cumbersome. The perturbed Hamiltonian on the strip reads:

$$H^{\mathrm{pert}}_{\mathrm{strip}} = H^{\mathrm{CFT}}_{\mathrm{strip}} + \lambda \cdot \int_0^R \phi(x)dx + \mu \cdot \psi(0) \tag{7.7}$$

Transformed to the UHP this becomes:

$$\begin{aligned} H^{\mathrm{pert}} &= \tfrac{\pi}{R}\Big(L_0 - \tfrac{c}{24} + \lambda \cdot \big(\tfrac{R}{\pi}\big)^{2-\Delta_\phi} \int_0^\pi \phi(e^{i\theta}, e^{-i\theta})d\theta + \mu \cdot \big(\tfrac{R}{\pi}\big)^{1-h_\psi}\, \psi^{(aa)}(1)\Big) \\ &= \big(\tfrac{R}{\pi}\big)^{-1} H^{\mathrm{CFT}} + \lambda\big(\tfrac{R}{\pi}\big)^{1-\Delta_\phi} H^{\mathrm{bulk}} + \mu\big(\tfrac{R}{\pi}\big)^{-h_\psi} H^{\mathrm{bnd}} \end{aligned} \tag{7.8}$$

The perturbed Hamiltonian H^{pert} acts on the Hilbert space $\mathcal{H}_{ab}$. To work out its matrix elements one has to compute $\langle v_1|H^{\mathrm{pert}}|v_2\rangle$ for all $v_1, v_2 \in \mathcal{H}_{ab}$. Looking at (7.8) we see that this involves calculating correlators with three boundary fields or with one bulk and two boundary fields. The first is just a chiral 3pt function, whereas the second requires a chiral 4pt function. If this is not available analytically, it can be obtained by numerically solving the differential equation, by explicitly

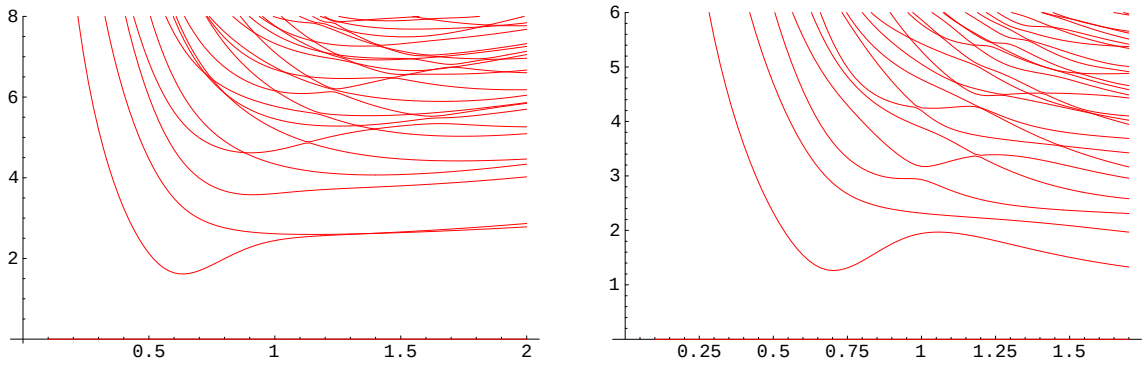


Figure 7.1: TCSA results (level 14, 135 states) for a bulk perturbation in the A-series model M(5,6) on a strip with $\mathbb{1}$-boundary condition on either side. $\tilde{E}_n - \tilde{E}_0$ is plotted against r. (a) the perturbation by $\phi_{1,2}$ looks integrable and (b) the perturbation by $\phi_{2,3}$ does not.

computing the power series according to (4.11), or by using a recursion relation for conformal blocks due to Zamolodchikov [Zam84].

To accomplish the numerical diagonalisation we introduce an energy cutoff $E_{\max}$. All states with *unperturbed* energy higher than $E_{\max}$ are neglected and we are left with a finite, truncated matrix $\hat{H}^{\text{pert}}$. Note that the energy cutoff approximately translates into a level cutoff on the conformal Hilbert space $\mathcal{H}_{ab}$, i.e. in each representation we work only with states up to some level $N_{\max}$.

One can now, for example, plot the spectrum of the truncated perturbed Hamiltonian. To make the graphs reproducible one has to specify the normalisation of the perturbing fields and give the results in terms dimensionless quantities. In the following we will normalise perturbing fields $\varphi(x)$ s.t. the two point function takes the form $\langle\varphi(x)\varphi(y)\rangle = \pm|x-y|^{\nu}$, for some power ν. The minus sign is chosen whenever the plus sign does not lead to real coupling constants. If the field $\varphi(x)$ has self-coupling we can fix the remaining sign ambiguity by demanding that $C>0$ in $\langle\varphi(u)\varphi(v)\varphi(w)\rangle = C\cdot|(u-v)(u-w)(v-w)|^{\eta}$.

In the case of a bulk perturbation one can for example use the parameters $r = \frac{R}{\pi}\cdot\lambda^{1/(2-\Delta_\phi)}$ and $\tilde{\mu} = \left(\frac{R}{\pi}\right)^{1-h_\psi}\mu$. If the model develops a mass gap one can plot E_n/M, where M is the mass of the lightest particle. If M is not known one can resort to $\tilde{E}_n = \lambda^{-1/(2-\Delta_\phi)}E_n$. Here E_n is the energy of the n'th excited state.

As an example the spectrum of $M(5,6)$ perturbed by (1,2) and (2,3) in the bulk is shown in fig. 7.1. An interesting point is that by an heuristic argument one can try to read off integrability of the perturbed theory. If it stays integrable, there are infinitely many conserved quantities and the energy levels in the spectrum just cross, without seeing each other, as they reorganise from the UV to the IR fixed point spectrum. An example is show in fig. 7.1a. For a nonintegrable perturbation

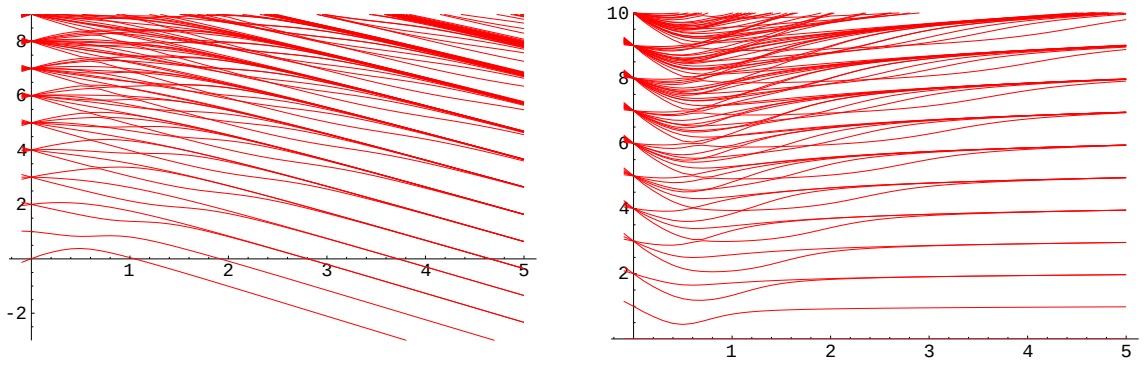


Figure 7.2: TCSA results (level 16, 504 states) for a boundary perturbation by $\psi_{3,3}$ in the A-series model M(4,5) on a strip with boundary conditions (1,1) and (2,2). (a) scaling functions F_n and (b) differences $F_n - F_0$ plotted against $\tilde{r}$.

some of the lines will seem to repel each other, as in fig. 7.1b. One has to be careful though not to confuse non-integrability and level repulsion due to truncation effects.

For massless perturbations all states will form a gapless continuum as $R \to \infty$. In this case it is better to plot the scaling functions F_n, which are defined in terms of the energy levels E_n by

$$F_n(R, \lambda, \mu) = \frac{R}{\pi} E_n(R, \lambda, \mu) \; . \tag{7.9}$$

If we are perturbing only the boundary we can use $\tilde{r} = \frac{R}{\pi} \cdot \mu^{1/(1-h_\psi)}$ as dimensionless variable. As an example the perturbation of $M(4,5)$ on a strip with boundary conditions (1,1) and (2,2) by the boundary field $\psi_{3,3}$ is shown in fig. 7.2. In particular we see in fig. 7.2b how the degeneracy of spectrum reorganises from its UV-value, in this case given by the character

$$q^{c/24-h_{2,2}} \chi_{2,2}(q) = 1\, q^0 + 1\, q^1 + 2\, q^2 + 3\, q^3 + 4\, q^4 + 6\, q^5 + 8\, q^6 + \ldots \tag{7.10}$$

to its IR-value, given by the character

$$q^{c/24-h_{3,1}} \chi_{3,1}(q) = 1\, q^0 + 1\, q^1 + 2\, q^2 + 2\, q^3 + 3\, q^4 + 4\, q^5 + 6\, q^6 + \ldots \tag{7.11}$$

Thus the $\psi_{3,3}$ perturbation changes the (2,2)–boundary condition to the (3,1)–boundary condition.

There are several points worth mentioning. If the perturbation were to need UV regularisation (i.e. $\Delta \geq 1$ or $h \geq \frac{1}{2}$), the ground state energy would not converge as we took the truncation level $N_{\max}$ to infinity. In this case only energy differences have physical meaning. Also, the energy levels (fig. 7.2b) should become horizontal

lines for $R \to \infty$, representing the spectrum of the IR–fixed point. But at some point the numerical data from the truncated Hamiltonian becomes unreliable and we see a departure from the expected behaviour.

Writing a program

Here is a list of ingredients for a TCSA program. It can be written in the order presented, but this is by no means the only way. For simplicity we assume that the perturbing bulk field ϕ is diagonal, i.e. $h_\phi{=}\bar{h}_\phi$ and that we are only perturbing the right boundary on the strip.

(i) First of all we fix the model we are working in, $M(p,q)$ say. Let $\mathcal{R}$ be the list of all Vir–highest weight representations occuring in the model.

(ii) Next we need a basis for each Vir–module $i \in \mathcal{R}$, up to the truncation level $N_{\max}$. Let $\{b(i)_k\}$ be such a basis. For convenience we take each element to have the form $b(i)_k{=}L_{-n_1}\dots L_{-n_m}|i\rangle$ with $n_1{>}\dots{>}n_m{>}0$. To compute the basis one can try to find a maximal set of $b(i)_k$ s.t. the inner product matrix

$$g(i)_{k\ell} = \frac{\langle b(i)_k | b(i)_\ell \rangle}{\langle i|i\rangle} \tag{7.12}$$

has nonzero determinant. Let $G(i) = g(i)^{-1}$ be the inverse matrix.

(iii) For the boundary perturbation we need the chiral three-point functions. Let $i,j,k \in \mathcal{R}$. For each ℓ, m in the respective basis, compute the matrix

$$M(i,j,k)_{\ell m} = \frac{\langle b(i)_\ell | \phi_j(1) | b(k)_m \rangle}{\langle i|\phi_j(1)|k\rangle} \ . \tag{7.13}$$

One way to obtain the matrix elements M is to first move all L_n's to the right by using the commutator $[L_n - x^n L_0, \phi(x)]{=}nh\phi(x)$ in the form

$$L_n\phi_j(1) = L_0\phi_j(1) + \phi_j(1)(nh_j + L_0 - L_n) \ . \tag{7.14}$$

We are left with a sum of terms of the form $\langle i|\phi_j(1)L_{-n_1}\dots L_{-n_m}|k\rangle$. The L_{-n}'s can now be annihilated on the left using

$$\langle i|\phi_j(1)L_{-n} = (nh_j - h_i)\langle i|\phi_j(1) + \langle i|\phi_j(1)L_0 \ . \tag{7.15}$$

With this procedure the 3pt-matrix element $\langle b(i)_\ell|\phi_j(1)|b(k)_m\rangle$ is eventually reduced to $const \cdot \langle i|\phi_j(1)|k\rangle$.

We have now gathered all the information $(b(i), G(i), M(i,j,k))$ that can be obtained from the representation theory alone.

(iv) The next step is to build a basis for the Hilbert space. We have to fix the strip-boundary conditions a, b and compute which representations occur in the Hilbert space $\mathcal{H}_{ab} = \sum_k n_{kb}{}^a \mathcal{H}_k$ as described in section 6.1. We can choose a basis $\{v_x | x{=}1, 2, \dots\}$ of $\mathcal{H}_{ab}$ of the form $v_x = |b(r_x)_{k_x}, \alpha_x\rangle$. Here r_x is the Vir–representation v_x lives in and α_x is a multiplicity index to distinguish states in equivalent Vir–representations.

(v) The matrix form H_{xy} of the Hamiltonian H^{pert} of eqn. (7.8) is defined as follows: Let $\sum_y c_y v_y$ be a state in $\mathcal{H}_{ab}$. Then

$$H^{\text{pert}}\big(\sum_y c_y v_y\big) = \sum_x \big(\sum_y H_{xy} c_y\big) v_x \ . \tag{7.16}$$

Taking the inner product with another state $\langle v_z|$ gives the relation

$$\langle v_z | H^{\text{pert}} | v_y \rangle = \sum_x \langle v_z | v_x \rangle H_{xy} \quad \Rightarrow \quad H_{xy} = \sum_z G_{xz} \cdot \langle v_z | H^{\text{pert}} | v_y \rangle \ . \tag{7.17}$$

where G_{xz} is the inverse matrix to $g_{zx}{=}\langle v_z | v_x \rangle$. In terms of (7.12) it takes the form $G_{xy}{=}\delta_{i_x,i_y} G(r_x)_{k_x,k_y} / \langle i_x | i_x \rangle$. Here i_x denotes the field (r_x, α_x) and i_y the field (r_y, α_y), i.e. the highest weight states for v_x, v_y. Relation (7.17) can be evaluated for H^{CFT}, H^{bulk}, H^{bnd}, as defined in (7.8). For H^{CFT} this just gives a diagonal matrix

$$H^{\text{CFT}}_{xy} = \delta_{x,y} \cdot \frac{\langle v_x | L_0 - \frac{c}{24} | v_x \rangle}{\langle v_x | v_x \rangle} \ . \tag{7.18}$$

For H^{bnd} one can use the relation

$$\begin{aligned} \langle v_z | H^{\text{bnd}} | v_y \rangle &= \frac{\langle b(r_z)_{k_z}, \alpha_z | \, \psi^{(aa)}(1) | b(r_y)_{k_y}, \alpha_y \rangle}{\langle i_z | \, \psi^{(aa)}(1) | i_y \rangle} \langle i_z | \, \psi^{(aa)}(1) | i_y \rangle \\ &= M(r_z, \psi, r_y)_{k_z k_y} \cdot C^{(aaa)i_z}_{\psi i_y} \cdot \langle i_z | i_z \rangle \ , \end{aligned} \tag{7.19}$$

where again i_y denotes the field (r_y, α_y) and i_z the field (r_z, α_z). The boundary 2pt–function cancels in (7.17) and the result is

$$H^{\text{bnd}}_{xy} = C^{(aaa)i_x}_{\psi i_y} \sum_\ell G(r_x)_{k_x \ell} M(r_x, \psi, r_y)_{\ell k_y} \ . \tag{7.20}$$

(vi) The most difficult part of the procedure is to obtain the matrix elements of

the bulk perturbation H^{bulk}. We have to compute

$$B_{zy} := \int_0^\pi d\theta \, \langle v_z|\phi(e^{i\theta}, e^{-i\theta})|v_y\rangle \ , \quad H^{\text{bulk}}_{xy} = \sum_z G_{xz} \cdot B_{zy} \ . \tag{7.21}$$

First we need rules to commute L_n's past the bulk field to reduce the correlator to one involving only primary fields:

$$\begin{aligned}
&\int_0^\pi d\theta \langle\alpha|L_n\phi(e^{i\theta}, e^{-i\theta})|\beta\rangle \\
&= \tfrac{n(2h_\phi-1)+|\alpha|-|\beta|}{n(1-2h_\phi)+|\alpha|-|\beta|}\Big(\int_0^\pi \langle\alpha|L_{-n}\phi|\beta\rangle - \int_0^\pi \langle\alpha|\phi L_{-n}|\beta\rangle\Big) + \int_0^\pi \langle\alpha|\phi L_n|\beta\rangle \ , \qquad (7.22)
\end{aligned}$$

$$\begin{aligned}
&\int_0^\pi f(\theta)\langle i|\phi(e^{i\theta}, e^{-i\theta})L_{-n}|\beta\rangle d\theta \\
&= \big(n(2h_\phi-1)-h_i+|\beta|\big)\int_0^\pi \cos(n\theta) f(\theta)\langle i|\phi|\beta\rangle - \int_0^\pi \sin(n\theta) f'(\theta)\langle i|\phi|\beta\rangle \qquad (7.23)
\end{aligned}$$

Here $n{>}0$, $|\alpha\rangle, |\beta\rangle$ are arbitrary states and $|i\rangle$ is a highest weight state. Note that (7.22) cannot be applied if the denominator is zero. In this case, if $n{\neq}2$ we can replace $L_n{=}\frac{1}{n-2}(L_{n-1}L_1{-}L_1L_{n-1})$. If $n{=}2$ (this can only happen if $4h_\phi$ is an integer), one can resort to (A.101). The rules (7.22)–(7.23) are derived in appendix A.8. For each matrix element we are left with an integral of the form

$$\int_0^\pi f(\theta)\langle i|\phi(e^{i\theta}, e^{-i\theta})|j\rangle \, d\theta \ , \tag{7.24}$$

where i, j are highest weight states and the function $f(\theta)$ can be expressed as a finite linear combination of $(\sin\theta)^n$ and $\cos\theta(\sin\theta)^n$. Let S_n and C_n be defined as

$$S_n := \int_0^\pi (\sin\theta)^n \langle i|\phi(1,\theta)|j\rangle d\theta \ , \qquad C_n := \int_0^\pi \cos\theta \ (\sin\theta)^n \langle i|\phi(1,\theta)|j\rangle d\theta \ . \tag{7.25}$$

As described in appendix A.8, the null state of the perturbing bulk field can be used to derive a recursion relation for S_n and C_n. As an example the

calculation for $\phi{\equiv}\phi_{1,2}$ is shown, with the results, for $h_\phi{<}\frac{1}{2}$,

$$\begin{aligned} h_i \neq h_j: \quad & S_n = C\cdot 2^{\frac{1+2h_\phi}{3}}\sqrt{\pi}\frac{\Gamma(\frac{2h_\phi+3n+4}{6})}{\Gamma(\frac{2h_\phi+3n+7}{6})}\ , \quad C_n = 0\ . \\ h_i = h_j = h: \quad & S_n = \frac{(2h_\phi{-}n{+}1)(2h_\phi{+}3n{-}2)}{4h_\phi(h_\phi{+}4h{+}n{-}1)+8h-n(3n{-}2)}\cdot S_{n-2}\ , \\ & C_n = \frac{(2h_\phi{-}n{+}1)(2h_\phi{+}3n{-}2)}{4h_\phi(h_\phi{+}4h{+}n)+8h-1-n(3n{+}4)}\cdot C_{n-2}\ . \end{aligned} \tag{7.26}$$

Here C is defined via $\langle i|\phi(1,\theta)|j\rangle \underset{\theta\to 0}{\sim} C\cdot(2\theta)^{h_{13}-2h_{12}}$. In the case $\phi{\equiv}\phi_{1,2}$, for $h_i{\neq}h_j$ the conformal block takes an easy enough form and the integrals S_n, C_n can be given explicitly, whereas for $h_i{=}h_j$ we have a recursion relation. The initial values S_0, S_1, C_0, C_1 can be found e.g. by numerically integrating (A.117).

Another useful case that can be calculated explicitly is when both $|i\rangle$,$|j\rangle$ are equal to the vacuum $|0\rangle$. Then from $\langle i|\phi(e^{i\theta},e^{-i\theta})|j\rangle = C\cdot(2\sin\theta)^{-2h_\phi}$ we get, for $h_\phi{<}\frac{1}{2}$,

$$h_i = h_j = 0: \quad S_n = C\cdot 2^{-2h_\phi}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}{-}h_\phi{+}\frac{n}{2})}{\Gamma(1{-}h_\phi{+}\frac{n}{2})}\ , \quad C_n = 0\ . \tag{7.27}$$

The TCSA program is now complete. All that remains to be done is to add the truncated matrices (7.18), (7.20) and (7.21) for different values of R, λ, μ as in eqn. (7.8), diagonalise $H_{xy}^{\rm pert}$ and plot the spectra.

7.3 Conformal perturbation theory

In this approach we expand the perturbed action as a power series in the coupling constants. The object we are interested in is the $\mathcal{G}$–function of the perturbed theory, defined as

$$\mathcal{G}_\alpha = \frac{\langle\alpha|\Omega\rangle}{\langle\Omega|\Omega\rangle^{1/2}}\ , \tag{7.28}$$

where $|\alpha\rangle$ is the (possibly perturbed) boundary state and $|\Omega\rangle$ is the ground state of the perturbed theory. Typically these functions will not be equal to the g-functions g_α but will differ by the extensive free–energy terms as in eqn. (7.4):

$$\log\mathcal{G}_\alpha = \log g_\alpha \ - \ L\, f_\alpha^{\rm pcft}\ . \tag{7.29}$$

$\mathcal{G}$-functions on the cylinder

Consider a boundary CFT on a cylinder of length R and circumference L. Now take the pure CFT action $\mathcal{A}_{\rm CFT}$ to be perturbed by a bulk field ϕ and the conformal boundary conditions on the left and right end of the cylinder by boundary fields $\psi_{\ell/r}$, resulting in an action of the form

$$\mathcal{A}_{\rm pert} = \mathcal{A}_{\rm CFT} + \mu_\ell \int \psi_\ell + \lambda \iint \phi + \mu_r \int \psi_r \ . \tag{7.30}$$

If we take Euclidean time to flow along the cylinder then the Hamiltonian in the perturbed theory is given by an integral around the cylinder

$$H = \frac{1}{2\pi} \int_0^L ds\, T_{yy}(s) + \lambda \int_0^L ds\, \phi(s) \tag{7.31}$$

and the boundary conditions are described by a state in the Hilbert space:

$$\langle B| = \langle \alpha(\mu)| := \langle \alpha| \exp\Big\{ -\mu \int \psi \Big\} \tag{7.32}$$

The partition function of the perturbed CFT can now be expressed as:

$$Z_{\rm cyl}(R,L) = \langle B_\ell | e^{-RH} | B_r \rangle = \sum_{n=0}^{\infty} \frac{\langle B_\ell | E_n(\lambda)\rangle e^{-RE_n(\lambda)} \langle E_n(\lambda)| B_r\rangle}{\langle E_n(\lambda)|E_n(\lambda)\rangle} \tag{7.33}$$

where $E_n(\lambda)$ denotes the n'th eigenvalue/vector of the (perturbed) Hamiltonian and we take $E_0(\lambda)$ to be the ground state.

We extend the definition (7.28) of $\mathcal{G}$–functions to excited states as follows:

$$\mathcal{G}_\alpha^{(n)}(\mu,\lambda) = \frac{\langle \alpha(\mu)|E_n(\lambda)\rangle}{|\langle E_n(\lambda)|E_n(\lambda)\rangle|^{1/2}} \tag{7.34}$$

If we introduce the dimensionless variables

$$x = \lambda \left(\tfrac{L}{2\pi}\right)^{2-x_\phi} \ , \qquad y = \mu \left(\tfrac{L}{2\pi}\right)^{1-h_\psi} \ , \tag{7.35}$$

we can write the power series expansion of (7.34) as

$$\mathcal{G}_\alpha^{(n)}(\mu,\lambda) = \mathcal{G}_\alpha^{(n)}(0,0) \cdot \big(1 + a_{10}\cdot y + a_{20}\cdot y^2 + a_{01}\cdot x + a_{11}\cdot xy + \dots\big) \ , \tag{7.36}$$

where the coefficients a_{mn} are constants. The aim of the remainder of this section is to compute some of these coefficients using CPT. For quick reference, the resulting formulas in the various cases we are going to consider are (7.52), (7.54), (7.61),

(7.66).

Transformation to the unit disc

To make use of the Hilbert space description of CFT in the complex plane we apply the conformal map $z \to w = \exp(2\pi i z/L)$. This takes the cylinder to an annulus. The corresponding Hamiltonian and boundary states defined in the Hilbert space of the complex plane are:

$$\begin{aligned} H = H_0 + \lambda H_p &= \frac{2\pi}{L}\left\{ L_0 + \bar{L}_0 - \frac{c}{12} + x \cdot \int_0^{2\pi} d\theta\, \phi(e^{i\theta}) \right\} \\ \langle \alpha(\mu)| &= \langle \alpha| \exp\Big\{ -\big(\tfrac{L}{2\pi}\big)^{1-h_\psi}\, \mu \int_0^{2\pi} d\theta\, \psi(e^{i\theta}) \Big\} \\ &= \langle \alpha| \Big(1 - y \cdot \int_0^{2\pi} d\theta\ \psi(e^{i\theta}) + \tfrac{1}{2} y^2 \cdot \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta'\ \psi(e^{i\theta})\psi(e^{i\theta'}) + \dots \Big)\,. \end{aligned} \tag{7.37}$$

To obtain the first order approximation to the perturbed eigenstates we have to solve the equation

$$(H_0 + \lambda H_p)\big(|E_n\rangle + \lambda |e_n\rangle\big) = (E_n + \lambda e_n)\big(|E_n\rangle + \lambda |e_n\rangle\big) + O(\lambda^2) \tag{7.38}$$

together with the constraint that the perturbation should not change the norm of the eigenvector $\langle E_n(\lambda)|E_n(\lambda)\rangle = \langle E_n|E_n\rangle$. The result is:

$$\begin{aligned} E_n(\lambda) &= E_n + \lambda \frac{\langle E_n|H_p|E_n\rangle}{\langle E_n|E_n\rangle} + O(\lambda^2) \\ |E_n(\lambda)\rangle &= |E_n\rangle + \lambda Q_n H_p |E_n\rangle + O(\lambda^2) \end{aligned} \tag{7.39}$$

where Q_n is given by:

$$Q_n = -(1 - P_n)\frac{1}{H_0 - E_n}(1 - P_n) \tag{7.40}$$

and P_n is the projector on $|E_n\rangle$.

To handle the inverse operator we use a relation of the type $x^{-1} = \int_0^1 q^{x-1} dq$ which holds for $x > 0$. Consequently we need to treat positive and negative eigenvalues of $H_0 - E_n$ differently. Define projectors $P_{<n}$ and $P_{\le n}$ to project on the sum of the H_0-eigenspaces with eigenvalues $E < E_n$, resp. $E \le E_n$. Then

$$Q_n = -\int_0^1 \frac{dq}{q}(1 - P_{\le n}) q^{H_0 - E_n}(1 - P_{\le n}) + P_{<n}\frac{1}{E_n - H_0}P_{<n} \tag{7.41}$$

Using that and the fact that L_0 generates dilations we get:

$$\lambda Q_n H_p |E_n\rangle = x\Big\{ \int_0^1 \frac{dq}{q} \left[-q^{x_\phi} \int_0^{2\pi} d\theta\ \phi(qe^{i\theta})|E_n\rangle + \sum_{E_k \le E_n} |E_k\rangle q^{\frac{L}{2\pi}(E_k - E_n)} A_k \right] + \sum_{E_k < E_n} |E_k\rangle (E_n - E_k)^{-1} \frac{2\pi}{L} A_k \Big\} \tag{7.42}$$

where

$$A_k = 2\pi \frac{\langle E_k|\phi(1)|E_n\rangle}{\langle E_k|E_k\rangle} \tag{7.43}$$

Evaluating inner products using disc-amplitudes

To evaluate an inner product of the form $\langle\alpha(\mu)|E_n\rangle$ we interpret it as a unit-disc diagram in boundary CFT with conformal boundary condition α perturbed by the boundary field $\psi(e^{i\theta})$ and with a field Φ_n inserted at the origin. Our notation for this identification will be, e.g.

$$\langle\alpha| \int_0^{2\pi} d\theta\ \psi(e^{i\theta})|E_n\rangle = \langle \int_0^{2\pi} d\theta\ \psi(e^{i\theta})\Phi_n(0)\rangle^\alpha_{\text{disc}}\ . \tag{7.44}$$

Möbius invariance then fixes the form of the simplest unit disc amplitudes to be, for a primary boundary field ψ and a primary bulk field ϕ:

$$\begin{aligned}
\langle\phi(r)\rangle^\alpha_{\text{disc}} &= \langle 1\rangle^\alpha_{\text{disc}}\ {}^\alpha B_\phi{}^1 \cdot (1-r^2)^{-x_\phi} \\
\langle\psi(e^{i\theta_1})\psi(e^{i\theta_2})\rangle^\alpha_{\text{disc}} &= \langle 1\rangle^\alpha_{\text{disc}}\ C^{(\alpha\alpha\alpha)1}_{\psi\psi} \left|2\sin\frac{\theta_{12}}{2}\right|^{-2h_\psi} \\
\langle\psi(e^{i\theta})\phi(r)\rangle^\alpha_{\text{disc}} &= \langle 1\rangle^\alpha_{\text{disc}}\ {}^\alpha B_\phi{}^\psi\ C^{(\alpha\alpha\alpha)1}_{\psi\psi} \cdot \left|1-2r\cos\theta+r^2\right|^{-h_\psi} \left|1-r^2\right|^{h_\psi - x_\phi} \\
\langle\psi(e^{i\theta_1})\psi(e^{i\theta_2})\psi(e^{i\theta_3})\rangle^\alpha_{\text{disc}} &= \langle 1\rangle^\alpha_{\text{disc}}\ C^{(\alpha\alpha\alpha)\psi}_{\psi\psi}\ C^{(\alpha\alpha\alpha)1}_{\psi\psi} \left|8\sin\frac{\theta_{12}}{2}\sin\frac{\theta_{13}}{2}\sin\frac{\theta_{23}}{2}\right|^{-3h_\psi}
\end{aligned} \tag{7.45}$$

In general the energy eigenstate $|E_n\rangle$ will not be given by a primary field but by a descendent thereof. Typically we will want to evaluate a unit disc amplitude of the form

$$\langle\alpha| \int\psi \int\psi \cdots \phi \cdots L_{-n}L_{-m} \cdots (\bar{L}\text{'s})|\gamma\rangle\ . \tag{7.46}$$

where α denotes a conformal boundary state, ψ is the perturbing boundary field, ϕ the perturbing bulk field and $|\gamma\rangle$ a bulk highest weight state.

Before we turn to analysing (7.46) it is helpful to introduce some notation. As a

first step we define the operators

$$F_n := L_n - \bar{L}_{-n} \tag{7.47}$$

Their commutator is $[F_n, F_m] = (n-m)F_{n+m}$. The F_n have the property that they annihilate the conformal boundary state of the unit disc $\forall n : \langle\alpha|F_n = 0$. Their commutators with a boundary field $\psi(e^{i\theta})$ and a bulk field $\phi(z,\bar{z})$ are given by

$$\begin{aligned}
[F_n, \psi(e^{i\theta})] &= e^{in\theta}\Big(hn - i\tfrac{\partial}{\partial\theta}\Big)\psi(e^{i\theta}) \\
[F_n, \phi(z,\bar{z})] &= \Big\{h(n{+}1)z^n + h(n{-}1)\bar{z}^{-n} + \tfrac{1}{2}(z^n - \bar{z}^{-n})r\partial_r\Big\}\phi(z,\bar{z}) \\
&\quad + \tfrac{1}{2}(z^n + \bar{z}^{-n})[F_0, \phi(z,\bar{z})]\ .
\end{aligned} \tag{7.48}$$

Next we define an operator denoting a boundary field with a phase factor integrated around the boundary:

$$D_n = \int_0^{2\pi} d\theta\, e^{in\theta}\psi(e^{i\theta}) \tag{7.49}$$

Note that the D's all commute since e.g. D_nD_m and D_mD_n both stand for the same boundary integral $\iint \exp(i(n\theta + m\theta'))\psi(e^{i\theta})\psi(e^{i\theta'})d\theta d\theta'$. Applying (7.48) we obtain the commutation relation

$$[F_m, D_n] = ((h_\psi{-}1)m - n)\, D_{n+m}\ . \tag{7.50}$$

We can now check that $[F_m, e^{-yD_0}]{=}(1{-}h_\psi)mD_m y e^{-yD_0}$ or, expressed using the perturbed boundary state $\langle\alpha(y)|{=}\langle\alpha|\exp(-yD_0)$,

$$\langle\alpha(y)|F_n = -n\,y\,(1{-}h_\psi)\cdot\langle\alpha(y)|D_n\ . \tag{7.51}$$

We can now replace the leftmost L_{-n} in (7.46) with $F_{-n} + \bar{L}_n$ and use (7.50) to commute F_{-n} to the left till it annihilates on the boundary. The $\bar{L}_n$ can be commuted to the right. Repeating that process often enough gets rid of all L's and for the $\bar{L}$'s one can apply the same procedure with $\bar{L}_{-n}{=}{-}F_n{+}L_n$. In the end we are left with an amplitude involving only primary fields as desired.

Note that when expanding $\langle\alpha(\mu)|E_n(\lambda)\rangle$ for most energy eigenstates $|E_n\rangle$ we can only go to first order in μ if we do not want to use anything more complicated than the disc amplitudes given in (7.45). The only exceptions are energy eigenstates based on the vacuum $|0\rangle$, for which we can go to order μ^3 and $\lambda\mu$ respectively.

In the first case, when $|E_n\rangle$ is given by a primary field $|\gamma\rangle$ we get, using the

dimensionless quantities (7.35):

$$\mathcal{G}_\alpha^{(\gamma)}(\mu,\lambda)=\frac{\langle\alpha(\mu)|\gamma(\lambda)\rangle}{|\langle\gamma|\gamma\rangle|^{1/2}}=\mathcal{G}_\alpha^{(\gamma)}(0,0)(1-y\cdot\frac{{}^{\alpha}B_\gamma{}^{\psi}}{{}^{\alpha}B_\gamma{}^{1}}\,C_{\psi\psi}^{(\alpha\alpha\alpha)1}\,2\pi+\cdots) \tag{7.52}$$

For descendent fields we use (7.50) to reduce the amplitude to the above situation. E.g. for the states $|\gamma'\rangle{:=}L_{-1}\bar{L}_{-1}|\gamma\rangle$ and $|\gamma''\rangle{:=}{L_{-1}}^2\bar{L}_{-1}^2|\gamma\rangle$ we get the relations (recall (7.35))

$$\begin{aligned}\mathcal{G}_\alpha^{(\gamma')}(\mu,\lambda)&=\mathcal{G}_\alpha^{(\gamma)}(0,0)\Big(1-y\cdot\frac{{}^{\alpha}B_\gamma{}^{\psi}}{{}^{\alpha}B_\gamma{}^{1}}\,C_{\psi\psi}^{(\alpha\alpha\alpha)1}\,2\pi\Big\{1+\frac{h_\psi(1-h_\psi)}{2h_\gamma}\Big\}+\dots\Big)\\ \mathcal{G}_\alpha^{(\gamma'')}(\mu,\lambda)&=\mathcal{G}_\alpha^{(\gamma)}(0,0)\Big(1-y\cdot\frac{{}^{\alpha}B_\gamma{}^{\psi}}{{}^{\alpha}B_\gamma{}^{1}}\,C_{\psi\psi}^{(\alpha\alpha\alpha)1}\,2\pi\Big\{1+\frac{h_\psi(1-h_\psi)}{2h_\gamma}\big(1+\frac{h_\psi(1-h_\psi)}{2(1+2h_\gamma)}\big)\Big\}+\dots\Big)\end{aligned} \tag{7.53}$$

If $|E_n\rangle$ is a descendent of the vacuum vector $|0\rangle$ we can go slightly further. For $|0\rangle$ itself we get:

$$\begin{aligned}\mathcal{G}_\alpha^{(n)}(\mu,\lambda)=\mathcal{G}_\alpha^{(n)}(0,0)\cdot\Big\{&1+y^2\cdot\Big(\pi^{2h_\psi-\frac{1}{2}}\,C_{\psi\psi}^{(\alpha\alpha\alpha)1}\,\frac{\Gamma(\frac{1}{2}-h_\psi)}{\Gamma(1-h_\psi)}\Big)\\ &+x\cdot\Big(-\tfrac{1}{4}\pi^{x_\phi-\frac{3}{2}}\,{}^{\alpha}B_\phi{}^{1}\,\Gamma(\tfrac{1}{2}-\tfrac{x_\phi}{2})\Gamma(\tfrac{x_\phi}{2})\Big)\\ &+y^3\cdot\Big(-\tfrac{1}{6}(2\pi)^{3h_\psi-2}\,C_{\psi\psi}^{(\alpha\alpha\alpha)\psi}\;C_{\psi\psi}^{(\alpha\alpha\alpha)1}\;\cdot I_1\Big)\\ &+xy\cdot\Big(-(2\pi)^{h_\psi+x_\phi-1}\,{}^{\alpha}B_\phi{}^{\psi}\;C_{\psi\psi}^{(\alpha\alpha\alpha)1}\;\cdot I_2\Big)\\ &+\cdots\Big\}\end{aligned} \tag{7.54}$$

where

$$\begin{aligned}I_1&=\int_0^{2\pi}\!\!d\theta\int_0^{2\pi}\!\!d\theta'\,\Big|8\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\sin\frac{\theta-\theta'}{2}\Big|^{-3h_\psi}\\ I_2&=\int_0^1\frac{dq}{q}\Big[q^{x_\phi}+q^{-x_\phi}-q^{x_\phi}(1-q)^{h_\psi-x_\phi}(1+q)^{-h_\psi-x_\phi}F\big(\tfrac{1}{2},h_\psi;1;4\tfrac{q}{(1+q)^2}\big)\Big]\end{aligned} \tag{7.55}$$

The (1,2)-conformal block

In the case where $|E_n\rangle$ is given by a primary field with Kac-labels $(1,2)$ we can make use of the fact that the chiral 4-point functions involving $\Phi_n(z,\bar{z})$ can be expressed through hypergeometric functions. This enables us to compute the terms linear in λ and quadratic in μ in the $\mathcal{G}$-function expansion.

To fix notation, in the minimal model $M(p,q)$ we take the Kac-labels (r,s) to have ranges $r=1\dots p{-}1$ and $s=1\dots q{-}1$. Define $t=p/q$ and $d_{rs}=r{-}st$. Then

the highest weight of the representation (r,s) is given by $h_{rs} = \frac{1}{4t}({d_{rs}}^2 - {d_{11}}^2)$. Let $\phi_{1,2}(z,\bar{z})$ be a primary bulk field with conformal weights $(h_{1,2}, h_{1,2})$ and $\psi_{r,s}(x)$ be a boundary field with weight $h_{r,s}$.

The coefficient of the y^2-term in the expansion of the $\mathcal{G}$-function (7.36) is

$$a_{20} = \frac{\pi}{\langle\alpha|\phi_{1,2}\rangle} \int_0^{2\pi} \langle\psi_{r,s}(e^{i\theta})\psi_{r,s}(1)\phi_{1,2}(0)\rangle^{\alpha}_{\text{disc}}\, d\theta\ . \tag{7.56}$$

The correlator can be obtained from the two chiral 4-point blocks, transformed to unit disc geometry:

$$\begin{aligned}\langle\psi_{r,s}(e^{i\theta})\psi_{r,s}(1)\phi_{1,2}(0)\rangle^{\alpha}_{\text{disc}} &= \langle 1\rangle^{\alpha}_{\text{disc}}\ {}^{\alpha}B_{\phi_{1,2}}{}^{1}\,(\cos\tfrac{\theta}{2})^{2h_{1,2}-2h_{r,s+1}} \\ &\left(C^{(\alpha\alpha\alpha)1}_{\psi_{r,s}\psi_{r,s}}\, f^{1,1}(\tan\tfrac{\theta}{2}) + \frac{{}^{\alpha}B_{\phi_{1,2}}{}^{\psi_{1,3}}}{{}^{\alpha}B_{\phi_{1,2}}{}^{1}}\, C^{(\alpha\alpha\alpha)\psi_{1,3}}_{\psi_{r,s}\psi_{r,s}}\ C^{(\alpha\alpha\alpha)1}_{\psi_{1,3}\psi_{1,3}}\, f^{1,3}(\tan\tfrac{\theta}{2})\right)\end{aligned} \tag{7.57}$$

where (see also (4.54))

$$\begin{aligned} f^{1,1}(x) &= |2x|^{-2h_{r,s}} F(\tfrac{1}{2} - \tfrac{t}{2} - \tfrac{d_{r,s}}{2}, 1 - \tfrac{t}{2} - \tfrac{d_{r,s}}{2}; \tfrac{3}{2} - t; -x^2) \\ f^{1,3}(x) &= |2x|^{h_{1,3}-2h_{r,s}} F(\tfrac{t}{2} - \tfrac{d_{r,s}}{2}, \tfrac{1}{2} + \tfrac{t}{2} - \tfrac{d_{r,s}}{2}; \tfrac{1}{2} + t; -x^2)\end{aligned} \tag{7.58}$$

Using transformations on the hypergeometric functions as well as the fact that the expression (7.57) is symmetric under the reflection $\theta \to -\theta$ we can reformulate the amplitude as:

$$\begin{aligned}\langle\psi_{r,s}(e^{i\theta})\psi_{r,s}(1)\phi_{1,2}(0)\rangle^{\alpha}_{\text{disc}} &= \langle 1\rangle^{\alpha}_{\text{disc}}\ {}^{\alpha}B_{\phi_{1,2}}{}^{1}\ C^{(\alpha\alpha\alpha)1}_{\psi_{r,s}\psi_{r,s}}\ \cdot A \\ &\cdot (2\sin\tfrac{\theta}{2})^{-2h_{r,s}} F(\tfrac{1}{2} - \tfrac{t}{2} - \tfrac{d_{r,s}}{2}, \tfrac{1}{2} - \tfrac{t}{2} + \tfrac{d_{r,s}}{2}; \tfrac{1}{2}; (\cos\tfrac{\theta}{2})^2)\end{aligned} \tag{7.59}$$

with

$$A = \frac{\Gamma(\frac{3}{2} - t)\Gamma(\frac{1}{2})}{\Gamma(1 - \frac{t}{2} + \frac{d_{r,s}}{2})\Gamma(1 - \frac{t}{2} - \frac{d_{r,s}}{2})} \cdot \left(1 - \frac{\sin\pi(\frac{1}{2} + \frac{t}{2} + \frac{d_{r,s}}{2}) \cdot \sin\pi(\frac{1}{2} + \frac{t}{2} - \frac{d_{r,s}}{2})}{\sin\pi(\frac{t}{2} + \frac{d_{r,s}}{2}) \cdot \sin\pi(\frac{t}{2} - \frac{d_{r,s}}{2})}\right) \tag{7.60}$$

Remarkably in this form only one hypergeometric function appears and all structure constants describe the coupling to the identity and thus have a relatively easy form while still leaving (7.59) independent of the normalisation of the fields.

We can now perform the integral in (7.56) and obtain

$$a_{20} = \pi\, C^{(\alpha\alpha\alpha)1}_{\psi_{r,s}\psi_{r,s}}\ \cdot A \cdot I_\theta \tag{7.61}$$

where the integral I_θ converges for $h_{r,s} < \frac{1}{2}$ and Mathematica gives the answer as:

$$\begin{aligned} I_\theta &= 2\int_0^\pi d\theta \left(2\sin\tfrac{\theta}{2}\right)^{-2h_{r,s}} F\big(h_{r,s+1}{-}h_{r,s}{-}h_{1,2}, h_{r,s-1}{-}h_{r,s}{-}h_{1,2}; \tfrac{1}{2}; (\cos\tfrac{\theta}{2})^2\big) \\ &= \sqrt{\pi}\; 2^{1-2h_{r,s}} \cdot \frac{\Gamma(\frac{1}{2}{-}h_{r,s})\Gamma(1{+}h_{r,s}{-}h_{r,s+1}{-}h_{r,s-1}{+}2h_{1,2})}{\Gamma(1{-}h_{r,s-1}{+}h_{1,2})\Gamma(1{-}h_{r,s+1}{+}h_{1,2})} \ . \end{aligned} \tag{7.62}$$

The coefficient of the x-term in the $\mathcal{G}$-function expansion follows from (7.42). For simplicity we restrict ourselves to a perturbation by the bulk field $\phi_{1,3}$. The expression given below is valid for $t < \frac{1}{3}$, this condition insures that $h_{1,4} < h_{1,2}$ (i.e. $E_{1,4} < E_{1,2}$ for the sum in (7.42)). We get

$$\begin{aligned} a_{01} = 2\pi\Big\{ \int_0^1 \frac{dq}{q}\Big[&- q^{2h_{1,3}} \frac{\langle\phi_{1,2}(0)\phi_{1,3}(q)\rangle^\alpha_{\rm disc}}{\langle\alpha|\phi_{1,2}\rangle} + \frac{\langle\alpha|\phi_{1,4}\rangle}{\langle\alpha|\phi_{1,2}\rangle} q^{2h_{1,4}-2h_{1,2}}\, C_{\phi_{1,2}\phi_{1,3}}{}^{\phi_{1,4}} \\ &+ C_{\phi_{1,2}\phi_{1,3}}{}^{\phi_{1,2}}\Big] + \frac{\langle\alpha|\phi_{1,4}\rangle}{\langle\alpha|\phi_{1,2}\rangle} \frac{1}{2(h_{1,2}{-}h_{1,4})}\, C_{\phi_{1,2}\phi_{1,3}}{}^{\phi_{1,4}} \Big\} \end{aligned} \tag{7.63}$$

where $\phi_{1,2}$ is the primary bulk field representing the unperturbed energy eigenstate and $\phi_{1,3}$ is the perturbing bulk field. The correlator involving two bulk fields is given by

$$\begin{aligned} \langle\phi_{1,2}(0)\phi_{r,s}(r)\rangle^\alpha_{\rm disc} &= \langle 1\rangle^\alpha_{\rm disc}\, {}^\alpha B_{\phi_{1,2}}{}^1\, (1-r^2)^{-2h_{r,s}+2h_{1,2}} \\ &\left(\frac{{}^\alpha B_{\phi_{r,s+1}}{}^1}{{}^\alpha B_{\phi_{1,2}}{}^1}\, C_{\phi_{r,s}\phi_{1,2}}{}^{\phi_{r,s+1}}\, f^+(r^2) + \frac{{}^\alpha B_{\phi_{r,s-1}}{}^1}{{}^\alpha B_{\phi_{1,2}}{}^1}\, C_{\phi_{r,s}\phi_{1,2}}{}^{\phi_{r,s-1}}\, f^-(r^2)\right) \end{aligned} \tag{7.64}$$

where

$$\begin{aligned} f^+(x) &= x^{h_{r,s+1}-h_{r,s}-h_{1,2}}\,(1-x)^{t/2} F(t, t-d_{r,s}; 1-d_{r,s}; x) \\ f^-(x) &= x^{h_{r,s-1}-h_{r,s}-h_{1,2}}\,(1-x)^{t/2} F(t, t+d_{r,s}; 1+d_{r,s}; x)\ . \end{aligned} \tag{7.65}$$

We get:

$$a_{01} = 2\pi\Big\{ C_{\phi_{1,2}\phi_{1,3}}{}^{\phi_{1,2}} \cdot I^- + \frac{{}^\alpha B_{\phi_{1,4}}{}^1}{{}^\alpha B_{\phi_{1,2}}{}^1}\, C_{\phi_{1,2}\phi_{1,3}}{}^{\phi_{1,4}} \cdot I^+ \Big\} \tag{7.66}$$

where the integrals I^- and I^+ are given by Mathematica as

$$\begin{aligned} I^- &= \tfrac{1}{2}\int_0^1 x^{-1}\big(1-(1-x)^{1-2t}F(t,1{-}2t;2{-}3t;x)\big)dx \\ &= \frac{t}{4(t{-}1)(3t{-}2)}\cdot {}_3F_2(1,2{-}2t,1{+}t;3{-}3t,3{-}2t;1) \\ &\quad + \frac{2-5t+3t^2}{2(t{-}1)(3t{-}2)}\big(\psi(2{-}4t)-\psi(2{-}3t)+\psi(2{-}2t)-\psi(1)\big) \\ I^+ &= \tfrac{1}{2}\int_0^1 x^{3t-2}\big(1-(1-x)^{1-2t}F(t,4t{-}1;3t;x)\big)dx + \frac{1}{2(h_{1,2}{-}h_{1,4})} \\ &= -\pi^{-1}\cos(\pi t)\sin(2\pi t)\Gamma(2{-}4t)\Gamma(1{-}2t)\Gamma(3t)\Gamma(3t{-}1) \end{aligned} \tag{7.67}$$

Here $\psi(z)=\partial/\partial z\ \ln\Gamma(z)$ denotes the digamma function.

7.4 Thermodynamic Bethe ansatz

We will describe the thermodynamic Bethe ansatz (TBA) only very briefly, as it is outside of the main line of argument in this thesis. The TBA equations for the g–functions were proposed in [LMSS95] and are put to test in the next section by comparing them to TCSA data for the Lee-Yang model. In essence, the finding is that the proposed TBA method fails to extract the correct g–function in the case of massive perturbations. In this section we want to recall some of the assumptions that enter the construction to see where the reasons for this problem may lie.

The starting point in deriving the TBA method is the partition function of a collection of particles moving on a line segment of length R at temperature $1/L$. We restrict ourselves to the case with only a single kind of particle of mass m. Their energy and momentum are parametrised by the rapidity θ: $E{=}m\cosh\theta$ and $p{=}m\sinh\theta$.

The crucial ingredient into TBA is the assumption that there are enough conserved charges to make the scattering integrable, that is all processes can be decomposed into two particle processes (see e.g. [Dor98] for an introduction to integrable scattering). In particular this means that while all perturbations of CFT are accessible via TCSA, only some of them have enough conserved charges for TBA to be applicable, typically those by a primary field in the list $\{\phi_{1,2},\phi_{1,3},\phi_{1,5},\phi_{2,1},\phi_{3,1},\phi_{5,1}\}$ which is relevant in a given minimal model [FFr93].

The partition function is computed as the sum over all allowed particle configurations, weighted by e^{-LE}. The condition whether a configuration is allowed is that the wave function for each particle closes up to a multiple of 2π when the particle returns to its original position. On its way the i'th particle will pick up

phases $S(\theta_i - \theta_j)$ from scattering with other particles and $R_\alpha(\theta_i)$ when bouncing off the boundary.

For a finite segment of length R, the energy spectrum of allowed configurations is discrete. As we increase R, the spectrum condenses into a continuum of states and it is a good approximation to replace a configuration by its particle density $\rho^r(\theta)$. We can also define the density of possible particles states $\rho(\theta)$. More precisely, for a given particle configuration with density $\rho^r(\theta)$, there are $R\,\rho(\theta_0)\,\Delta\theta$ rapidities in the interval $[\theta_0, \theta_0 + \Delta\theta]$ at which a new particle could be added to the system without violating the self-consistency condition. In terms of these densities, the requirement that the wave functions of all particles close translates into the condition[1]

$$2\pi\rho(\theta) = -2\pi\delta(\theta) + 2MR\cosh\theta + \phi_\alpha(\theta) + \phi_\beta(\theta) - 2\phi(2\theta) + \int_{-\infty}^{\infty} \rho^r(\theta')\phi(\theta-\theta')d\theta' \,,$$
$$\phi(\theta) = -i\frac{d}{d\theta}\log S(\theta) \,, \quad \phi_\alpha(\theta) = -i\frac{d}{d\theta}\log R_\alpha(\theta) \,. \tag{7.68}$$

Assuming that the number of allowed states with effective particle density $\rho^r(\theta)$ is $\mathcal{N}[\rho^r(\theta)]$, one can write the partition function as an integral

$$Z = \sum_{\text{states}} e^{-L\,E} = \int \mathcal{D}[\rho^r(\theta)]\,\mathcal{N}[\rho^r(\theta)]\,e^{-L\,E[\rho^r(\theta)]} \,, \tag{7.69}$$

where the energy of the configuration is $E = \int_0^\infty (M\cosh\theta)\,\rho^r(\theta)\,d\theta$. In the case of the Lee-Yang model (and in fact almost all known cases), to calculate $\mathcal{N}[\rho^r(\theta)]$ we can think of the particles as fermions in the sense that each allowed state can be occupied only by one particle. Consequently we take the number of allowed configurations in the interval $\Delta\theta$ with a fixed number of occupied states $R\,\rho^r(\theta)\,\Delta\theta$ to be

$$\frac{(R\rho(\theta)\Delta\theta)!}{(R\rho^r(\theta)\Delta\theta)!\,(R\rho(\theta)\Delta\theta - R\rho^r(\theta)\Delta\theta)!} \,, \tag{7.70}$$

and replace the factorials by the two leading terms $\log\Gamma(z) \sim z\log z - z + \ldots$ of Stirling's formula, so that the total number of configurations with a given effective

[1] The term $-2\pi\delta(\theta)$ was introduced in [LMSS95] to exclude the state with rapidity zero. Nepomechie observed [prN] that when keeping the subleading term in approximating sums by integrals (Euler-Maclaurin formula), the δ-term should be replaced by $-\phi(\theta)$. Correspondingly in (7.75) he obtains $-\phi(\theta)/2$ instead of $-\pi\delta(\theta)$. But initial checks indicate that even with this modification the problem with the TBA g-functions for massive flows remains. The precise implications of the new term have to be investigated in future work.

density $\rho^r(\theta)$ is

$$\mathcal{N}[\rho^r(\theta)] \sim \exp\Big[\int_0^\infty \big(\rho\log\rho - \rho^r\log\rho^r - (\rho-\rho^r)\log(\rho-\rho^r)\big)d\theta\Big] . \qquad (7.71)$$

The next step, the thermodynamic limit $R\to\infty$, is the main approximation in the TBA method. This consists of replacing the infinite sum over all configurations in (7.69) by just one term, i.e. the configuration with the largest contribution $\mathcal{N}[\rho^r(\theta)]\, e^{-L\,E[\rho^r(\theta)]}$. Physically this is justified since in the thermodynamic limit all configurations but the one with the minimal free energy have probability zero. Mathematically we are approximating Z in the limit $R\to\infty$ by the saddle point method, giving the leading behaviour

$$\log Z \sim -RE_0^{\rm circ}(L) + \log(g_\alpha(L)\, g_\beta(L)) , \qquad (7.72)$$

where (extending the range of θ by symmetry where necessary and setting $\varepsilon = \log(\rho/\rho^r - 1)$)

$$\begin{aligned} E_0^{\rm circ}(L) &= \int_{-\infty}^{\infty} m\cosh\theta\, L(\theta)\,\tfrac{d\theta}{2\pi} \\ \log(g_\alpha(L)\, g_\beta(L)) &= \int_{-\infty}^{\infty} \Big(\phi_\alpha(\theta)+\phi_\beta(\theta)-2\phi(2\theta)-2\pi\delta(\theta)\Big)\, L(\theta)\,\tfrac{d\theta}{4\pi} \end{aligned} \qquad (7.73)$$

The quantities $L(\theta) = \log(1+e^{-\varepsilon(\theta)})$ and $\varepsilon(\theta)$ solve the equation

$$\varepsilon(\theta) = mL\cosh\theta - \int_{-\infty}^{\infty}\phi(\theta-\theta')\, L(\theta')\,\tfrac{d\theta'}{2\pi} . \qquad (7.74)$$

The g–functions are then identified as

$$\log(g_\alpha(L)) \;=\; \int_{-\infty}^{\infty}\Big(\,\phi_\alpha(\theta) \;-\; \phi(2\theta) \;-\; \pi\delta(\theta)\,\Big)\, L(\theta)\,\tfrac{d\theta}{4\pi} . \qquad (7.75)$$

Notice that the g–functions are given by the first subleading term in (7.72). In the thermodynamic limit the approximations made become exact and hence the leading term in (7.72) is exact as well. With this approach it is however not clear why the subleading terms should contain any information about the original system at all, especially in the light of the saddle point approximation, where an infinity of terms has been neglected against the leading one. From this point of view it is more surprising that the g–functions found by the TBA-method agree, in a way made precise in the next section, very well with the TCSA data.

7.5 The Lee-Yang model

As a concrete example we will consider the Lee-Yang model. This sections lists some of the findings in [DRTW99]. The technical details in obtaining the TCSA data have been left out, along with all the subtleties involved in making the TBA approach work. We will concentrate on CPT and TCSA and merely state the findings of the TBA–investigations.

The Lee-Yang model is the simplest non-unitary conformal field theory, $M_{2,5}$, and has central charge $-22/5$. There are only two representations of the Virasoro algebra of interest, of weight 0 and $-1/5$, which we will denote $\mathbb{1}$ and ϕ, respectively. The fusion rules are

$$\mathbb{1} \times \mathbb{1} = \mathbb{1} \,, \quad \mathbb{1} \times \phi = \phi \,, \quad \phi \times \phi = \mathbb{1} + \phi \,. \tag{7.76}$$

The modular S-matrix is given by

$$S = \begin{pmatrix} S_{11} & S_{1\phi} \\ S_{\phi 1} & S_{\phi\phi} \end{pmatrix} = \tfrac{2}{\sqrt{5}} \begin{pmatrix} -\sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ \sin\frac{\pi}{5} & \sin\frac{2\pi}{5} \end{pmatrix} = \begin{pmatrix} -0.8506.. & 0.5257.. \\ 0.5257.. & 0.8506.. \end{pmatrix} \,. \tag{7.77}$$

The Lee-Yang model is an A–series theory and so the boundary field content is given by the fusion rules. There are two pure conformal boundary conditions, which we label $\mathbb{1}$ and Φ, and three primary boundary fields other than the identity, and all have weight h=1/5. Two of these interpolate the two different conformal boundary conditions and one lives on the Φ boundary

$$\psi \equiv \psi^{(\Phi\mathbb{1})}_{-1/5} \,, \quad \psi^\dagger \equiv \psi^{(\mathbb{1}\Phi)}_{-1/5} \,, \quad \phi \equiv \psi^{(\Phi\Phi)}_{-1/5} \,. \tag{7.78}$$

The bulk theory contains two primary fields, the identity $\mathbb{1}$ of weight 0, and φ of weight $x_\varphi = -2/5$.

Structure constants

We give here the structure constants appearing in all the operator products of interest, that is the bulk OPE

$$\varphi(z,\bar z)\,\varphi(w,\bar w) = C_{\varphi\varphi}{}^{\mathbb{1}}\,|z-w|^{4/5} \;+\; C_{\varphi\varphi}{}^{\varphi}\,|z-w|^{2/5}\,\varphi(w,\bar w) \;+\; \dots \,,$$

the boundary OPEs,

$$\begin{aligned}\phi(z)\,\phi(w) &= C_{\phi\phi}{}^{\mathbf{1}}\,|z-w|^{2/5} \;+\; C_{\phi\phi}{}^{\phi}\,|z-w|^{1/5}\,\phi(w) \;+\; \dots\;,\\ \phi(z)\,\psi(w) &= C_{\phi\psi}{}^{\psi}\,|z-w|^{1/5}\,\psi(w) \;+\; \dots\;,\end{aligned}$$

and the two bulk–boundary OPEs

$$\begin{aligned}\varphi(z)\,\big|_{\mathbf{1}} &= {}^{\mathbf{1}}B_{\varphi}{}^{\mathbf{1}}\,|2(z-w)|^{2/5} \;+\; \dots\;,\\ \varphi(z)\,\big|_{\Phi} &= {}^{\Phi}B_{\varphi}{}^{\mathbf{1}}\,|2(z-w)|^{2/5} \;+\; {}^{\Phi}B_{\varphi}{}^{\phi}\,|2(z-w)|^{1/5}\,\phi(w) \;+\; \dots\;.\end{aligned}$$

We want all these structure constants to be real, and a suitable[2] choice is

$$\begin{aligned}
&C_{\varphi\varphi}{}^{\mathbf{1}} = C_{\phi\phi}{}^{\mathbf{1}} = -1\;,\\
&C_{\varphi\varphi}{}^{\varphi} = -\left|\tfrac{2}{1+\sqrt5}\right|^{1/2}\cdot\alpha^2 = -1.91131..\;, \qquad {}^{\mathbf{1}}B_{\varphi}{}^{\mathbf{1}} = -\left|\tfrac{2}{1+\sqrt5}\right|^{1/2} = -0.78615..\;,\\
&C_{\phi\phi}{}^{\phi} = -\left|\tfrac{1+\sqrt5}{2}\right|^{1/2}\cdot\alpha = -1.98338..\;, \qquad {}^{\Phi}B_{\varphi}{}^{\mathbf{1}} = \left|\tfrac{1+\sqrt5}{2}\right|^{3/2} = 2.05817..\;,\\
&C_{\phi\psi}{}^{\psi} = -\left|\tfrac{2}{1+\sqrt5}\right|^{1/2}\cdot\alpha = -1.22579..\;, \qquad {}^{\Phi}B_{\varphi}{}^{\phi} = \left|\tfrac{5+\sqrt5}{2}\right|^{1/2}\cdot\alpha = 2.96585..\;,
\end{aligned}$$

$$\alpha = \left|\tfrac{\Gamma(1/5)\,\Gamma(6/5)}{\Gamma(3/5)\,\Gamma(4/5)}\right|^{1/2}\;. \tag{7.79}$$

The state of lowest conformal dimension in the bulk theory is

$$|\Omega\rangle = |\varphi\rangle\;, \tag{7.80}$$

and we choose to normalise the bulk highest-weight states as

$$\langle 0|0\rangle = -1 \quad\Rightarrow\quad \langle\varphi|\varphi\rangle = 1\;. \tag{7.81}$$

With this, and demanding $g_\alpha^\varphi > 0$, the coefficients appearing in the boundary states are

$$\begin{aligned}
g_{\mathbf{1}}^{0} &= \left|\tfrac{\sqrt5+1}{2\sqrt5}\right|^{1/4}\;, & g_{\mathbf{1}}^{\varphi} &= \left|\tfrac{\sqrt5-1}{2\sqrt5}\right|^{1/4}\\
g_{\Phi}^{0} &= -\left|\tfrac{\sqrt5-2}{\sqrt5}\right|^{1/4}\;, & g_{\Phi}^{\varphi} &= \left|\tfrac{2+\sqrt5}{\sqrt5}\right|^{1/4}
\end{aligned} \tag{7.82}$$

[2] It is convenient for calculations to have the 2-point structure constants equal to ±1. This normalisation is related to the one chosen in chapter 6 by a rescaling with a real factor. The choice in chapter 6 has the benefit that the general formulas take an easier form and are free of square roots.

The zero-point functions are then given as

$$Z_{\mathbb{1}} = \langle 1\rangle^{\mathbb{1}}_{\text{disc}} = \langle \mathbb{1}|0\rangle = -g^0_{\mathbb{1}} \,, \quad Z_\Phi = \langle 1\rangle^{\Phi}_{\text{disc}} = \langle \Phi|0\rangle = -g^0_\Phi \,. \tag{7.83}$$

Finally, the g-functions are given by $g_\alpha = g^\Phi_\alpha \sqrt{\langle\varphi|\varphi\rangle} = g^\Phi_\alpha$, i.e.

$$\begin{aligned} \log g_{\mathbb{1}} &= \tfrac{1}{4}\log\left|\tfrac{\sqrt{5}-1}{2\sqrt{5}}\right| = -\tfrac{1}{4}\log\left|\tfrac{1+\sqrt{5}}{2}\right| - \tfrac{1}{8}\log 5 = -0.321482..\,, \\ \log g_\Phi &= \tfrac{1}{4}\log\left|\tfrac{2+\sqrt{5}}{\sqrt{5}}\right| = \tfrac{3}{4}\log\left|\tfrac{1+\sqrt{5}}{2}\right| - \tfrac{1}{8}\log 5 = \quad 0.159729..\,. \end{aligned} \tag{7.84}$$

Correlation functions

The unit disc correlation functions resulting from chiral three–point functions were already listed in (7.45). We will also need correlators given in terms of chiral four point functions. They are given in (7.59), (7.64) and specialised to the Lee-Yang model they read:

$$\begin{aligned} \langle\phi(e^{i\theta})\phi(1)\varphi(0)\rangle^\Phi_{\text{disc}} &= \langle 1\rangle^\Phi_{\text{disc}}\, {}^\Phi B_\varphi{}^1\, C_{\phi\phi}{}^1\, \tfrac{\sqrt{\pi}(1-\sqrt{5})\Gamma(\frac{11}{10})}{\Gamma(\frac{7}{10})\Gamma(\frac{9}{10})} \cdot (2\sin\tfrac{\theta}{2})^{\frac{2}{5}} \cdot F\big(\tfrac{1}{5},\tfrac{2}{5};\,\tfrac{1}{2};\,(\cos\tfrac{\theta}{2})^2\big) \\ \langle\varphi(0)\varphi(r)\rangle^\alpha_{\text{disc}} &= \langle 1\rangle^\alpha_{\text{disc}} \cdot r^{\frac{2}{5}}(1-r^2)^{\frac{1}{5}} \cdot \big\{\, {}^\alpha B_\varphi{}^1\, C_{\varphi\varphi}{}^\varphi \cdot F\big(\tfrac{1}{5},\,\tfrac{2}{5}\,;\,\tfrac{4}{5}\,;\,r^2\big) \\ &\qquad + C_{\varphi\varphi}{}^1 \cdot r^{\frac{2}{5}} \cdot F\big(\tfrac{2}{5},\,\tfrac{3}{5}\,;\,\tfrac{6}{5}\,;\,r^2\big)\big\} \end{aligned} \tag{7.85}$$

$\mathcal{G}$–functions and conformal perturbation theory

We will consider the Lee-Yang model on a cylinder of length R and circumference L. The perturbed action is given by

$$\mathcal{A}_{\text{pert}} = \mathcal{A}_{\text{CFT}} + \lambda \cdot \int \varphi(z,\bar z) d^2z + h \cdot \int \phi(x) dx \,. \tag{7.86}$$

We take the left boundary condition to be fixed to $\mathbb{1}$ and the right boundary to be either $\mathbb{1}$, in which case there is no relevant boundary perturbation and (7.86) only makes sense for h=0, or $\Phi(h)$, i.e. the conformal Φ–boundary perturbed by $h\cdot\int\phi$.

With CPT we can calculate the perturbative expansion of $\log\mathcal{G}_\alpha$, the ground–

state $\mathcal{G}$–function as defined in (7.28). Define the coefficients d_n and c_{mn} via

$$\begin{aligned}\log \mathcal{G}_{\mathbf{1}}(\lambda, L) &= \sum_{n=0}^{\infty} d_n \, (\lambda L^{12/5})^n = \log g_{\mathbf{1}} \; + \; d_1 \, (\lambda L^{12/5}) \; + \; \dots \; , \\ \log \mathcal{G}_{\Phi(h)}(\lambda, L) &= \sum_{m,n=0}^{\infty} c_{mn} \, (hL^{6/5})^m (\lambda L^{12/5})^n \\ &= \log g_{\Phi} \; + \; c_{10} \, (hL^{6/5}) \; + \; c_{20} \, (hL^{6/5})^2 \; + \; c_{01} \, (\lambda L^{12/5}) \; + \; \dots \end{aligned} \tag{7.87}$$

The connection to the constants a_{mn} in the expansion (7.36) is

$$\begin{aligned} c_{01} &= (2\pi)^{x_\phi - 2} \cdot a_{01} \; , \quad c_{10} = (2\pi)^{h_\psi - 1} \cdot a_{10} \; , \\ c_{20} &= (2\pi)^{2h_\psi - 2} \cdot \big(a_{20} - \tfrac{1}{2} (a_{10})^2 \big) \; . \end{aligned} \tag{7.88}$$

With the expressions (7.52), (7.61) and (7.66) we find

$$\begin{aligned} d_1 &= (2\pi)^{-7/5} \big\{ \, C_{\varphi\varphi}{}^{\varphi} \cdot I^- - \tfrac{5^{1/4}(\sqrt{5}-1)}{8\pi} \Gamma(\tfrac{1}{5})^2 \Gamma(\tfrac{2}{5}) \Gamma(\tfrac{6}{5}) \big\} = -0.253117.. \; , \\ c_{01} &= (2\pi)^{-7/5} \Big\{ \, C_{\varphi\varphi}{}^{\varphi} \cdot I^- + \tfrac{5^{1/4}}{4\pi} \big(\tfrac{\sqrt{5}-1}{\sqrt{5}+1} \big)^{3/2} \Gamma(\tfrac{1}{5})^2 \Gamma(\tfrac{2}{5}) \Gamma(\tfrac{6}{5}) \Big\} = 0.0797648.. \; , \\ c_{10} &= (2\pi)^{-1/5} 5^{1/4} \tfrac{2}{1+\sqrt{5}} \cdot \alpha = 0.99777.. \; , \\ c_{20} &= \big(\tfrac{1}{\sqrt{5}} - \tfrac{1}{2} \big) (c_{10})^2 = -0.0525515.. \end{aligned} \tag{7.89}$$

where

$$I^- = \tfrac{5}{24} \cdot {}_3F_2\big(1, \tfrac{6}{5}, \tfrac{7}{5}; \tfrac{9}{5}, \tfrac{11}{5}; 1\big) + \tfrac{1}{2} \big(\psi(\tfrac{2}{5}) - \psi(\tfrac{4}{5}) + \psi(\tfrac{6}{5}) - \psi(1) \big) = 0.0839379.. \tag{7.90}$$

The coefficients c_{10} and c_{20} have also been calculated using the free-field construction in [BLZ96], with the same results as given above.

$\mathcal{G}$–functions in TCSA

The aim of this section is to provide numerical data to verify the coefficients obtained above and data from the TBA method, to be discussed in the next section. Note that the coefficients d_1, c_{01}, c_{10}, c_{20} can be checked by perturbing only the bulk or only the boundary. So we will restrict the TCSA investigation to these two cases, which will simplify the treatment. The first coefficient that would require a simultaneous perturbation of bulk and boundary is c_{11}.

We can use the TCSA method to compute the partition function $Z(R, L, \lambda, h)$ of a cylinder of length R and circumference L. To do so take the (perturbed) Hamiltonian $H(R, \lambda, h)$ to run from side to side on the cylinder. The perturbed

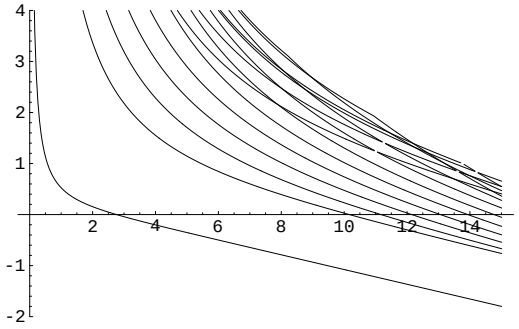


Fig. 7.3a: The first 15 eigenvalues of H/M for the $(\mathbf{1},\mathbf{1})$ system plotted against r from TCSA with 29 states.

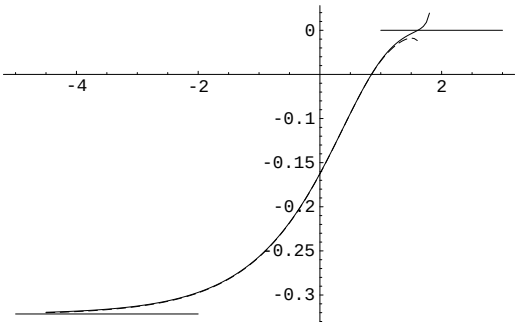


Fig. 7.3b: $\log g_{\mathbf{1}}(\ell)$ vs. $\log \ell$ from TCSA with 19 (dashed line) and 106 (solid line) states. Also shown are the exact UV and IR values.

spectrum is computed as described in section 7.2 and the partition function can be approximated by taking the sum over all TCSA energy levels E_i^{TCSA} at the given truncation level

$$Z(R,L,\lambda,h) = \sum_i \exp(-LE_i^{\mathrm{TCSA}}) \ . \tag{7.91}$$

From (7.1) we expect the large R asymptotics to be of the form

$$\begin{aligned} &Z(R,L,\lambda,h) \sim -a(L,\lambda)\cdot R + b(L,\lambda,h) \ , \\ &a(L,\lambda) = E_0^{cyl} \ , \quad b(L,\lambda,h) = \log \mathcal{G}_{\mathbf{1}}(L,\lambda) + \log \mathcal{G}_{\Phi(h)}(L,\lambda) \ . \end{aligned} \tag{7.92}$$

So to obtain a and b from TCSA we compute the partition function, go to large R and check if Z displays the behaviour (7.92). One has to be careful though not to make R too large, otherwise the truncation effects from TCSA render the data meaningless.

In a next step we have to separate the supposedly universal part of $\mathcal{G}_\alpha$ from the boundary free energy as in (7.4), that is we try to find a parameter region in which the numerical estimate of $\log \mathcal{G}_\alpha(L,\lambda)$ behaves as

$$\log \mathcal{G}_\alpha(L,\lambda) \sim -Lf_\alpha(\lambda) + \log g_\alpha(L,\lambda) \ . \tag{7.93}$$

It is the quantity $\log g_\alpha(L,\lambda)$ which we later compare to the TBA results.

Bulk perturbation – massive case

Here we consider a pure bulk perturbation $\lambda\neq 0$, $h=0$ in (7.86). The perturbed spectrum of the Hamiltonian develops a mass gap. The mass scale is set by the mass

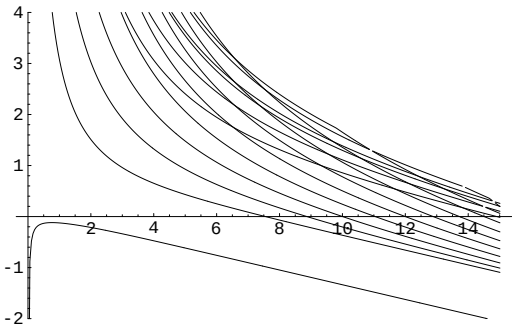


Fig. 7.4a: The first 15 eigenvalues of H/M for the $(\mathbb{1}, \Phi(0))$ system plotted against r from TCSA with 81 states.

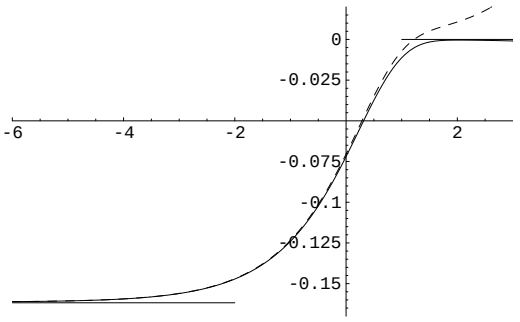


Fig. 7.4b: $\log(g_{\mathbb{1}}(\ell)\, g_{\Phi(0)}(\ell))$ vs. $\log \ell$ from the TCSA with 140 states using two different extrapolation methods[3] (dashed and solid line). Also shown are the exact UV and IR values.

M of the single massive particle of the scaling Lee-Yang model, which is related to λ by [Zam90, Zam95]

$$M = \kappa \lambda^{5/12} \,, \quad \kappa = 2^{19/12}\sqrt{\pi}\, \frac{\left(\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})\right)^{5/12}}{5^{5/16}\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})} = 2.642944.. \,, \tag{7.94}$$

It is convenient to reexpress (7.92) in terms of the dimensionless quantities

$$r = M\, R \,, \quad \ell = M\, L \,, \tag{7.95}$$

and to normalise the energy s.t. the mass gap is one, i.e. to look at the ratio E/M.

In fig. 7.3 and fig. 7.4 the plots of the energy spectra and the resulting flow of the g–function is shown for the strip with $(\mathbb{1}, \mathbb{1})$ and $(\mathbb{1}, \Phi(0))$ boundary conditions. Since the perturbed model is massive (and has a unique ground state) the infrared value of the groundstate degeneracy has to be one: $\log g|_{\rm IR}$=0. The starting point of the flow, i.e. $\log g|_{\rm UV}$, is the value computed within CFT, given in (7.84). This is seen in fig. 7.3b and fig. 7.4b[3].

One can make fits to the TCSA estimates of the $\mathcal{G}$–functions to extract the coefficients of the power series expansion, computed via CPT in (7.89). We see that the numerics agrees very well with the theoretical predictions:

[3] The two extrapolation methods refer to the extensive free energy term in (7.4). For the solid line it has been estimated from the TCSA data and for the dashed line the exact value, which can be obtained from TBA, has been substituted.

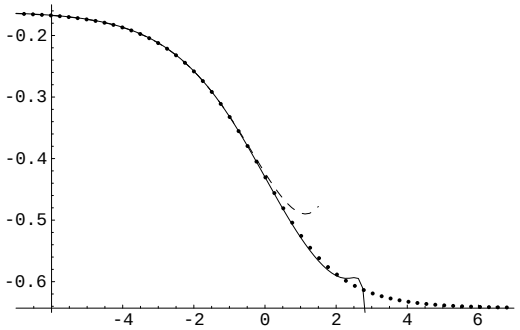


Figure 7.5: Graphs of $\log(g_\Phi(\tilde{h}))$ vs. $\log(\tilde{h})$ from the TCSA with 140 states using two different extrapolation methods[3] (dashed and solid line) together with the TBA results (points).

	TCSA (106 states)	exact
d_0	$-0.3215..$	$-0.321482..$
d_1	$-0.2526..$	$-0.253115..$
d_2	$0.0775..$	—

	TCSA (140 states)	exact
$d_0 + c_{00}$	$-0.1613..$	$-0.161753..$
$d_1 + c_{01}$	$-0.1746..$	$-0.173352..$
$d_2 + c_{02}$	$0.0667..$	—

Boundary perturbation – massless case

In the massless case the bulk mass M cannot be used to form dimensionless quantities. Instead we consider $Z(R, L, 0, h)$ as a function of

$$\tilde{r} = R/L\ , \quad \tilde{h} = h\, L^{6/5}\ . \tag{7.96}$$

As described in detail in [DPTW97] the $(\mathbb{1}, \Phi(h))$ boundary condition flows from $(\mathbb{1}, \Phi(0))$ in the UV to $(\mathbb{1}, \mathbb{1})$ in the IR. This is what we expect intuitively since there are fewer fields on the $\mathbb{1}$–boundary than on the Φ–boundary, so that the perturbation effectively removes some degrees of freedom. Whereas for the bulk this line of thought can be made rigorous for unitary models, in the form of Zamolodchikov's c–theorem [Zam86], in the present case it has to be treated with care. First of all the Lee-Yang model is non–unitary and secondly no "g–theorem" exists for general boundary flows. In fig. 7.5 we see how $g(\tilde{h})$ drops along the flow from it UV value $\log g_{\mathbb{1}} + \log g_\Phi = -0.1617..$ to its IR value $2\log g_{\mathbb{1}} = -0.6415...$

Again we can compare the TCSA estimates for the quantities in (7.89). We see that the quality of the fit drops off quickly as the order increases:

	TCSA (140 states)	exact
c_{00}	$-0.1617..$	$-0.161753..$
c_{10}	$0.9977..$	$0.997728..$
c_{20}	$-0.0450..$	$-0.052551..$
c_{30}	$0.0084..$	—

Comparison to TBA

As already mentioned we will not enter into the details of the TBA computation. The methods leading to the results are described in great detail in a series of papers [DTa96, DTa97, DPTW97, DRTW99]. Complete agreement was found between TBA and CFT results for quantities that can be extracted from the leading order of the partition function in the thermodynamic limit. In particular the energy spectrum along the flow agrees in both methods.

As outlined briefly in section 7.4, the g–functions are extraced from a subleading contribution in the thermodynamic limit. TBA predicts, for massless and massive flows in the LY-model alike:

$$\log(\, g_\alpha \,)\Big|_{\rm IR} = 0 \;, \qquad \log(\, g_\alpha \,)\Big|_{\rm UV} = \tfrac{n}{4}\, \log(\tfrac{1+\sqrt{5}}{2}) \;, \quad n \in \mathbb{Z} \;. \tag{7.97}$$

The parameter n refers to the choice of integration contour in the TBA method.

It was already suspected in (7.1) that the g–functions themselves are not universal, but only their differences. There are two distinct physical situations: massless bulk and massive bulk.

1. **Massless bulk:** There is a single identifiable massless flow with $\tilde{h}{>}0$, from Φ to $\mathbb{1}$, for which (7.84) gives

$$\log(g_\Phi) \;-\; \log(g_{\mathbb{1}}) = \log(\tfrac{1+\sqrt{5}}{2}) \;, \tag{7.98}$$

   and which is in agreement with (7.97) for $n{=}4$. In fact we can plot the suitably shifted massless TBA flow over the corresponding TCSA data, as in fig. 7.5. We see that in the massless case the TBA equations (7.75) describe the difference of g–functions correctly.

2. **Massive bulk:** For the massive flows from $\mathbb{1}$ and from $\Phi(h)$, $g|_{UV}$ is given by conformal field theory, and our expectation that for the massive theory

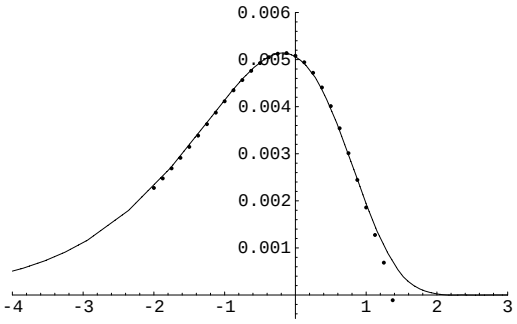


Figure 7.6: The difference $\log(g_{\Phi(h_1)})-\log(g_{\Phi(h_2)})$ vs. $\log l$ for the massive LY model[4]. The solid line is the TBA result and the points are from the TCSA to 117 states.

$g|_{IR} = 1$ leads to the results

$$\begin{aligned} \log(g_{\mathbf{1}})\Big|_{\rm UV} - \log(g)\Big|_{\rm IR} &= -\tfrac{1}{4}\log\left|\tfrac{1+\sqrt{5}}{2}\right| - \tfrac{1}{8}\log 5\,, \\ \log(g_{\Phi(h)})\Big|_{\rm UV} - \log(g)\Big|_{\rm IR} &= \tfrac{3}{4}\log\left|\tfrac{1+\sqrt{5}}{2}\right| - \tfrac{1}{8}\log 5\,. \end{aligned} \tag{7.99}$$

These cannot be reproduced by (7.97). Thus we see that the TBA equations (7.75) do not agree with the TCSA even at the level of the overall change in the g–functions without investigating finer detail. We note however that the *ratio* $\log(g_\Phi(h)/g_{\mathbf{1}})$ changes by $\log(\frac{1}{2}(\sqrt{5}+1))$ along a massive flow, in agreement with (7.97). So the TBA equations do have the possibility to describe the ratio correctly. The plot in fig. 7.6[4] confirms this by showing that the ratio of two g–functions at different values of h along the massive flow agrees with TCSA.

We learn that the subleading terms of the thermodynamic limit in the TBA expressions do still carry physical information. The evidence shown supports the idea that the g–functions extracted from TBA do no longer carry all the universal information present in the "original" g–functions. To accomodate massive and massless flows alike we can only consider differences of *ratios* of g–functions, for example the quantity

$$\log\big(\frac{g_{\Phi(h)}(\lambda)}{g_{\mathbf{1}}(\lambda)}\big) - \log\big(\frac{g_{\Phi(0)}(0)}{g_{\mathbf{1}}(0)}\big) \tag{7.100}$$

These should then give the same answer, irrespective of whether they are computed with TCSA or TBA.

[4] In the TBA equations the boundary condition is described by a parameter b. It is related to h via $h = -|h_c|\sin\big(\pi(b+\frac{1}{2})/5\big)M^{6/5}$ for some constant h_c (see [DRTW99]). The plot is for $h_1 = h(b=-1.1), h_2 = h(b=-1.2)$.

Chapter 8.

Outlook

The research presented in this thesis can be the starting point for several directions of future work. Some interesting continuations are presented in the following.

In chapter 6 we found the bulk and boundary structure constants. It became apparent that once the boundary field content and structure constants were known, all structure constants involving bulk fields could be read off directly from the sewing constraints. If one had tried to go the opposite way, typically the sewing constraints would relate squares of structure constants, e.g. (5.27) relates C^2 to C^2 and (5.25) relates B^2 to C, so that there are sign ambiguities at each step. It would be interesting to apply the methods outlined in chapters 4, 5 together with the idea of progressing from the boundary to the bulk to CFTs other than minimal models, e.g. to $SU(2)$ and $SU(3)$ WZW models.

With the boundary TCSA program presented in chapter 7 it is possible to predict the endpoints of all integrable and non-integrable boundary flows. By investigating enough models, one could arrive at a conjecture for how the various boundary conditions of minimal models are connected by the renormalisation group, i.e. a "phase diagram" for boundary conditions. If one wants to investigate a physical system at its critical point, this phase diagram will help to identify boundary conditions of the system with their CFT analogue. The phase diagram can also guide the construction of a proof of what the IR endpoints of perturbed boundary conditions are.

With the help of the minimal model structure constants one can investigate boundary flows in models with central charge close to one. Some of these then become perturbative, i.e. the endpoint moves arbitrarily close to the starting point as $c \to 1$ [RRS00]. Alternatively one can try to define a $c=1$ theory as a limit of minimal models for $c \to 1$. Whereas the bulk fields acquire infinite multiplicity in this limit, the primary boundary field content remains finite. One can now investigate non-perturbative flows at $c=1$ and conjecture that the corresponding flows for minimal models with c close to one are just small "perturbations" of these flows [GRWip].

A class of 1+1 dimensional integrable quantum field theories are the affine Toda

field theories. These are defined in terms of particles moving on a line and whose S-matrix factorises into two-particle S-matrices. If one introduces a boundary, i.e. confines the particles to the half line, the main problem consists in finding all integrable boundary conditions and the associated reflection factors for the particles in the theory (see e.g. [Cor96, DGa99]). It would be very useful to understand which conformal boundary condition corresponds to the UV-limit of a given ATFT boundary condition. To investigate this, one could use methods similar to those described in the case of the Lee-Yang model in section 7.5. As a result one will also gain some information on whether the complete set of integrable boundary conditions for the ATFT has been found.

Boundary CFT is also very useful in the investigation of D-branes in string theory. D-branes are hypersurfaces in the target space of string theory on which open strings are allowed to end. The CFT lives on the world sheet of the open string and its boundary conditions describe the possible D-brane configurations (see e.g. [RSc97, FSc97b]). The important point is that the CFT approach does not rely on the notion of a classical target space. This is exploited in [ARS99] where boundary CFT is used to investigate the non-commutativity of the brane world-volume in the case of $SU(2)$-WZW models. It would be very interesting to follow this line of thought in the case where the world sheet CFT is an $N{=}2$ superconformal field theory and apply the results to D-branes in superstring theory.

The author hopes to have shown that boundary conformal field theory is an area of research with very varied applications and that the results and methods presented in this thesis are useful for many of them.

Chapter A.

Appendix

A.1 OPEs to subleading order

Here we present a short calculational scheme to work out the bulk–bulk, bulk–boundary and boundary–boundary OPE to subleading orders. The general formulae become quickly too long, so that for higher orders one has to work with the explicit values of c and h.

Quasi primary states

When expressing the OPE it is convenient to work with quasi primary states, i.e. states $|\mu\rangle$ which obey $L_1|\mu\rangle=0$.

The three $sl(2,\mathbb{C})$ generators $\{L_1, L_0, L_{-1}\}$ form a subalgebra of Vir. Consequently we can try to decompose a Vir–irrep into $sl(2,\mathbb{C})$ representations. Quasi primary states $|\mu\rangle$ are by definition $sl(2,\mathbb{C})$ highest weight states. The corresponding $sl(2,\mathbb{C})$–module $M(\mu)$ is spanned by the states $\{(L_{-1})^n|\mu\rangle | n\in\mathbb{Z}^+\}$.

Let $Q(h,c)$ be an orthogonal basis of all quasi-primary states in the Vir–highest weight irrep $M(h,c)$. One can show that after discarding null vectors, the states

$$\left\{\ (L_{-1})^n|\mu\rangle \mid n\in\mathbb{Z}^+\,,\ \mu\in Q(h,c)\ \right\}\,. \tag{A.1}$$

form an orthogonal basis of $M(h,c)$. In particular, as a vector space $M(h,c)$ can be written as the direct sum

$$M(h,c) = \bigoplus_{\mu\in Q(h,c)} M(\mu)\ . \tag{A.2}$$

For the case that $M(h,c)$ does not contain null vectors, the first few quasi primary states $|\mu\rangle\in M(h,c)$ are listed in table A.1.

The norm of elements of $M(\mu)$ is given by

$$\langle\mu|(L_1)^n(L_{-1})^n|\mu\rangle = n!(2h)_{(n)}\langle\mu|\mu\rangle\ , \tag{A.3}$$

level	q.p. state $\lvert\mu\rangle$	norm $\langle\mu\lvert\mu\rangle/\langle h\lvert h\rangle$
0	$\lvert h\rangle$	1
2	$\lvert\alpha\rangle = \big(L_{-2} - \frac{3}{2(2h+1)}L_{-1}L_{-1}\big)\lvert h\rangle$	$\frac{c(2h+1)+2h(8h-5)}{2(2h+1)}$
3	$\lvert\beta\rangle = \big(L_{-3} - \frac{2}{h+2}L_{-1}L_{-2} + \frac{1}{(h+1)(h+2)}(L_{-1})^3\big)\lvert h\rangle$	$\frac{2h(c(h+1)+h(3h-7)+2)}{(h+1)(h+2)}$

Table A.1: The first three quasi-primary states in a Vir–highest weight irrep $M(h,c)$ without null vectors.

where $(a)_{(n)}$ is the Pochammer symbol $(a)_{(n)}{=}a(a{+}1)\cdots(a{+}n{-}1)$ and $(a)_{(0)}{=}1$. We see that $M(\mu)$ is finite dimensional if $h_\mu \in \mathbb{Z}_0^-/2$. An example is the vacuum representation $|0\rangle$ for which we have $L_{-1}|0\rangle = 0$. The first few elements of $Q(0,c)$ are given in table A.2. A general method to construct all quasi-primary states in $M(h,c)$ has been described by Kausch and Watts in [KWa91].

The OPE of two chiral quasi-primary fields ϕ_i, ϕ_j can be written down explicitly. The general expression is

$$\phi_i(z)\phi_j(0) = \sum_{k \text{ q.p.}} A_{ijk} \sum_{n=0}^{\infty} a_n \cdot z^{h_k-h_i-h_j+n} \cdot (L_{-1})^n \phi_k(0) \; . \tag{A.4}$$

The constants a_n are fixed by requiring that acting with $\langle k|(L_1)^n$ gives the same answer on both sides. The resulting values can be found e.g. in [Bow91] to be

$$a_n = \frac{(h_i{-}h_j{+}h_k)_{(n)}}{(2h_k)_{(n)} \cdot n!} \; . \tag{A.5}$$

Using this we can write down the following formal shorthand for (A.4)

$$\phi_i(z)\phi_j(w) = \sum_{k \text{ q.p.}} A_{ijk} \cdot (z{-}w)^{h_k-h_i-h_j} \cdot F_{ij}^k(z{-}w)\, \phi_k(w) \; , \tag{A.6}$$

with

$$\begin{aligned} F_{ij}^k(x) &= {}_1F_1\big(\, h_i{-}h_j{+}h_k \,;\, 2h_k \,;\, x\, L_{-1} \,\big) \\ &= 1 + \tfrac{h_i-h_j+h_k}{2h_k} \cdot xL_{-1} + \tfrac{(h_i-h_j+h_k)(h_i-h_j+h_k+1)}{4h_k(2h_k+1)} \cdot x^2 L_{-1}L_{-1} + \ldots \; . \end{aligned} \tag{A.7}$$

An example calculation in the case of the bulk–boundary OPE below will show how

level	q.p. state $\lvert\mu\rangle$	norm $\langle\mu\lvert\mu\rangle/\langle 0\lvert 0\rangle$
0	$\lvert 0\rangle$	1
2	$\lvert\alpha\rangle = L_{-2}\lvert 0\rangle$	$\frac{c}{2}$
4	$\lvert\gamma\rangle = \big(L_{-4} - \frac{5}{3}L_{-2}L_{-2}\big)\lvert 0\rangle$	$\frac{5c(5c+22)}{18}$
6	$\lvert\varepsilon_1\rangle = \big(L_{-6} + \frac{14}{5}L_{-4}L_{-2} - \frac{7}{4}L_{-3}L_{-3}\big)\lvert 0\rangle$	$\frac{63c(70c+29)}{100}$
	$\lvert\varepsilon_2\rangle = \big(6(10c{+}13)L_{-6} + 3(41c{+}67)L_{-4}L_{-2} - 93L_{-3}L_{-3} - (70c{+}29)(L_{-2})^3\big)\lvert 0\rangle$	$\frac{3c(2c-1)(5c+22)(7c+68)(70c+29)}{4}$

Table A.2: A choice for the first elements of $Q(0,c)$. If the norm vanishes for certain values of c the state is not part of $Q(0,c)$.

(A.5) and (A.6) can be obtained. But let us first look at the boundary–boundary OPE.

Boundary–boundary OPE

Because the boundary fields only transform in one copy of Vir, their OPE is most directly related to the above discussion. Let $\psi_\mu^{(ac)}(y)$ be the boundary field corresponding to the quasi primary state $\mu \in Q(h_k, c)$. Then the OPE can be written as

$$\psi_i^{(ab)}(x)\,\psi_j^{(bc)}(y) = \sum_k C_{ij}^{(abc)k} \sum_{\mu\in Q(h_k,c)} (x-y)^{h_\mu - h_i - h_j} \cdot A_{ij}^\mu \cdot F_{ij}^\mu(x-y)\,\psi_\mu^{(ac)}(y)\ . \quad \text{(A.8)}$$

Let $|\mu, ab\rangle = \psi_\mu^{(ab)}(0)$, etc. To determine the factors A_{ij}^μ we can set $x=1$, $y=0$ and take the inner product with $\langle \mu, ca|$ on both sides. Since we have chosen all quasi primary states in $Q(h_k, c)$ to be orthogonal, the resulting condition is

$$A_{ij}^\mu = \frac{\langle k|k\rangle}{\langle \mu|\mu\rangle} \cdot \frac{\langle \mu, ac|\ \psi_i^{(ab)}(1)\ |j, bc\rangle}{\langle k, ac|\ \psi_i^{(ab)}(1)\ |j, bc\rangle} \qquad ;\ \text{where } \mu \in Q(h_k, c)\ . \quad \text{(A.9)}$$

The three point function involving the quasi primary state can be reduced to the three point function of primary fields appearing in the denominator by using the commutation relation

$$L_n\psi(1) = \psi(1)L_n + L_0\psi(1) - \psi(1)L_0 + nh_\psi\psi(1)\ . \quad \text{(A.10)}$$

E.g. for the quasi-primary states listed in table A.1 we obtain

$$\begin{aligned} A_{ij}^{h_k} &= 1 \\ A_{ij}^{\alpha} &= \frac{2(2h_k{+}1)(h_k{+}2h_j{-}h_i) - 3(h_k{+}h_j{-}h_i)(h_k{+}h_j{-}h_i{+}1)}{c + 2h_k(8h_k{+}c{-}5)} \\ A_{ij}^{\beta} &= \frac{(h_j{-}h_i)\big\{(h_k{+}h_j{-}h_i)(h_k{+}h_i{-}h_j) + (h_i{+}h_j{-}h_k)(1{+}h_k)\big\}}{2h_k\big\{2 + c + h_k(3h_k{+}c{-}7)\big\}} \end{aligned} \quad \text{(A.11)}$$

The first subleading terms of the boundary–boundary OPE are

$$\begin{aligned}
&\psi_i^{(ab)}(x)\,\psi_j^{(bc)}(y) = \\
&\quad\sum_k C_{ij}^{(abc)k}\,(x{-}y)^{h_k-h_i-h_j}\cdot\Big\{\ \psi_k^{(ac)}(y)\ +\ (x{-}y)\cdot\tfrac{h_i-h_j+h_k}{2h_k}\cdot(L_{-1}\,\psi_k^{(ac)})(y) \\
&\qquad +\ (x{-}y)^2\cdot\big(\tfrac{(h_i-h_j+h_k)(h_i-h_j+h_k+1)}{4h_k(2h_k+1)}\cdot(L_{-1}L_{-1}\,\psi_k^{(ac)})(y) + A_{ij}^{\alpha}\,\psi_\alpha^{(ac)}(y)\big)\ +\ \ldots\Big\}\,.
\end{aligned} \tag{A.12}$$

In the case of $h_k{=}0$, i.e. the coupling to the identity, we necessarily have $h_i{=}h_j{=:}h$ and the formulas look somewhat simpler. For the states in table A.2 we find

$$A_{ii}^0 = 1\,,\quad A_{ii}^{\alpha} = \frac{2h}{c}\,,\quad A_{ii}^{\gamma} = -\frac{6h(5h+1)}{5c(5c+22)}\,, \tag{A.13}$$

and the OPE becomes

$$\begin{aligned}
&\psi_i^{(aa)}(x)\,\psi_i^{(aa)}(y) = C_{ii}^{(aaa)1}\,(x{-}y)^{-2h} \\
&\quad\cdot\Big\{\ \mathbb{1}\ +\ (x{-}y)^2\cdot\tfrac{2h}{c}\cdot\alpha(y)\ +\ (x{-}y)^3\cdot\tfrac{h}{c}\cdot(L_{-1}\alpha)(y) \\
&\qquad +\ (x{-}y)^4\cdot\big(\tfrac{3h}{10c}\cdot(L_{-1}L_{-1}\alpha)(y) - \tfrac{6h(5h+1)}{5c(5c+22)}\cdot\gamma(y)\big)\ +\ \ldots\Big\} + \ldots\,.
\end{aligned} \tag{A.14}$$

In this formula the dots stand for higher terms and conformal families other than the identity.

Bulk–boundary OPE

The bulk–boundary OPE is slightly different in structure to the other two, so calculations are written out more explicitly. The general bulk–boundary OPE on the UHP, in terms of quasi-primary states can again be written as

$$\phi(x{+}iy) = \sum_k {}^aB_\phi{}^k\,(2y)^{h_k-\Delta_\phi}\sum_{\mu\in Q(h_k,c)}\sum_{n=0}^{\infty} c_{\phi,k}^{\mu}(n)\cdot y^{h_\mu-h_k+n}(L_{-1})^n\,\psi_\mu^{(aa)}(x)\,. \tag{A.15}$$

In this expression we need to determine the coefficients $c_{\phi,k}^{\mu}(n)$. Set $x{=}0$. Since again all states on the rhs are orthogonal, we need to evaluate the conditions

$$\frac{\langle\mu|(L_1)^n\phi(iy)|0\rangle}{\langle k|\phi(iy)|0\rangle} = c_{\phi,k}^{\mu}(n)\cdot\frac{\langle\mu|(L_1)^n(L_{-1})^n|\mu\rangle}{\langle\mu|\mu\rangle}\,. \tag{A.16}$$

The three-point function on the lhs can be simplified with the commutation relation

$$L_n\phi(iy)|0\rangle = (iy)^n\Big\{\frac{1+(-1)^n}{2}\big(n\Delta_\phi + L_0\big) + \frac{1-(-1)^n}{2}\big((n+1)J_\phi + iy\cdot L_{-1}\big)\Big\}\phi(iy)|0\rangle\ . \tag{A.17}$$

We can obtain a recursion relation for the coefficients $c^\mu_{\phi,k}(n)$ if we employ (A.3) and use the commutator

$$[(L_1)^n, L_{-1}] = 2n\big(L_0 + \tfrac{n-1}{2}\big)(L_1)^{n-1}\ . \tag{A.18}$$

After a little algebra we obtain

$$c_{n+2} = \frac{1}{(n+2)(2h_\mu+n+1)}\big(2iJ_\phi\cdot c_{n+1} - c_n\big)\ . \tag{A.19}$$

Consider the generating function $G(t)=\sum_{k=0}^\infty c(k)t^k$. The recursion relation on the coefficients translates into the differential equation

$$t\,G''(t) + 2h_\mu\,G'(t) + (t-2iJ_\phi)G(t) = 0\ ;\quad G(0)=1,\ G'(0) = iJ_\phi/h_\mu\ . \tag{A.20}$$

Given the initial conditions, the differential equation is solved by

$$\begin{aligned} G^\mu_\phi(t) &= e^{-it}\cdot {}_1F_1\big(\,h_\mu{+}J_\phi\,;\,2h_\mu\,;\,2i\,t\,\big) \\ &= 1 + \tfrac{iJ_\phi}{h_\mu}\cdot t - \tfrac{h_\mu+2{J_\phi}^2}{2h_\mu(2h_\mu+1)}\cdot t^2 - \tfrac{iJ_\phi(2{J_\phi}^2+3h_\mu+1)}{6h_\mu(h_\mu+1)(2h_\mu+1)}\cdot t^3 + \dots\ . \end{aligned} \tag{A.21}$$

We can thus write the OPE (A.15) in the form

$$\phi(x{+}iy) = \sum_k {}^aB_\phi{}^k\,(2y)^{h_k-\Delta_\phi}\sum_{\mu\in Q(h_k,c)} b^\mu_\phi\cdot y^{h_\mu-h_k}\cdot G^\mu_\phi(yL_{-1})\,\psi^{(aa)}_\mu(x)\ . \tag{A.22}$$

The coefficients b^μ_ϕ of the quasi primary fields are determined by (A.16).

If we compare the form of G in (A.21) to the earlier result for F in (A.7) we see that they are consistent. The only difference is that the bulk–boundary OPE corresponds to the OPE of the chiral half of the bulk field ϕ with its mirror image, taken in the midpoint between the two fields. Hence the translation factor $e^{-iyL_{-1}}$ in (A.21).

For a generic representation with weight h_k the coefficients b^μ_ϕ for the quasi pri-

mary states in table A.1 take the form

$$\begin{aligned} b_\phi^{h_k} &= 1 \\ b_\phi^{\alpha} &= \frac{4(3{J_\phi}^2 - \Delta_\phi + h_k - 2\Delta_\phi h_k - {h_k}^2)}{c - 10h_k + 2c\,h_k + 16{h_k}^2} \\ b_\phi^{\beta} &= \frac{4iJ_\phi(\Delta_\phi - h_k + \Delta_\phi\, h_k - {J_\phi}^2)}{h_k(2 + c - 7h_k + c\,h_k + 3{h_k}^2)} \end{aligned} \tag{A.23}$$

For the identity h_k=0 representation we obtain

$$b_\phi^0 = 1\ , \quad b_\phi^\alpha = -\frac{4\Delta_\phi}{c}\ , \quad b_\phi^\gamma = -\frac{24\Delta_\phi(5\Delta_\phi+2)}{5c(5c+22)}\ . \tag{A.24}$$

This leads to the following bulk–boundary OPE, restricted to the coupling to the identity representation on the boundary,

$$\begin{aligned} \phi(x+iy) = {}^aB_\phi{}^{\mathbf{1}} \cdot (2y)^{-\Delta_\phi} \Big\{ \mathbb{1} - y^2 \cdot \tfrac{4\Delta_\phi}{c} \cdot T(x) + y^4 \cdot \big(\tfrac{2\Delta_\phi}{5c} \cdot T''(x) \\ - \tfrac{24\Delta_\phi(5\Delta_\phi+2)}{5c(5c+22)} \cdot \gamma(x) \big) + \dots \Big\} + \dots \end{aligned} \tag{A.25}$$

Here $T(x) = (L_{-2}\mathbb{1})(x)$ is the stress tensor and $T''(x)$ its second derivative. The dots stand for higher terms and conformal families other than the identity.

Bulk–bulk OPE

The bulk-bulk OPE is now easy to calculate, since the chiral and anti-chiral part produce the same formulas as in the boundary–boundary OPE. This leads to the expression

$$\begin{aligned} \phi_i(z,\bar z)\phi_j(w,\bar w) = \sum_k C_{ij}{}^k \sum_{\mu\in Q(h_k,c)} \sum_{\bar\mu\in Q(\bar h_k,c)} (z-w)^{h_\mu - h_i - h_j} (\bar z - \bar w)^{\bar h_\mu - \bar h_i - \bar h_j} \\ \cdot A_{ij}^\mu\ A_{\bar\imath\bar\jmath}^{\bar\mu} \cdot F_{ij}^\mu(z-w)\ F_{\bar\imath\bar\jmath}^{\bar\mu}(\bar z - \bar w)\ \phi_k(w,\bar w)\ . \end{aligned} \tag{A.26}$$

The coefficients A_{ij}^μ are those in (A.11).

A.2 Contour integration on the cylinder

In this section of the appendix the calculation leading to (3.25) is presented.

As in the main text construct a cylinder of circumference L by identifying two vertical lines on the UHP, one through the real point $-L/2$ and the other through $L/2$. Consider the following analytic function $f(w)$ from the cylinder to the UHP

$$z = f(w) = \tan\left(\tfrac{\pi}{L}w\right) , \quad f'(w) = \tfrac{\pi}{L}\big(\cos\tfrac{\pi w}{L}\big)^{-2} , \quad \tfrac{c}{12}\{f;w\} = \tfrac{c}{24}\big(\tfrac{2\pi}{L}\big)^2 \qquad (A.27)$$

Let $\beta(0)$ be the identity field or a descendent thereof. We have the UHP correlator

$$\langle T(z)\beta(0)\rangle_{\text{UHP}} = C_\beta z^{-2-|\beta|} . \qquad (A.28)$$

Here C_β is some constant and $|\beta|$ is the conformal weight of $\beta(0)$. Next we want to express the two-point function $\langle T(w)\beta(0)\rangle_{\text{cyl}}$ in terms of UHP-correlators. Let $N = |\beta|$. The transformation of $\beta(0)$ to the UHP will generate the $\mathbb{1}$–descendents $\gamma_0, \ldots, \gamma_N$ where $|\gamma_k| = k$ and $\gamma_0 = (const)\cdot\mathbb{1}$. Thus

$$\langle T(w)\beta(0)\rangle_{\text{cyl}} = \langle \big[f'(w)^2 T(f(w)) + \tfrac{c}{24}\big(\tfrac{2\pi}{L}\big)^2\big] \cdot \big[\sum_{k=0}^{N}\gamma_k(0)\big]\rangle_{\text{UHP}} . \qquad (A.29)$$

In the limit $w \to i\infty$ the lhs will tend towards the constant $\langle 0|T(w)|0\rangle\langle 0|\beta\rangle = \langle T(w)\rangle_{\text{cyl}}\langle\beta(0)\rangle_{\text{cyl}}$. The rhs will approach the constant $\frac{c}{24}\big(\frac{2\pi}{L}\big)^2 \cdot \langle\gamma_0(0)\rangle_{\text{UHP}}$. In particular we see that $\gamma_0{=}\langle\beta(0)\rangle_{\text{cyl}} \cdot \mathbb{1}$. Altogether we can rewrite (A.29) as

$$\langle T(w)\beta(0)\rangle_{\text{cyl}} = \tfrac{c}{24}\big(\tfrac{2\pi}{L}\big)^2\langle\beta(0)\rangle_{\text{cyl}} + \sum_{k=0}^{N-2} C_k\big(\tfrac{\pi}{L}\big)^2\big(\sin\tfrac{\pi w}{L}\big)^{-4}\big(\tan\tfrac{\pi w}{L}\big)^{-k} \qquad (A.30)$$

for some constants C_k. Note that we shifted $k \to k{-}2$ from (A.29) as $\gamma_1(0){=}0$.

The aim is to evaluate the contour integral

$$\int_\zeta \Big|\circlearrowleft_0\Big| h(\zeta)^{n+1}\langle T(\zeta)\beta(0)\rangle_{\text{cyl}} = \int_\zeta \Big|\square_0\Big| h(\zeta)^{n+1}\langle T(\zeta)\beta(0)\rangle_{\text{cyl}} \qquad (A.31)$$

where $h(w) = \frac{L}{2\pi i}\big(e^{\frac{2\pi i}{L}w} - 1\big)$ was defined in (3.23). We will treat the upper and the lower contour separately and assume $n{\leq}1$.

Upper contour: Let $w = x{+}iy$. In this case we have $x \in [-\frac{L}{2}, \frac{L}{2}]$ and $y{>}0$. The

integral is

$$-\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{2\pi i} h(w)^{n+1} \langle T(w)\beta(0)\rangle_{\text{cyl}}\big|_{w=x+iy} = A \cdot \tfrac{c}{24}\big(\tfrac{2\pi}{L}\big)^2 \langle \beta(0)\rangle_{\text{cyl}} + \sum_{k=0}^{N-2} B_k \cdot C_k \big(\tfrac{\pi}{L}\big)^2 \tag{A.32}$$

with

$$\begin{aligned} A &= -\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{2\pi i} h(w)^{n+1} = \tfrac{-1}{2\pi i}\big(\tfrac{L}{2\pi i}\big)^{n+1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \big(e^{\frac{2\pi i}{L}w} - 1\big)^{n+1} dx = \cdots = \Big(\frac{Li}{2\pi}\Big)^{n+2} \\ B_k &= -\int_{-\frac{L}{2}}^{\frac{L}{2}} h(w)^{n+1} \big(\sin \tfrac{\pi w}{L}\big)^{-4} \big(\tan \tfrac{\pi w}{L}\big)^{-k} = \cdots = 0 \end{aligned} \tag{A.33}$$

Lower contour: In this case we get the same expression as (A.32), up to an overall minus sign from the direction of the integration and with $y < 0$. After a similar calculation we obtain

$$A = B_k = 0 \qquad ; \quad \text{for } n \leq -2. \tag{A.34}$$

The only contribution for $n \leq -2$ thus comes from A in the upper contour, leading to the first case in equation (3.25):

$$\int_{\zeta \circlearrowleft 0} h(\zeta)^{n+1} \langle T(\zeta)\chi(0)\rangle_{\text{cyl}} = -\tfrac{c}{24}\big(\tfrac{iL}{2\pi}\big)^n \cdot \langle \chi(0)\rangle_{\text{cyl}} \qquad ; \quad \text{for } n \leq -2. \,. \tag{A.35}$$

The second case of (3.25) is for boundary fields $\chi(x)$ which are not descendents of the identity. Specifically let $\chi(x)$ be in the conformal family of the primary boundary field $\psi(x)$. On the UHP any correlator $\langle \chi(0)\rangle_{\text{UHP}}$, $\langle T(z)\chi(0)\rangle_{\text{UHP}}$ will just vanish since it can be reduced to a differential operator acting on $\langle \psi(0)\rangle_{\text{UHP}}$, which is zero. When transforming $\langle T(z)\chi(0)\rangle_{\text{cyl}}$ to the UHP, the descendent field $\chi(0)$ only generates fields in its own family $[\psi]$. But correlators of the form $\langle \chi(0)\rangle_{\text{UHP}}$, $\langle T(z)\chi(0)\rangle_{\text{UHP}}$ vanish, thus $\langle T(z)\chi(0)\rangle_{\text{cyl}} = 0$.

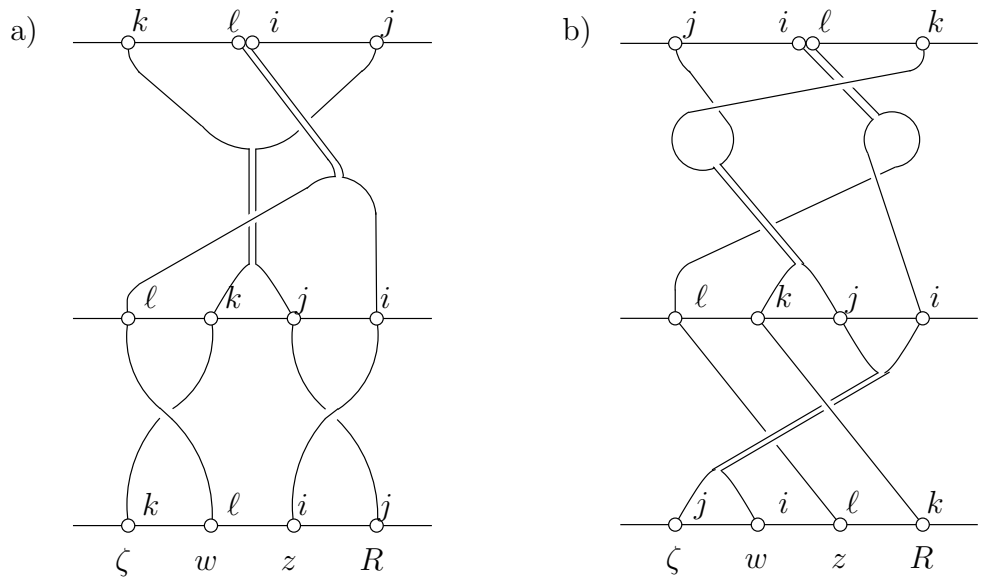


Figure A.1: Two sequences of moves expressing two different F-matrix elements in terms of the same known one. Both sequences are monodromy free ((b) might require a bit of patience to see that).

A.3 Properties of the F-matrix

First of all it is useful to find how F-matrix elements obtained from each other by swapping indices are related. Consider the analytic continuations in fig. A.1. We will work through the first diagram and just state the result of the second.

Figure A.1a corresponds to the following series of basis transformations:

$$\begin{array}{c|c|c|c|c}
 & j^\vee & i,a & \ell,b & k \\
0 & j & p & k & 0 \\
\hline
 & R & z & w & \zeta
\end{array}$$

$$= \Omega^p_{ij,a}(+)\Omega^p_{k\ell,b}(-)
\begin{array}{c|c|c|c|c}
 & i & j^\vee,a & k,b & \ell \\
0 & i^\vee & p & \ell & 0 \\
\hline
 & z & R & \zeta & w
\end{array}$$

$$= \sum_{q,c,d} \Omega^p_{ij,a}(+)\Omega^p_{k\ell,b}(-)\mathsf{F}_{pq}
\begin{bmatrix} j^\vee & k \\ i^\vee & \ell \end{bmatrix}^{cd}_{ab}
\begin{array}{c|c|c|c|c}
 & i & q,c & j^\vee,d & \ell \\
0 & i^\vee & R-\zeta & k & 0 \\
\hline
 & z & \zeta & w &
\end{array}$$

$$= \sum_{q,c,d} \Omega^p_{ij,a}(+)\Omega^p_{k\ell,b}(-)\mathsf{F}_{pq}
\begin{bmatrix} j^\vee & k \\ i^\vee & \ell \end{bmatrix}^{cd}_{ab}
\Omega^i_{q\ell,c}(+)
\begin{array}{c|c|c|c|c}
 & i & l,c & q & j^\vee,d \\
0 & i^\vee & q & R-\zeta & k \\
\hline
 & z & w & \zeta &
\end{array}$$

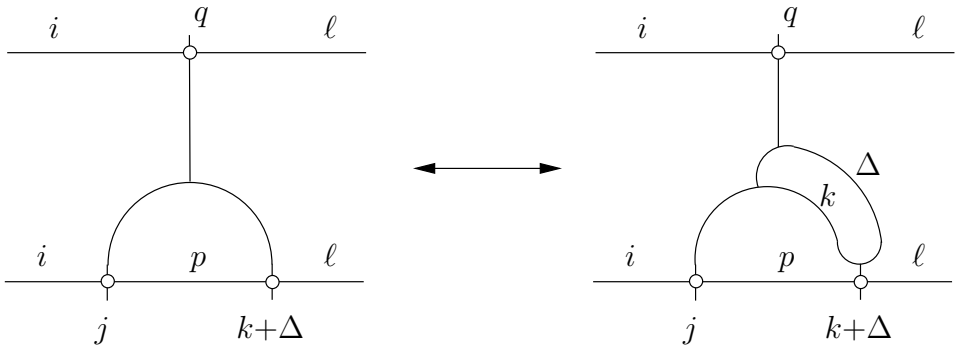
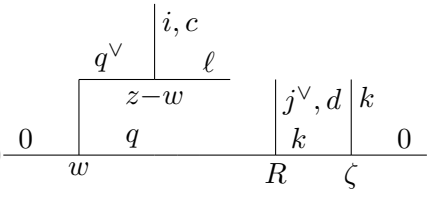
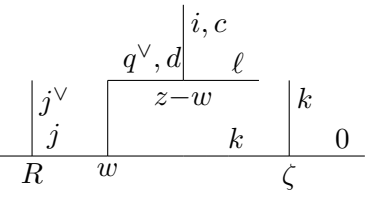


Figure A.2: A sequence of moves on a 5-point conformal block that leads to a recursion relation for the F-matrix

$$= \sum_{q,c,d} \Omega^p_{ij,a}(+)\Omega^p_{k\ell,b}(-)\mathsf{F}_{pq}\begin{bmatrix} j^\vee & k \\ i^\vee & \ell \end{bmatrix}^{cd}_{ab} \Omega^i_{q\ell,c}(+)$$



$$= \sum_{q,c,d} \Omega^p_{ij,a}(+)\Omega^p_{k\ell,b}(-)\Omega^i_{q\ell,c}(+)\Omega^k_{jq,d}(-)\mathsf{F}_{pq}\begin{bmatrix} j^\vee & k \\ i^\vee & \ell \end{bmatrix}^{cd}_{ab}$$



(A.36)

The phase factors of the Ω-coefficients cancel up to the signs ξ; the final result is:

$$\xi^a_{ijp}\xi^b_{klp}\xi^c_{ilq}\xi^d_{jkq}\mathsf{F}_{pq^\vee}\begin{bmatrix} i & \ell \\ j & k \end{bmatrix}^{dc}_{ab} = \mathsf{F}_{pq}\begin{bmatrix} j^\vee & k \\ i^\vee & \ell \end{bmatrix}^{cd}_{ab} \tag{A.37}$$

The analogous calculation for fig. A.1b gives:

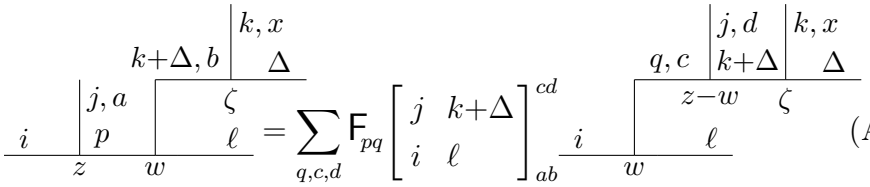
$$\mathsf{F}_{p^\vee q^\vee}\begin{bmatrix} \ell & i^\vee \\ k^\vee & j \end{bmatrix}^{dc}_{ba} = \mathsf{F}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}^{cd}_{ab} \tag{A.38}$$

Next we would like a recursion relation allowing us to work out a general F-matrix element in terms of a few generating ones. The analytic continuation in fig. A.2 tells us how to do this.

Let Δ be a "simple" chiral field (i.e. representation) in the theory we are studying, in the sense that we can by some means work out all F–matrices involving Δ. If there are enough such fields that repeated fusion among them generates all representations present in the theory we can set up a recursion procedure. Let $k{+}\Delta$ denote a

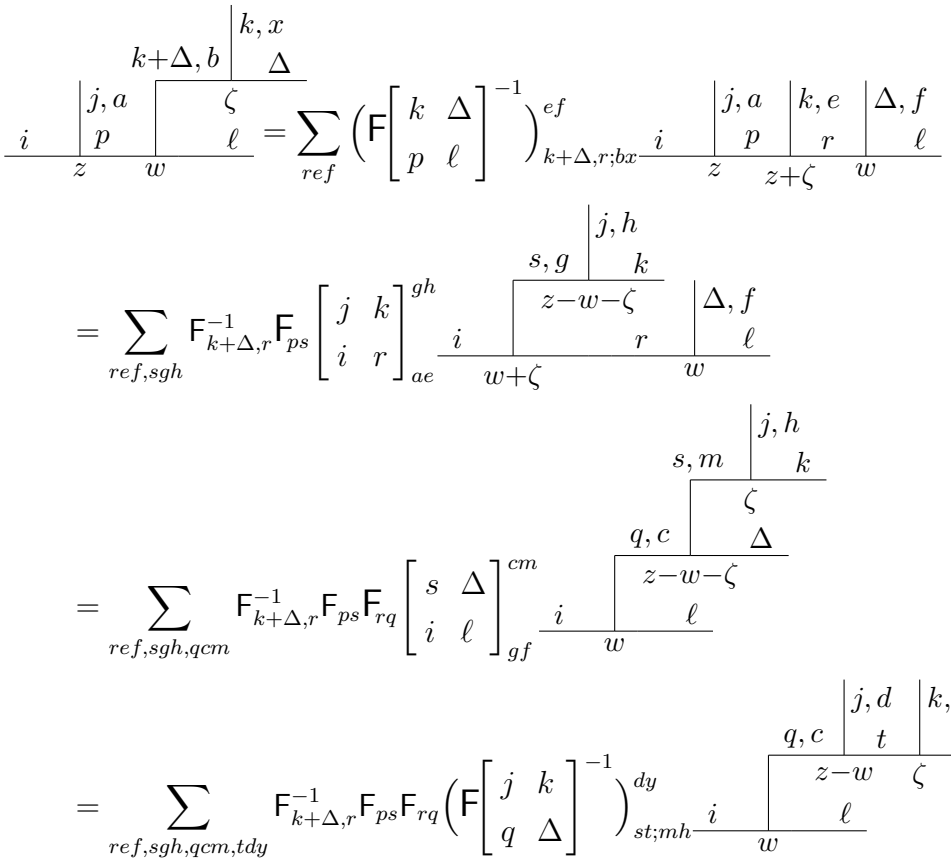
representation occurring in the fusion k and Δ. The notation suggests that $k{+}\Delta$ is in some sense "higher up" than k, as needed for the recursion, but it could be any representation.

First of all, we have:



(A.39)

Now implementing the moves in figure A.2 we find



(A.40)

In the last line we can pick the coefficient in front of the block with $t = k{+}\Delta$ and $y{=}x$. This must be the same number as the F-matrix entry in (A.39). In fact for (A.40) to be equal to (A.39), for all other choices for t, y the F-matrices have to sum to zero. We get, together with the relation for the inverse of the fusion matrix (4.51):

$$\mathsf{F}_{pq}\begin{bmatrix} j & k{+}\Delta \\ i & \ell \end{bmatrix}^{cd}_{ab} = \sum_{r,s;efghm} \mathsf{F}_{(k+\Delta)^\vee,r}\begin{bmatrix} \Delta & \ell \\ k^\vee & p^\vee \end{bmatrix}^{ef}_{xb} \mathsf{F}_{ps}\begin{bmatrix} j & k \\ i & r \end{bmatrix}^{gh}_{ae} \mathsf{F}_{rq}\begin{bmatrix} s & \Delta \\ i & \ell \end{bmatrix}^{cm}_{gf} \mathsf{F}_{s^\vee,\,k+\Delta}\begin{bmatrix} k & \Delta \\ j^\vee & q^\vee \end{bmatrix}^{dx}_{hm}$$

A.4 Explicit F-matrix

Consider the minimal model $M(p,q)$. Let $t{=}p/q$ and $d_{rs}{=}r{-}st$. We want to find the matrix connecting the conformal blocks occurring in the $x{\to}0$ and $x{\to}1$ expansion of the chiral correlator $\langle\phi_I|\phi_J(1)\phi_K(x)|\phi_L\rangle$. The indices are given by Kac-labels $I{=}(r_I,s_I)$, $J{=}(r_J,s_J)$, etc. Let correspondingly $d_I{=}r_I{-}s_It$, $d_J{=}r_J{-}s_Jt$, etc.
From [DFa85](A.35) we find:

$$\begin{aligned}
b_{xy}(\alpha,\beta;\rho) &= \prod_{g=1}^{y}\frac{\Gamma(g\rho)\Gamma(\alpha{+}g\rho)\Gamma(\beta{+}g\rho)}{\Gamma(\rho)\Gamma(\alpha{+}\beta{-}2x{+}(y{+}g)\rho)}\\
m_{xy}(\alpha,\beta) &= t^{2xy}\prod_{g=1}^{x}\prod_{h=1}^{y}\Big\{(ht{-}g)(\alpha{+}ht{-}g)(\beta{+}ht{-}g)(\alpha{+}\beta{+}(y{+}h)t{-}(x{+}g))\Big\}^{-1}\\
j(x,y;\alpha,\beta) &= m_{xy}(\alpha,\beta)\cdot b_{yx}(-\tfrac{1}{t}\alpha,-\tfrac{1}{t}\beta;\tfrac{1}{t})\cdot b_{xy}(\alpha,\beta;t)
\end{aligned}\qquad(A.41)$$

From [FGP90](3.5) we find:

$$\begin{aligned}
a(s;x,y;\alpha,\beta,\gamma,\delta;\rho) = \sum_{h=\max(x,y)}^{\min(s,x+y-1)} &\frac{\prod_{g=1}^{s-h}\sin\pi(\delta{+}(x{-}1{+}g)\rho)\prod_{g=1}^{h-y}\sin\pi(-\alpha{+}(s{-}x{+}g)\rho)}{\prod_{g=1}^{s-y}\sin\pi(-\alpha{+}\delta{+}(s{-}y{+}g)\rho)}\\
&\times\frac{\prod_{g=1}^{y-1-(h-x)}\sin\pi(\beta{+}(s{-}x{+}g)\rho)\prod_{g=1}^{h-x}\sin\pi(\gamma{+}(x{-}1{+}g)\rho)}{\prod_{g=1}^{y-1}\sin\pi(\beta{+}\gamma{+}(y{-}1{+}g)\rho)}\\
&\times\prod_{g=1}^{h-x}\frac{\sin\pi((x{+}y{-}h{-}1{+}g)\rho)}{\sin\pi(g\rho)}\prod_{g=1}^{s-h}\frac{\sin\pi((h{-}y{+}g)\rho)}{\sin\pi(g\rho)}
\end{aligned}\qquad(A.42)$$

Putting together [DFa85](4.1) and [FGP90](3.1) we find:

$$\begin{aligned}
&\mathsf{F}_{PQ}\begin{bmatrix}J & K\\ I & L\end{bmatrix} =\\
&\frac{j\big(\frac12(r_L{-}r_I{-}1{+}r_Q),\frac12(s_L{-}s_I{-}1{+}s_Q);{-}d_I,d_L\big)\cdot j\big(\frac12(r_J{+}r_K{-}1{-}r_Q),\frac12(s_J{+}s_K{-}1{-}s_Q);d_J,d_K\big)}{j\big(\frac12(r_J{-}r_I{-}1{+}r_P),\frac12(s_J{-}s_I{-}1{+}s_P);{-}d_I,d_J\big)\cdot j\big(\frac12(r_K{+}r_L{-}1{-}r_P),\frac12(s_K{+}s_L{-}1{-}s_P);d_K,d_L\big)}\\
&\cdot a\big(\tfrac12(-r_I{+}r_J{+}r_K{+}r_L);\tfrac12(r_K{+}r_L{+}1{-}r_P),\tfrac12(r_J{+}r_K{+}1{-}r_Q);-\tfrac1t d_I,-\tfrac1t d_J,-\tfrac1t d_K,-\tfrac1t d_L;\tfrac1t\big)\\
&\cdot a\big(\tfrac12(-s_I{+}s_J{+}s_K{+}s_L);\tfrac12(s_K{+}s_L{+}1{-}s_P),\tfrac12(s_J{+}s_K{+}1{-}s_Q);d_I,d_J,d_K,d_L;t\big)
\end{aligned}\qquad(A.43)$$

A.5 Some F-matrix identites

The following constitutes a collection of F-matrix identies for Virasoro minimal models used in this thesis. They are either directly taken from [MSb89a] or special cases thereof. Note that for the S–matrix we have ${S_1}^i = {S_1}^{i^*}$ where the $*$-operation was defined as $i^* = (r, q-s)$ if i has Kac-labels (r,s) and we are in the minimal model $M(p,q)$ with p odd.

$$\mathsf{B}^{(\varepsilon)}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} = e^{i\pi\varepsilon(h_i+h_\ell-h_p-h_q)}\mathsf{F}_{pq}\begin{bmatrix} j & l \\ i & k \end{bmatrix} \tag{A.44}$$

$$\mathsf{F}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} = \mathsf{F}_{pq}\begin{bmatrix} i & \ell \\ j & k \end{bmatrix} = \mathsf{F}_{pq}\begin{bmatrix} \ell & i \\ k & j \end{bmatrix} \tag{A.45}$$

$$\sum_r \mathsf{F}_{pr}\begin{bmatrix} b & c \\ a & d \end{bmatrix}\mathsf{F}_{rq}\begin{bmatrix} d & c \\ a & b \end{bmatrix} = \delta_{p,q} \qquad \mathsf{F}_{b^*x_i^*}\begin{bmatrix} x_i & \omega_o \\ a & b \end{bmatrix}\mathsf{F}_{x_i^*b^*}\begin{bmatrix} b & \omega_o \\ a & x_i \end{bmatrix} = 1 \tag{A.46}$$

$$\mathsf{F}_{1k}\begin{bmatrix} i & j \\ i & j \end{bmatrix}\mathsf{F}_{k1}\begin{bmatrix} i & i \\ j & j \end{bmatrix} = \frac{{S_1}^1 \cdot {S_1}^k}{{S_1}^i \cdot {S_1}^j} \qquad \mathsf{F}_{1x_o}\begin{bmatrix} x_e & \omega_o \\ x_e & \omega_o \end{bmatrix}\cdot\mathsf{F}_{x_o1}\begin{bmatrix} \omega_o & \omega_o \\ x_e & x_e \end{bmatrix} = 1 \tag{A.47}$$

$$\mathsf{F}_{k1}\begin{bmatrix} i & i \\ j & j \end{bmatrix} = \frac{{S_1}^k}{{S_1}^j}\cdot\mathsf{F}_{j1}\begin{bmatrix} k & k \\ i & i \end{bmatrix} \qquad \mathsf{F}_{x_o1}\begin{bmatrix} \omega_o & \omega_o \\ x_e & x_e \end{bmatrix} = \mathsf{F}_{x_e1}\begin{bmatrix} \omega_o & \omega_o \\ x_o & x_o \end{bmatrix} \tag{A.48}$$

$$\mathsf{F}_{11}\begin{bmatrix} i & i \\ i & i \end{bmatrix} = \frac{{S_1}^1}{{S_1}^i} \qquad \mathsf{F}_{11}\begin{bmatrix} \omega_o & \omega_o \\ \omega_o & \omega_o \end{bmatrix} = 1 \tag{A.49}$$

$$\mathsf{F}_{pi}\begin{bmatrix} j & k \\ n & \ell \end{bmatrix}\mathsf{F}_{n1}\begin{bmatrix} i & i \\ \ell & \ell \end{bmatrix} = \mathsf{F}_{nk}\begin{bmatrix} i & j \\ \ell & p \end{bmatrix}\mathsf{F}_{p1}\begin{bmatrix} k & k \\ \ell & \ell \end{bmatrix} \tag{A.50}$$

$$\sum_s \mathsf{F}_{qs}\begin{bmatrix} j & k \\ p & b \end{bmatrix}\mathsf{F}_{p\ell}\begin{bmatrix} i & s \\ a & b \end{bmatrix}\mathsf{F}_{sr}\begin{bmatrix} i & j \\ \ell & k \end{bmatrix} = \mathsf{F}_{pr}\begin{bmatrix} i & j \\ a & q \end{bmatrix}\mathsf{F}_{q\ell}\begin{bmatrix} r & k \\ a & b \end{bmatrix} \tag{A.51}$$

$$\begin{aligned}\sum_s e^{i\pi(h_p+h_q+2h_i-2h_s-\frac{1}{2}(h_k+h_\ell))}\mathsf{F}_{ks}\begin{bmatrix} \bar{i} & p \\ i & q \end{bmatrix}\mathsf{F}_{s\ell}\begin{bmatrix} q & p \\ i & \bar{i} \end{bmatrix} \\ = \sum_m e^{i\pi(-h_p-h_q-2\bar{h}_i+2h_m+\frac{1}{2}(h_k+h_\ell))}\mathsf{F}_{km}\begin{bmatrix} p & i \\ q & \bar{i} \end{bmatrix}\mathsf{F}_{m\ell}\begin{bmatrix} p & q \\ i & \bar{i} \end{bmatrix}\end{aligned} \tag{A.52}$$

$$S_i{}^j(p) = {S_1}^1 e^{-i\pi h_p}\frac{\mathsf{F}_{i1}\begin{bmatrix} i & i \\ p & p \end{bmatrix}}{\mathsf{F}_{11}\begin{bmatrix} p & p \\ p & p \end{bmatrix}\mathsf{F}_{p1}\begin{bmatrix} j & j \\ j & j \end{bmatrix}\mathsf{F}_{p1}\begin{bmatrix} i & i \\ i & i \end{bmatrix}}\sum_r \mathsf{B}^{(-)}_{pr}\begin{bmatrix} i & j \\ i & j \end{bmatrix}\mathsf{B}^{(-)}_{r1}\begin{bmatrix} j & i \\ i & j \end{bmatrix} \tag{A.53}$$

A.6 Derivation of the five-point constraint

This appendix is included to illustrate how the notation and techniques for calculating with conformal blocks described in [MSb89a] were used to rederive the sewing constraints in [Lew92] presented in section 5.2.

The example we consider is the derivation of the constraint resulting from taking different limits in the correlator involving two bulk fields ϕ_{i_α}, ϕ_{j_β} and one boundary field $\psi^{(aa)}_{k_\gamma}$ on the upper half plane with the boundary condition labelled a.

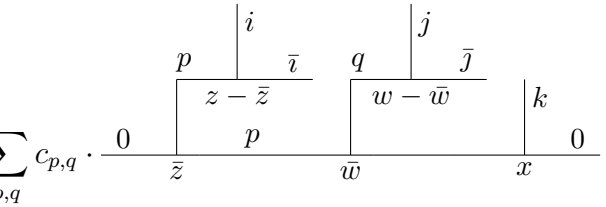
In the first limit we take the two bulk fields to the boundary and are left with a three-point function on the boundary. In the second limit we start by taking the OPE of the two bulk fields and then take the remaining bulk field to the boundary. Let $z{=}x_z{+}iy_z$ and $w{=}x_w{+}iy_w$. The asymptotic behaviour in the two limiting cases is then given by:

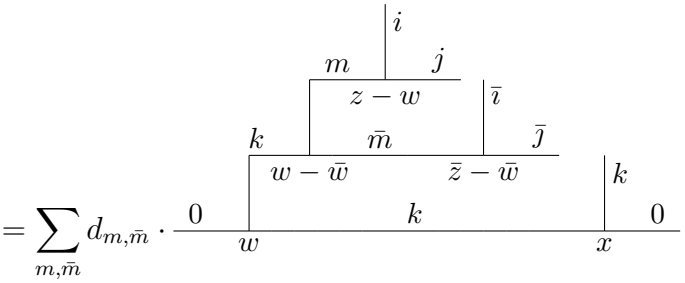
$$\begin{aligned}
&\langle\phi_{i_\alpha}(z,\bar z)\phi_{j_\beta}(w,\bar w)\,\psi^{(aa)}_{k_\gamma}(x)\rangle\\
&\quad\underset{\substack{y_z\to0\\ y_w\to0}}{\sim}\sum_{p,q}\left(\sum_{\nu,\varepsilon}{}^aB_{i_\alpha}{}^{p_\nu}\;{}^aB_{j_\beta}{}^{q_\varepsilon}\;C^{(aaa)k_\gamma}_{p_\nu q_\varepsilon}\;C^{(aaa)1}_{k_\gamma k_\gamma}\,\langle1\rangle^a_{\mathrm{UHP}}\right)\\
&\qquad\cdot(2y_z)^{h_p-h_i-\bar h_i}(2y_w)^{h_q-h_j-\bar h_j}(x_z-x_w)^{h_k-h_p-h_q}\\
&\qquad\quad\cdot(x_z-x)^{h_q-h_p-h_k}(x_w-x)^{h_p-h_k-h_q}\\
&\quad\underset{\substack{z\to w\\ \bar z\to\bar w}}{\sim}\sum_{m,\bar m}\left(\sum_{\rho}C_{i_\alpha j_\beta}{}^{m_\rho}\;{}^aB_{m_\rho}{}^{k_\gamma}\;C^{(aaa)1}_{k_\gamma k_\gamma}\,\langle1\rangle^a_{\mathrm{UHP}}\right)\\
&\qquad\cdot(z-w)^{h_m-h_i-h_j}(\bar z-\bar w)^{\bar h_m-\bar h_i-\bar h_j}(2y_w)^{h_k-h_m-\bar h_m}(x_w-x)^{-2h_k}
\end{aligned}\tag{A.54}$$

On the other hand the correlator can be expressed as a linear combination of conformal blocks. We use two sets of conformal blocks to express the correlator in two different ways, one associated to each asymptotic behaviour in (A.54). We obtain the following two linear combinations:

$$\langle\phi_{i_\alpha}(z,\bar z)\phi_{j_\beta}(w,\bar w)\,\psi^{(aa)}_{k_\gamma}(x)\rangle$$

$$= \sum_{p,q} c_{p,q}\cdot$$





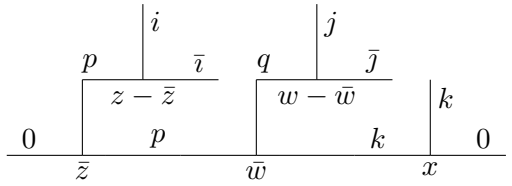
$$= \sum_{m,\bar{m}} d_{m,\bar{m}} \cdot \qquad (A.55)$$

Taking the limits calculated in (A.54) for the exact expressions in terms of conformal blocks (A.55) relates the coefficients $c_{p,q}$ and $d_{m,\bar{m}}$ to the products of structure constants obtained by applying the OPE. The precise relation is:

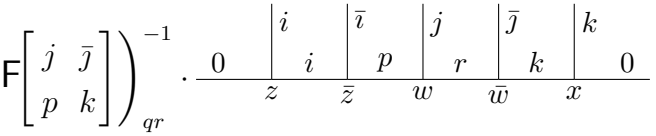
$$\begin{aligned} c_{p,q} \cdot e^{i\frac{\pi}{2}(h_p+h_q-h_i-\bar{h}_i-h_j-\bar{h}_j)} &= \sum_{\nu,\varepsilon} {}^{a}B_{i_\alpha}{}^{p_\nu}\ {}^{a}B_{j_\beta}{}^{q_\varepsilon}\ C^{(aaa)k_\gamma}_{p_\nu q_\varepsilon}\ C^{(aaa)1}_{k_\gamma k_\gamma}\ \langle 1\rangle^{a}_{\mathrm{UHP}} \\ d_{m,\bar{m}} \cdot e^{i\frac{\pi}{2}(h_k-h_m-\bar{h}_m)} &= \sum_{\rho} C_{i_\alpha j_\beta}{}^{m_\rho}\ {}^{a}B_{m_\rho}{}^{k_\gamma}\ C^{(aaa)1}_{k_\gamma k_\gamma}\ \langle 1\rangle^{a}_{\mathrm{UHP}} \end{aligned} \qquad (A.56)$$

The phase factors originate from relating $z-\bar{z} = 2iy_z$ to $2y_z$ etc. The sums over ν, ε and ρ in (A.56) take care of fields with multiplicities. Fields that transform in the same representation of the Virasoro algebra show the same asymptotic behaviour and cannot be discriminated by conformal blocks. Hence their structure constants occur as a sum in front of the corresponding conformal block.

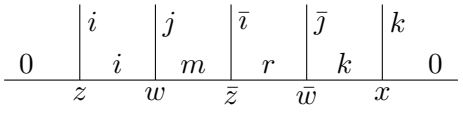
The two sets of conformal blocks in (A.55) are related by a basis transformation. This transformation can be carried out in several steps making use of two basic moves of braiding and fusion implemented by the B– and F–matrix (see section 4.3 for details). One possible way to perform the basis transformation is as follows:



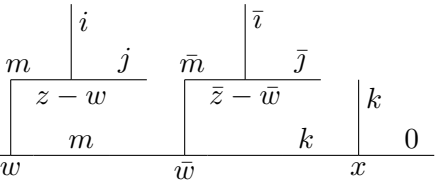
$$= \sum_{r} \left(\mathsf{F}\begin{bmatrix} j & \bar{j} \\ p & k \end{bmatrix} \right)^{-1}_{qr} \cdot$$



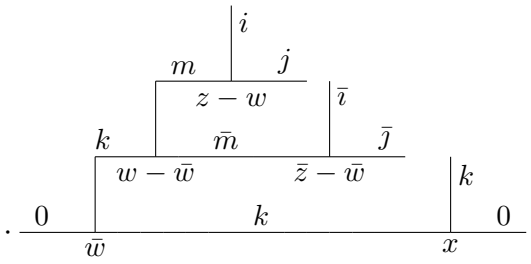
$$= \sum_{r,m} \mathsf{F}_{qr}\begin{bmatrix} k & \bar{j} \\ p & j \end{bmatrix} \mathsf{B}^{(-)}_{pm}\begin{bmatrix} \bar{\imath} & j \\ i & r \end{bmatrix} \cdot$$



$$= \sum_{r,m,\bar{m}} \mathsf{F}_{qr}\begin{bmatrix} k & \bar{\jmath} \\ p & j \end{bmatrix} \mathsf{B}^{(-)}_{pm}\begin{bmatrix} \bar{\imath} & j \\ i & r \end{bmatrix} \mathsf{F}_{r\bar{m}}\begin{bmatrix} \bar{\imath} & \bar{\jmath} \\ m & k \end{bmatrix} \cdot$$



$$= \sum_{r,m,\bar{m}} e^{-i\pi(h_i+h_r-h_p-h_m)} \mathsf{F}_{qr}\begin{bmatrix} k & \bar{\jmath} \\ p & j \end{bmatrix} \mathsf{F}_{pm}\begin{bmatrix} \bar{\imath} & r \\ i & j \end{bmatrix} \mathsf{F}_{r\bar{m}}\begin{bmatrix} \bar{\imath} & \bar{\jmath} \\ m & k \end{bmatrix}$$

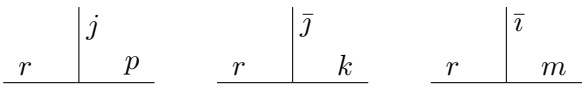


(A.57)

Putting (A.55), (A.56) and (A.57) together we recover the sewing constraint (5.26):

$$\begin{aligned} &\sum_{\rho} C_{i_\alpha j_\beta}{}^{m_\rho}\ {}^{a}B_{m_\rho}{}^{k_\gamma}\ C^{(aaa)1}_{k_\gamma k_\gamma}\ \langle 1\rangle^{a}_{\mathrm{UHP}} \\ &\quad = \sum_{p,q}\left(\sum_{\nu,\varepsilon} {}^{a}B_{i_\alpha}{}^{p_\nu}\ {}^{a}B_{j_\beta}{}^{q_\varepsilon}\ C^{(aaa)k_\gamma}_{p_\nu q_\varepsilon}\ C^{(aaa)1}_{k_\gamma k_\gamma}\ \langle 1\rangle^{a}_{\mathrm{UHP}}\right) \\ &\qquad \cdot \sum_{r} e^{i\frac{\pi}{2}(h_k+h_p-h_q-2h_r+h_m-\bar{h}_m-h_i+\bar{h}_i+h_j+\bar{h}_j)} \\ &\qquad\quad \cdot \mathsf{F}_{qr}\begin{bmatrix} k & \bar{\jmath} \\ p & j \end{bmatrix} \mathsf{F}_{pm}\begin{bmatrix} \bar{\imath} & r \\ i & j \end{bmatrix} \mathsf{F}_{r\bar{m}}\begin{bmatrix} \bar{\imath} & \bar{\jmath} \\ m & k \end{bmatrix} \end{aligned} \tag{A.58}$$

The summation range for r is principally over all entries in the Kac-table (quotiened by $\mathbb{Z}_2$) but the F–matrix entries will only be nonzero if the following three fusions are allowed:



(A.59)

A.7 Uniqueness of the boundary structure constants

A-series

To give necessary expressions for the boundary structure constants it is enough only to consider equation (5.20). The following table summarises the line of argument in this section by describing the boundary situations considered, which constant it fixes, and what the remaining freedom to rescale the boundary fields is:

boundary situation	s.c. fixed	eqn.	remaining freedom
$1\ \ b\ \ b\ \ j\ \ c\ \ j\ \ b\ \ b\ \ 1$	$C_{jj}^{(bcb)1}$	(A.65)	$b\neq c, j\neq 1 : \{\psi_j^{(bc)}, \psi_j^{(cb)}\} \to \{\lambda\cdot\psi_j^{(bc)}, \lambda^{-1}\cdot\psi_j^{(cb)}\}$ $b=c, j\neq 1 : \{\psi_j^{(bb)}\} \to \{\pm\psi_j^{(bb)}\}$
$1\ \ b\ \ b\ \ b\ \ 1\ \ d\ \ d\ \ d\ \ 1$	$C_{bd}^{(b1d)q}$	(A.67)	none
$a\ \ i\ \ b\ \ b\ \ 1\ \ d\ \ d\ \ \ell\ \ a$	$C_{\ell i}^{(dab)q}$	(A.69)	none

In order to simplify the notation when rescaling subsets of boundary operators it is helpful to introduce an ordering on the boundary conditions $(1) < (a_1) < (a_2) < \ldots$. The particular order one chooses is not important.

As a first step consider the relation coming from taking the two different OPE's in the 3-point function:

$$C_{ij}^{(abc)k}\ C_{kk}^{(aca)1} \;=\; C_{jk}^{(bca)i}\ C_{ii}^{(aba)1} \tag{A.60}$$

Setting $i{=}1, k{=}j, b{=}a$ in (A.60) leads to $C_{1j}^{(aac)j} = C_{11}^{(aaa)1}$. Similarly, setting $j{=}1$ resp. $k{=}1$ leads to $C_{i1}^{(abb)i} = C_{1i}^{(bba)i}$ resp. $C_{11}^{(aaa)1} = C_{11}^{(baa)i}$. It follows that consistent behaviour of the identity field on the 1-boundary $C_{11}^{(111)1} {=} 1$ already implies consistent behaviour of the identity on all other boundaries:

$$C_{1i}^{(aab)i} = 1 \qquad C_{i1}^{(abb)i} = 1 \qquad \forall a, b, i \tag{A.61}$$

Any given solution of the sewing constraints (5.20)–(5.23) must already fulfil (A.61).

To see how boundary operators couple to the identity, first consider (5.20) with $a{=}1$, $d{=}b$, $i{=}\ell{=}b$, $j{=}k$, $q{=}1$: The sum reduces to $p{=}c$ and with (A.60) in the form $C_{jb}^{(cb1)c}\ C_{cc}^{(1c1)1} = C_{cj}^{(1cb)b}\ C_{bb}^{(1b1)1}$ one obtains:

$$C_{jj}^{(bcb)1}\ \left(\mathsf{F}_{c1}\begin{bmatrix} j & j \\ b & b \end{bmatrix}\right)^{-1} \;=\; C_{bj}^{(1bc)c}\ C_{cj}^{(1cb)b} \tag{A.62}$$

Exchanging $b \leftrightarrow c$ leaves the RHS invariant. Transforming the F-matrix element using (A.48) we finally get:

$$S_1{}^c\, C_{jj}^{(cbc)1} = S_1{}^b\, C_{jj}^{(bcb)1} \tag{A.63}$$

Any solution has to fulfil this identity. In particular this implies that independent of the expectation values of the identity $\langle 1\rangle^a$ we cannot set both $C_{jj}^{(cbc)1}$ and $C_{jj}^{(bcb)1}$ to one.

We shall now make use of the freedom to rescale the fields. Note that the identity field 1 is fixed by the property $1\cdot 1{=}1$. For $b<c$ and $j{\neq}1$ we rescale $\psi_j^{(bc)} \to \lambda\cdot\psi_j^{(bc)}$, resulting in:

$$C_{jj}^{(cbc)1} \to \lambda\, C_{jj}^{(cbc)1} \qquad C_{jj}^{(bcb)1} \to \lambda\, C_{jj}^{(bcb)1} \tag{A.64}$$

For an appropriate λ we get, for all $b<c, j{\neq}1$:

$$C_{jj}^{(cbc)1} = \left(\mathsf{F}_{1j}\begin{bmatrix} c & b \\ c & b \end{bmatrix}\right)^{-1} \frac{S_1{}^1}{S_1{}^c} \tag{A.65}$$

In particular (A.64) implies that once the $C_{jj}^{(cbc)1}$ are adjusted for $b<c$ one is no longer free to rescale $C_{jj}^{(cbc)1}$ for $b>c$. However (A.63) implies that (A.65) holds also for $b>c, j{\neq}1$. For $b{=}c, j{\neq}1$ rescaling $\psi_j^{(bb)} \to \lambda\cdot\psi_j^{(bb)}$ gives $C_{jj}^{(bbb)1} \to \lambda^2\, C_{jj}^{(bbb)1}$, so that bringing $C_{jj}^{(bbb)1}$ to the form (A.65) only fixes $\psi_j^{(bb)}$ up to a sign. Setting $j{=}1$, together with (A.49) shows consistency with $C_{11}^{(bbb)1}{=}1$. Thus (A.65) is valid for all values of b, c, j. The scaling of the operators $\{\psi_j^{(bc)}, \psi_j^{(cb)}\}$ for $b{\neq}c, j{\neq}1$ is now fixed up to $\{\psi_j^{(bc)}, \psi_j^{(cb)}\} \to \{\lambda\cdot\psi_j^{(bc)}, \lambda^{-1}\cdot\psi_j^{(cb)}\}$, which leaves (A.65) invariant. $\psi_j^{(bb)}, j{\neq}1$ is fixed up to sign.

Taking (5.20) with $a{=}c{=}1, i{=}j{=}b, k{=}\ell{=}d, p{=}1$ and using (A.60) and (A.65) results in:

$$C_{bd}^{(b1d)q}\; C_{db}^{(d1b)q} = \left(\mathsf{F}_{1q}\begin{bmatrix} d & b \\ d & b \end{bmatrix}\right)^2 \tag{A.66}$$

For $b>d, q{\neq}1$ we rescale $\{\psi_q^{(db)}, \psi_q^{(bd)}\} \to \{\lambda\cdot\psi_q^{(db)}, \lambda^{-1}\cdot\psi_q^{(bd)}\}$ such that:

$$C_{bd}^{(b1d)q} = \mathsf{F}_{1q}\begin{bmatrix} d & b \\ d & b \end{bmatrix} \tag{A.67}$$

(A.66) now implies that (A.67) also holds for $b<d$. For $b{=}d, q{\neq}1$ we are still free to

choose the sign of $\psi_q^{(bb)} \to \pm\psi_q^{(bb)}$. This allows us to alter the sign of $C_{bb}^{(b1b)q}$ to match (A.67). The case $q{=}1$ can occur only for $b{=}d$ and we get $C_{bb}^{(b1b)1} = {S_1}^1/{S_1}^b$, consistent with the normalisation (A.65). Thus (A.67) holds for all b, d, q. The scaling of all boundary operators is now fixed.

Taking (5.20) with $c{=}1, j{=}b, k{=}d$ the sum reduces to $p{=}a$. Using (A.60) and rearranging terms one obtains:

$$C_{\ell i}^{(dab)q} = \mathsf{F}_{aq}\begin{bmatrix} b & d \\ i & \ell \end{bmatrix} \frac{C_{ii}^{(aba)1}\; C_{\ell\ell}^{(dad)1}}{C_{qq}^{(dbd)1}\; C_{aa}^{(a1a)1}} \frac{C_{ba}^{(b1a)i}\; C_{ad}^{(a1d)\ell}}{C_{bd}^{(b1d)q}} \tag{A.68}$$

All the C-terms cancel and after renaming indices one is left with:

$$C_{ij}^{(abc)k} = \mathsf{F}_{bk}\begin{bmatrix} a & c \\ i & j \end{bmatrix} \qquad \forall a, b, c, i, j, k \tag{A.69}$$

D-series

In this section we will construct a set of boundary structure constants that solve the sewing constraint (5.24) under the condition that there exists a solution at all (for the given field content). The results have already been summarised in section 6.2

Recall the assumptions we made in section 6.2 as starting point for the calculation

(a) the D-series field content obtained in section 6.1

(b) the assumption that all two-point functions $\langle\, \psi_{i_\alpha}^{(ab)}(x)\, \psi_{i_\alpha}^{(ba)}(y)\rangle$ are nonzero, i.e. $C_{i_\alpha i_\alpha}^{(aba)1} \neq 0$ for all a, b, i_α that are allowed by (a).

Nonzero boundary structure constants

First we investigate the consequence of (b), i.e. that all two-point functions are nonzero. Eqn. (5.24) can be rewritten so that the two-point structure constants cancel. To do so we set $\ell{=}1$ and obtain the three-point identity:

$$\underline{a\;\; \overset{\frown}{i_\alpha}\;\; b\;\; \overset{\frown}{j_\beta}\;\; c\;\; \overset{\frown}{k_\gamma}\;\; a} \quad\Rightarrow\quad C_{j_\beta k_\gamma}^{(bca)i_\alpha}\; C_{i_\alpha i_\alpha}^{(aba)1} = C_{i_\alpha j_\beta}^{(abc)k_\gamma}\; C_{k_\gamma k_\gamma}^{(aca)1} \tag{A.70}$$

Applying this to the r.h.s. of (5.24) we get:

$$\underline{a\;\; \overset{\frown}{i_\alpha}\;\; b\;\; \overset{\frown}{j_\beta}\;\; c\;\; \overset{\frown}{k_\gamma}\;\; d\;\; \overset{\frown}{\ell_\delta}\;\; a}$$

$$\Rightarrow \quad \sum_\varepsilon C_{j_\beta k_\gamma}^{(bcd)q_\varepsilon}\; C_{i_\alpha q_\varepsilon}^{(abd)\ell_\delta} = \sum_{p,\nu} C_{i_\alpha j_\beta}^{(abc)p_\nu}\; C_{p_\nu k_\gamma}^{(acd)\ell_\delta}\; \mathsf{F}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} \tag{A.71}$$

Note that there is no freedom the rescale the identity field 1 on any boundary. Its normalisation is already fixed by the condition that $1 \cdot \psi_{i_\alpha}^{(xy)} = \psi_{i_\alpha}^{(xy)}$, i.e. that $C_{1 i_\alpha}^{(xxy) j_\beta} = \delta_{i,j}\delta_{\alpha,\beta}$ and similar for $C_{i_\alpha 1}^{(xyy) j_\beta}$.

Define an operation '*' on the labels e, o, u as follows: ${x_e}^* = x_o$, ${x_o}^* = x_e$ and ${x_u}^* = x_u$, i.e. in terms of Kac-labels: $(r,s)^* = (r, q-s)$. Take $\underbrace{\omega\ \omega_o\ \omega\ x_i\ x\ x_i\ \omega\ \omega_o\ \omega}$ with $q{=}1$ and x any boundary condition. The sum on the r.h.s. of (A.71) reduces to $p_\nu = {x_i}^*$ because the representations x_i and ω_o can fuse only to ${x_i}^*$. We get:

$$C_{x_i x_i}^{(\omega x \omega) 1} = C_{\omega_o x_i}^{(\omega \omega x) {x_i}^*}\ C_{{x_i}^* x_i}^{(\omega x \omega)\omega_o}\ \mathsf{F}_{{x_i}^* 1}\begin{bmatrix} \omega_o & \omega_o \\ x_i & x_i \end{bmatrix} \neq 0 \tag{A.72}$$

The l.h.s. is nonzero due to assumption (b) and hence all terms on the r.h.s. have to be nonzero for a solution with properties (a) and (b).

In the same way, for an n-type boundary a and an arbitrary boundary x we can consider $\underbrace{\omega\ x_i\ x\ k_u\ a\ k_u\ x\ x_i\ \omega}$, again with $q{=}1$. The sum reduces to $p_\nu = a_u$ and we are left with:

$$C_{k_u k_u}^{(xax) 1} = C_{x_i k_u}^{(\omega x a) a_u}\ C_{a_u k_u}^{(\omega a x) x_i}\ \mathsf{F}_{a_u 1}\begin{bmatrix} x_i & x_i \\ k & k \end{bmatrix} \neq 0 \tag{A.73}$$

Again all terms on the r.h.s. have to be nonzero.

Mixed boundaries

Define the boundary condition label μ as $\mu{=}(1, \frac{q}{2})$, i.e. the first node of the A-diagram and the upper end node of the D-diagram in (6.6). Consider (A.71) with $\underbrace{\mu\ \mu_u\ \omega\ \omega_o\ \omega\ \omega_o\ \omega\ \mu_u\ \mu}$ and $q_\varepsilon{=}1$. The sum on the r.h.s. reduces to $p_\nu = \mu_u$ and we are left with:

$$C_{\omega_o \omega_o}^{(\omega\omega\omega) 1} = \big(C_{\mu_u \omega_o}^{(\mu\omega\omega)\mu_u}\big)^2\ \mathsf{F}_{\mu_u 1}\begin{bmatrix} \omega_o & \omega_o \\ \mu_u & \mu_u \end{bmatrix} \tag{A.74}$$

Using (A.70) in the form $\underbrace{\mu\ \mu_u\ \omega\ \omega_o\ \omega\ \mu_u\ \mu}$ implies $C_{\mu_u \omega_o}^{(\mu\omega\omega)\mu_u} = C_{\omega_o \mu_u}^{(\omega\omega\mu)\mu_u}$. We start by rescaling $\psi_{\omega_o}^{(\omega\omega)}$ such that:

$$C_{\omega_o \mu_u}^{(\omega\omega\mu)\mu_u} = 1 \tag{A.75}$$

From (A.74) it now follows that

$$C^{(\omega\omega\omega)1}_{\omega_o\omega_o} = A \qquad \text{where } A = \mathsf{F}_{\mu_u 1}\begin{bmatrix}\mu_u & \mu_u\\ \omega_o & \omega_o\end{bmatrix} \tag{A.76}$$

defining the constant A. We use the freedom to rescale $\psi^{(\omega x)}_{x_o}$ to fix $C^{(\omega\omega x)x_e}_{\omega_o x_o}=A$ for any i-type boundary x.

Let a be an n-type boundary. Taking $\underline{\ \omega\ \omega_o\ \omega\ \mu_u\ \mu\ \ell_u\ a\ a_u\ \omega\ }$ and $q_\varepsilon=a_u$ the sum reduces to $p_\nu=\mu_u$ and we see, using (A.75):

$$C^{(\omega\mu a)a_u}_{\mu_u\ell_u}\ C^{(\omega\omega a)a_u}_{\omega_o a_u} = C^{(\omega\mu a)a_u}_{\mu_u\ell_u}\,\mathsf{F}_{\mu_u a_u}\begin{bmatrix}\omega_o & a_u\\ \mu_u & \ell\end{bmatrix} \tag{A.77}$$

The representation ℓ_u lives on $\underline{\ \mu\ \ a\ }$ and the fusion $\mu_u, a_u \to \ell_u$ exists by construction of the field content. $C^{(\omega\mu a)a_u}_{\mu_u\ell_u}$ is nonzero (see (A.73)) and the F-matrix element has to be independent of the specific choice of ℓ_u we make. We denote this F-matrix element with B_a:

$$B_a = \mathsf{F}_{\mu_u a_u}\begin{bmatrix}\omega_o & a_u\\ \mu_u & \ell\end{bmatrix} \qquad \text{where } \underline{\ \mu\ \ell_u\ a\ } \tag{A.78}$$

Comparing to (A.77) we see $C^{(\omega\omega a)a_u}_{\omega_o a_u}=B_a$. It is also useful to define a constant C_x for any boundary x as follows:

$$x \text{ of i-type:}\quad C_x = \mathsf{F}_{x_e 1}\begin{bmatrix}\omega_o & \omega_o\\ x_o & x_o\end{bmatrix} \qquad x \text{ of n-type:}\quad C_x = \mathsf{F}_{x_u 1}\begin{bmatrix}\omega_o & \omega_o\\ x_u & x_u\end{bmatrix} \tag{A.79}$$

From (A.71) with $\underline{\ \omega\ \omega_o\ \omega\ \omega_o\ \omega\ a_u\ a\ a_u\ \omega\ }$, $q_\varepsilon=a_u$ and $p_\nu=1$ we get the following identity for any n-type boundary a:

$$C_a \cdot B_a = \frac{A}{B_a} \tag{A.80}$$

Let x and y be two boundaries s.t. one is of n-type. Then by rescaling $\psi^{(xy)}_{\ell_u}$ for $x\neq\omega$ and $\ell\neq 1$ we fix $C^{(\omega xy)y_\beta}_{x_\alpha\ell_u}=1$ where α, β stand for the labels e or u, as applicable for the present boundary conditions. Let x be an i-type boundary. We can use (A.71) to calculate the structure constants involving x_o instead of x_e:

$$C^{(\omega xa)a_u}_{x_o\ell_u} = \frac{A}{B_a}\mathsf{F}_{x_e a_u}\begin{bmatrix}\omega_o & a_u\\ x_o & \ell\end{bmatrix} \qquad C^{(\omega ax)x_o}_{a_u\ell_u} = \frac{B_a}{A}\mathsf{F}_{a_u x_o}\begin{bmatrix}\omega_o & x_e\\ a_u & \ell\end{bmatrix} \tag{A.81}$$

Using (A.71) and the identities in appendix A.5 one can verify that the definitions above give rise to symmetrical structure constants, i.e. $C^{(\omega xy)y_\gamma}_{x_\alpha \ell_u} = C^{(yx\omega)y_\gamma}_{\ell_u x_\alpha}$ where at least one of x, y is of n-type.

i-type boundaries

Let x, y be i-type boundaries. Consider the structure constant $C^{(\omega xy)y_\gamma}_{x_\alpha \ell_\beta}$ where the labels α, β, γ are either e or o (as u cannot occur in this situation). We will now try to regauge the boundary fields such that the following rule holds:

> The structure constant $C^{(\omega xy)y_\gamma}_{x_\alpha \ell_\beta}$ can be nonzero only if $\{\alpha, \beta, \gamma\}$ is one of the (unordered) sets $\{e, e, e\}$ or $\{e, o, o\}$.

Suppose that the representation ℓ on $\underline{\quad x \quad y \quad}$ occurs with multiplicity one. Recall from the analysis of the D-series boundary field content in section 6.1 that then the above rule is automatically true, since e.g. if ℓ gets label e then it occurs in the fusion of x_e, y_e or x_o, y_o but not in the fusion of x_e, y_o and x_o, y_e.

If the representation ℓ occurs with multiplicity two, we can use the freedom to form linear combinations of the two fields to make the above rule true. This is explained in more detail below.

Let ℓ be a representation that occurs with multiplicity two. Let $\psi^{(xy)}_{\ell_e}$ and $\psi^{(xy)}_{\ell_o}$ be the two fields. Recall that we normalised the two-point functions in such a way that only $\langle\, \psi^{(xy)}_{\ell_e}\, \psi^{(yx)}_{\ell_e} \rangle$ and $\langle\, \psi^{(xy)}_{\ell_o}\, \psi^{(yx)}_{\ell_o} \rangle$ are nonzero. When taking linear combinations of the primary fields one has to preserve this condition, or the form of the sewing constraints would change. One gets the following constraints:

$$\langle\, \psi^{(xy)}_{\ell_e}\, \psi^{(yx)}_{\ell_o} \rangle = 0 \qquad , \qquad \langle\, \psi^{(xy)}_{\ell_o}\, \psi^{(yx)}_{\ell_e} \rangle = 0 \tag{A.82}$$

Now consider the following change of basis:

$$\begin{pmatrix} \psi^{(xy)}_{\ell_e} \\ \psi^{(xy)}_{\ell_o} \end{pmatrix}_{\text{new}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi^{(xy)}_{\ell_e} \\ \psi^{(xy)}_{\ell_o} \end{pmatrix}_{\text{old}} \quad , \quad \begin{pmatrix} \psi^{(yx)}_{\ell_e} \\ \psi^{(yx)}_{\ell_o} \end{pmatrix}_{\text{new}} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \begin{pmatrix} \psi^{(yx)}_{\ell_e} \\ \psi^{(yx)}_{\ell_o} \end{pmatrix}_{\text{old}} \tag{A.83}$$

where the two matrices are invertible. The constraints (A.82) amount to:

$$a\tilde{c} \cdot C^{(xyx)1}_{\ell_e \ell_e, \text{old}} + b\tilde{d} \cdot C^{(xyx)1}_{\ell_o \ell_o, \text{old}} = 0 \qquad c\tilde{a} \cdot C^{(xyx)1}_{\ell_e \ell_e, \text{old}} + d\tilde{b} \cdot C^{(xyx)1}_{\ell_o \ell_o, \text{old}} = 0 \tag{A.84}$$

We see that we can choose the new basis of $\psi^{(xy)}$ arbitrarily and the above condition fixes the direction, but not the length of the new basis vectors for $\psi^{(yx)}$.

This change of basis can be used to bring the following 2×2–matrix composed of structure constants to diagonal form:

$$\begin{pmatrix} C^{(\omega xy)y_e}_{x_e\ell_e} & C^{(\omega xy)y_o}_{x_e\ell_e} \\ C^{(\omega xy)y_e}_{x_e\ell_o} & C^{(\omega xy)y_o}_{x_e\ell_o} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \tag{A.85}$$

We now redefined the structure constants $C^{(\omega xy)y_\gamma}_{x_\alpha\ell_\beta}$ in such a way that they obey our desired coupling relations $\{e,e,e\}$ or $\{e,o,o\}$.

The change of basis fixes the new $\psi^{(yx)}_{\ell_e}$ and $\psi^{(yx)}_{\ell_o}$ up to rescaling. However they do already obey the odd/even-coupling relation. To see this consider (A.71) in the form:

$$\begin{aligned} &\underbrace{\omega\ \overbrace{x_e\ x}\ \overbrace{\ell_e\ y}\ \overbrace{\ell_o\ x}\ \overbrace{x_e\ \omega}}\ , q_\varepsilon{=}1 \\ &\quad\Rightarrow\quad C^{(xyx)1}_{\ell_e\ell_o} = C^{(\omega xy)y_e}_{x_e\ell_e}\, C^{(\omega yx)x_e}_{y_e\ell_o}\, \mathsf{F}_{y_e 1}\begin{bmatrix} \ell & \ell \\ x_e & x_e \end{bmatrix} \end{aligned} \tag{A.86}$$

The l.h.s. of this equation is zero and on the r.h.s. both $C^{(\omega xy)y_e}_{x_e\ell_e}$ and the F-matrix entry are nonzero (this can be seen from evaluating $C^{(xyx)1}_{\ell_e\ell_e}{\neq}0$ instead of $C^{(xyx)1}_{\ell_e\ell_o}{=}0$ in (A.86)). Hence $C^{(\omega yx)x_e}_{y_e k_o}{=}0$ as we said.

From $C^{(\omega\omega x)x_e}_{\omega_o x_o}{=}A$ in the previous section we can determine $C^{(\omega\omega x)x_o}_{\omega_o x_e}$ by using (A.71) and (A.47):

$$\begin{aligned} &\underbrace{\omega\ \overbrace{\omega_o\,\omega}\,\overbrace{\omega_o\,\omega}\ \overbrace{x_e\ x}\ \overbrace{x_e\ \omega}}\ , q_\varepsilon{=}x_o \\ &\quad\Rightarrow\quad C^{(\omega\omega x)x_o}_{\omega_o x_e} = \frac{C^{(\omega\omega\omega)1}_{\omega_o\omega_o}}{C^{(\omega\omega x)x_e}_{\omega_o x_o}} \cdot \mathsf{F}_{1x_o}\begin{bmatrix} \omega_o & x_e \\ \omega_o & x_e \end{bmatrix} = \frac{1}{C_x} \end{aligned} \tag{A.87}$$

Next we fix the form of $C^{(\omega xy)y_\gamma}_{x_\alpha\ell_\beta}$. For $\ell{\neq}1$, $x{\neq}\omega$ we rescale $\psi^{(xy)}_{\ell_e}$ and $\psi^{(xy)}_{\ell_o}$ s.t.:

$$C^{(\omega xy)y_o}_{x_o\ell_e} = \mathsf{F}_{x_e y_o}\begin{bmatrix} \omega_o & y_e \\ x_o & \ell \end{bmatrix} \qquad C^{(\omega xy)y_o}_{x_e\ell_o} = \begin{cases} 1 & \text{if } x \le y \\ \mathsf{F}_{x_o y_o}\begin{bmatrix} \omega_o & y_e \\ x_e & \ell \end{bmatrix} & \text{if } x > y \end{cases} \tag{A.88}$$

One can verify that for $\ell{=}1$ indeed $C^{(\omega xx)x_o}_{x_o 1}{=}1$ and that for $x{=}\omega$ the above reduces to the expressions for $C^{(\omega\omega x)}$ given before.

As in the case of mixed boundaries one can now check that the definitions in (A.88) give rise to reflection symmetric structure constants, but this time there is an exception for $x{=}y$: In this case some of the structure constants are antisymmetric under reflection. As opposed to the A-series, for the D-series it is in general impossible to find a gauge in which all boundary structure constants have reflection

symmetry. To see this consider (A.71) in the form:

$$\begin{aligned}&\underbrace{\omega\ x_e\ x\ \ell_o\ x\ x_e\ \omega\ \omega_o\ \omega}\\ &\Rightarrow\quad C^{(xx\omega)x_o}_{\ell_o x_e}\ C^{(\omega x\omega)\omega_o}_{x_e x_o} = C^{(\omega xx)x_o}_{x_e\ell_o}\ C^{(\omega x\omega)\omega_o}_{x_o x_e}\,\mathsf{F}_{x_o x_o}\begin{bmatrix} x_e & \omega_o\\ \ell & x_e\end{bmatrix}\end{aligned} \tag{A.89}$$

Eqn. (A.46) forces the F-matrix entry in (A.89) to square to one. But in general it does take the values ±1, depending on ℓ_o. So no matter how we choose $C^{(\omega x\omega)\omega_o}_{x_e x_o}$ and $C^{(\omega x\omega)\omega_o}_{x_o x_e}$, in general we cannot avoid that if $C^{(\omega xx)x_o}_{x_e\ell_o}$ is symmetric for some values of ℓ_o, it will be antisymmetric for others.

One can verify that (A.88) implies that for even fields alone we get a solution that resembles the A-series, i.e. $C^{(\omega xy)y_e}_{x_e\ell_e}=1$, whereas the various structure constants involving odd fields are:

$$\begin{aligned}
\text{For } x\le y\text{:}\quad & C^{(\omega xy)y_o}_{x_e\ell_o} = 1\,; \quad C^{(\omega xy)y_o}_{x_o\ell_e} = \mathsf{F}_{x_e y_o}\begin{bmatrix}\omega_o & y_e\\ x_o & \ell\end{bmatrix};\\
& C^{(\omega xy)y_e}_{x_o\ell_o} = A\cdot\mathsf{F}_{x_e y_e}\begin{bmatrix}\omega_o & y_o\\ x_o & \ell\end{bmatrix}\cdot C_y\\
\text{For } x>y\text{:}\quad & C^{(\omega xy)y_o}_{x_e\ell_o} = \mathsf{F}_{x_o y_o}\begin{bmatrix}\omega_o & y_e\\ x_e & \ell\end{bmatrix}; \quad C^{(\omega xy)y_o}_{x_o\ell_e} = \mathsf{F}_{x_e y_o}\begin{bmatrix}\omega_o & y_e\\ x_o & \ell\end{bmatrix};\\
& C^{(\omega xy)y_e}_{x_o\ell_o} = A\cdot C_x
\end{aligned} \tag{A.90}$$

General boundary structure constants

Consider (A.71) in the form

$$\begin{aligned}&\underbrace{\omega\ x_\alpha\ x\ i_r\ y\ j_s\ z\ z_\gamma\ \omega}\ ,\, q_\varepsilon{=}k_t\\ &\Rightarrow\quad C^{(xyz)k_t}_{i_r j_s} = \sum_\beta \frac{C^{(\omega xy)y_\beta}_{x_\alpha i_r}\ C^{(\omega yz)z_\gamma}_{y_\beta j_s}}{C^{(\omega xz)z_\gamma}_{x_\alpha k_t}}\mathsf{F}_{y_\beta k}\begin{bmatrix}x_\alpha & z_\gamma\\ i & j\end{bmatrix}\end{aligned} \tag{A.91}$$

where the set $\{\alpha,t,\gamma\}$ has to be one of $\{e,e,e\}$, $\{e,o,o\}$ or $\{u,*,*\}$. Since all constants that can occur on the rhs have been computed in the previous two sections, the general boundary structure constants can be obtained from (A.91).

Suppose all boundaries are of i-type. Then choosing $\alpha{=}e$ and $\gamma{=}t$ reduces the sum in (A.91) to $\beta{=}r$. In the numerator on the rhs we now see the boundary structure constant $C^{(\omega yz)z_t}_{y_r j_s}$, which obeys the even/odd coupling rule as seen in the section dealing with i-type-boundaries. This implies that the even/odd coupling rule extends to all boundary structure constants, i.e. $C^{(xyz)k_t}_{i_r j_s}$ can be nonzero only if

$\{r,s,t\}$ is one of the sets $\{e,e,e\}$, $\{e,o,o\}$ or $\{u,*,*\}$.

In (A.91) we can distinguish two cases. First, for x,z of n-type and y of i-type the sum over β has to be carried out and we get:

$$C^{(xyz)k_u}_{i_u j_u} = \mathsf{F}_{y_e k}\begin{bmatrix} x_u & z_u \\ i & j \end{bmatrix} + \frac{B_x}{B_z}\cdot \mathsf{F}_{x_u y_o}\begin{bmatrix} y_e & \omega_o \\ i & x_u \end{bmatrix}\mathsf{F}_{y_e z_u}\begin{bmatrix} z_u & \omega_o \\ j & y_o \end{bmatrix}\mathsf{F}_{y_o k}\begin{bmatrix} x_u & z_u \\ i & j \end{bmatrix} \quad (A.92)$$

In all other cases the sum reduces to one term with the result:

$$C^{(xyz)k_t}_{i_r j_s} = \frac{C^{(\omega xy)y_\beta}_{x_\alpha i_r}\, C^{(\omega yz)z_\gamma}_{y_\beta j_s}}{C^{(\omega xz)z_\gamma}_{x_\alpha k_t}}\mathsf{F}_{y_\beta k}\begin{bmatrix} x_\alpha & z_\gamma \\ i & j \end{bmatrix} \quad (A.93)$$

The r.h.s. does not depend on the specific choice of α,β,γ as long as the combinations $\{\alpha,r,\beta\}$, $\{\beta,s,\gamma\}$ and $\{\alpha,t,\gamma\}$ are allowed by the even/odd coupling rule. The structure constant in the denominator is then automatically nonzero.

We can also investigate the behaviour of the boundary structure constants under reflection, i.e. given $C^{(xyz)k_t}_{i_r j_s}$, what is $C^{(zyx)k_t}_{j_s i_r}$? Using (A.70) we check that for $C^{(\omega xy)y_\gamma}_{x_\alpha \ell_\beta}$ reflection symmetry is equivalent to $\varepsilon{=}1$ in the following equation:

$$C^{(\omega xy)y_\gamma}_{x_\alpha \ell_\beta} = \varepsilon\cdot \frac{C^{(\omega x\omega)1}_{x_\alpha x_\alpha}}{C^{(\omega y\omega)1}_{y_\gamma y_\gamma}}\, C^{(\omega yx)x_\alpha}_{y_\gamma \ell_\beta} \quad (A.94)$$

As remarked in the previous two sections, using the explicit form of the structure constants derived there, we see that $\varepsilon{=}1$ in all cases except for one, and that is $\beta{=}o$ and $x{=}y$, where we can have $\varepsilon{=}\pm 1$. This can be made more precise in the light of (A.89), if we define:

$$\varepsilon(x,x,\ell_o) = \mathsf{F}_{x_o x_o}\begin{bmatrix} x_e & \omega_o \\ \ell & x_e \end{bmatrix} \quad \text{and} \quad \varepsilon(x,y,\ell_\beta) = 1 \text{ in all other cases} \quad (A.95)$$

Then (A.94) holds with $\varepsilon = \varepsilon(x,y,\ell_i)$ for all x,y,ℓ_i and we can use this relation to evaluate the reflection property of the general boundary structure constants obtained in (A.91). We get:

$$C^{(xyz)k_t}_{i_r j_s} = \frac{\varepsilon(x,y,i_r)\varepsilon(y,z,j_s)}{\varepsilon(x,z,k_t)}\cdot C^{(zyx)k_t}_{j_s i_r} \quad (A.96)$$

Note that the fact that some boundary structure constants are anti–symmetric under reflection is not a particularity of the gauge we chose. For $x{=}y$ in (A.94), one checks that rescaling, say, $\psi^{(\omega x)}_{x_\alpha}$ by a factor λ leads to $\varepsilon \to \lambda^2\cdot\varepsilon$. Altogether one cannot get rid of the sign and keep the structure constants real at the same time.

A.8 Calculations for the TCSA algorithm

Replacement rules for the bulk perturbation in TCSA

This section gives the calculation leading to the rules (7.22) and (7.23) to calculate matrix elements of the perturbing bulk fields.

It will be useful to work in polar coordinates $z = re^{i\theta}$:

$$\begin{aligned} r\tfrac{\partial}{\partial r} &= z\partial + z^*\bar\partial & z\partial &= \tfrac{1}{2}\big(r\tfrac{\partial}{\partial r} - i\tfrac{\partial}{\partial\theta}\big) \\ \tfrac{\partial}{\partial\theta} &= i\big(z\partial - z^*\bar\partial\big) & z^*\bar\partial &= \tfrac{1}{2}\big(r\tfrac{\partial}{\partial r} + i\tfrac{\partial}{\partial\theta}\big) \end{aligned} \tag{A.97}$$

When working out the commutator $[L_n, \phi(z, z^*)]$ one has to remember that on the UHP one gets a contribution from both, z and z^*:

$$[L_n, \phi(z, z^*)] = \big(h(n+1)(z^n + (z^*)^n) + z^{n+1}\partial + (z^*)^{n+1}\bar\partial\big)\phi(z, z^*) \ . \tag{A.98}$$

Recall that we have assumed that $h_\phi = \bar h_\phi$. In polar coordinates the commutator reads

$$[L_n, \phi(r,\theta)] = r^n\big(2(n+1)h_\phi \cos n\theta + \cos n\theta \cdot r\tfrac{\partial}{\partial r} + \sin n\theta \cdot \tfrac{\partial}{\partial\theta}\big)\phi(r,\theta) \ . \tag{A.99}$$

In particular, for $r{=}\,1$ we have

$$[L_n - \cos(n\theta)L_0, \phi(1,\theta)] = \big(2nh_\phi \cos(n\theta) + \sin(n\theta)\tfrac{\partial}{\partial\theta}\big)\phi(1,\theta) \ . \tag{A.100}$$

Consider two arbitrary states $|\alpha\rangle, |\beta\rangle \in \mathcal{H}_{ab}$. Define the numbers $|\alpha|$ and $|\beta|$ via $L_0|\alpha\rangle = |\alpha|\cdot|\alpha\rangle$ and $L_0|\beta\rangle = |\beta|\cdot|\beta\rangle$. Then partial integration[1] yields the formula

$$\int_0^\pi d\theta \langle\alpha|[L_n, \phi(1,\theta)]|\beta\rangle = \big(n(2h_\phi{-}1) + |\alpha| - |\beta|\big)\int_0^\pi d\theta \cos(n\theta)\langle\alpha|\phi(1,\theta)|\beta\rangle \ . \tag{A.101}$$

Note that the integral involving the $\cos(n\theta)$ term is the same for n and $-n$. Elimi-

[1] No boundary terms can appear in the partial integration if the integral is finite to start with. To illustrate the reason consider the right boundary. For $\theta \to 0$, the correlator $\langle i|\phi(1,\theta)|j\rangle$ will have leading behaviour $const\cdot\theta^\nu$. If the integral is finite, then $\nu > -1$. The boundary term would have the form $\sin(n\theta)\langle i|\phi(1,\theta)|j\rangle$, with asymptotics $const\cdot\theta^{\nu+1}$. Thus the boundary terms vanish.

nating the integral leads to our first replacement rule

$$\begin{aligned}&\int_0^\pi d\theta\langle\alpha|L_n\phi(e^{i\theta},e^{-i\theta})|\beta\rangle\\&=\tfrac{n(2h_\phi-1)+|\alpha|-|\beta|}{n(1-2h_\phi)+|\alpha|-|\beta|}\Big(\int_0^\pi\langle\alpha|L_{-n}\phi|\beta\rangle-\int_0^\pi\langle\alpha|\phi L_{-n}|\beta\rangle\Big)+\int_0^\pi\langle\alpha|\phi L_n|\beta\rangle\ .\end{aligned}\tag{A.102}$$

Note that the denominator in (A.102) can become singular for certain combinations of $n, h_\phi, |\alpha|, |\beta|$. From (A.101) we see that this occurs if $\int\langle\alpha|[L_{-n},\phi]|\beta\rangle=0$. As mentioned in the main text, in this case, if $n\neq2$ we can replace $L_n=\frac{1}{n-2}(L_{n-1}L_1-L_1L_{n-1})$ and if $n=2$ (which can only happen if $4h_\phi$ is an integer), one can resort to (A.101).

Eqn. (A.102) reduces the general element (7.21) to a sum of terms of the form $\int_0^\pi\langle i|\phi(e^{i\theta},e^{-i\theta})L_{-n_1}\dots L_{-n_m}|j\rangle$. To annihilate the L_{-n}'s on the left we use the commutator (A.100) to obtain the rule

$$\begin{aligned}&\int_0^\pi f(\theta)\langle i|\phi(e^{i\theta},e^{-i\theta})L_{-n}|\alpha\rangle\\&=\big(n(2h_\phi-1)+|\alpha|-h_i\big)\int_0^\pi\cos(n\theta)f(\theta)\langle i|\phi|\alpha\rangle-\int_0^\pi\sin(n\theta)f'(\theta)\langle i|\phi|\alpha\rangle\ .\end{aligned}\tag{A.103}$$

where $|i\rangle$ is a highest weight state and $f(\theta)$ is some function of θ.

By now we have brought (7.21) to the form $\int f(\theta)\langle i|\phi|j\rangle$. We still have to do one integration per matrix element of H^{bulk}. As it turns out the differential equations $\langle i|\phi|j\rangle$ fulfil allow us to set up a recursion for these integrals.

Recursion relation for bulk integrals

We will derive the differential equation associated to the null state of the perturbing field $\phi(z,z^*)$. For example, if ϕ has Kac-labels $(1,2)$, then $((L_{-2}-\frac{3}{2(2h+1)}L_{-1}L_{-1}).\phi)(z,z^*)=0$. As before, the notation $(L_{-n}.\chi)(z,z^*)$ stands for $[L_{-n}(z),\chi(z,z^*)]$, i.e. $(\zeta-z)^{n+1}T(\zeta)$ is integrated around a small circle containing z, but no other fields (in particular not z^*).

Suppose the field $\chi(z,z^*)$ is of the form $(L_{-n_1}\dots L_{-n_m}.\phi)(z,z^*)$. Then by contour deformation one can demonstrate the replacement rule

$$\begin{aligned}\langle i|(L_n.\chi)(z,z^*)|j\rangle=&(-1)^{n+1}\Big((n+1)\big(h_jz^n+h_\phi(z-z^*)^n\big)\\&+z^{n+1}(\partial+\bar\partial)-(z-z^*)^{n+1}\bar\partial\Big)\langle i|\chi(z,z^*)|j\rangle\ .\end{aligned}\tag{A.104}$$

Applying this several times allows us to turn a null-state condition into a differential

equation. With (A.97) the complex derivatives can be turned into ∂_θ and ∂_r. To get rid of the radial derivatives one can evaluate $\langle i|(L_0-h_i)\chi(z,z^*)|j\rangle=0$ with the result

$$r\tfrac{\partial}{\partial r}\langle i|\chi(r,\theta)|j\rangle = \big(h_i - h_j - h(\chi) - \bar{h}(\chi)\big)\langle i|\chi(r,\theta)|j\rangle \;, \tag{A.105}$$

where $h(\chi)$ and $\bar{h}(\chi)$ are the left/right conformal weights of the field $\chi(z,z^*)$. With this rule and $z=e^{i\theta}$ we can rewrite (A.104) as

$$\begin{aligned}\langle i|(L_n.\chi)(z,z^*)|j\rangle = (-1)^{n+1}\Big((n+1)\big(h_j e^{in\theta} + h_\phi(2i\sin\theta)^n\big) \\ + \tfrac{1}{2}\big(h_i-h_j-h(\chi)-\bar{h}(\chi)\big)\big(e^{in\theta} + e^{i(n+2)\theta} - e^{i\theta}(2i\sin\theta)^{n+1}\big) \\ + \tfrac{i}{2}\big(e^{i(n+2)\theta} - e^{in\theta} - e^{i\theta}(2i\sin\theta)^{n+1}\big)\tfrac{\partial}{\partial\theta}\Big)\langle i|\chi(z,z^*)|j\rangle \;.\end{aligned} \tag{A.106}$$

We end up with a linear differential equation in θ:

$$\mathcal{D}_\theta\langle i|\phi(1,\theta)|j\rangle = 0 \tag{A.107}$$

Since the correlator is a real function, this equation actually corresponds to two differential equations, resulting from taking only the real or imaginary part of $\mathcal{D}_\theta$. This is not too surprising, since alternatively we could have considered the additional null-state on the anti-holomorphic part of $\phi(z,z^*)$.

Inserting (A.107) into the integral $\int_0^\pi f(\theta)\langle i|\phi(1,\theta)|j\rangle d\theta$ and shifting the derivatives to $f(\theta)$ using partial integration gives a relation of the form

$$\int_0^\pi \big(\tilde{\mathcal{D}}_\theta f(\theta)\big)\langle i|\phi(1,\theta)|j\rangle d\theta = 0 \;. \tag{A.108}$$

The rule (A.103) only generates $\sin(n\theta)$ and $\cos(n\theta)$ terms. It is thus possible to express the function $f(\theta)$ as a finite linear combination of $(\sin\theta)^n$ and $\cos\theta(\sin\theta)^n$. Let S_n and C_n be defined as

$$S_n := \int_0^\pi (\sin\theta)^n\langle i|\phi(1,\theta)|j\rangle d\theta \;, \qquad C_n := \int_0^\pi \cos\theta\,(\sin\theta)^n\langle i|\phi(1,\theta)|j\rangle d\theta \tag{A.109}$$

Evaluating condition (A.108) for these expressions gives the desired recursion for matrix elements. In the following we list the resulting relation for the level two null state.

If the perturbing bulk field $\phi(z,z^*)$ has Kac-labels $(1,2)$ or $(2,1)$ then it has a

level two null state of the form

$$\Big(L_{-2} - \frac{3}{2(2h_\phi+1)}\Big)|\phi\rangle = 0\ . \tag{A.110}$$

The real and imaginary part of the second order differential equation are a bit cumbersome, at least when worked out with Mathematica. It turns out, though, that they are equivalent if $h_i{=}h_j$ and can be combined to a first order equation if $h_i{\neq}h_j$:

$$\begin{aligned} h_i{=}h_j{=}h:\quad & \Big\{\tfrac{3}{4(1+2h_\phi)}\sin\theta\cdot\partial_\theta\partial_\theta - \tfrac{1}{2}\cos\theta\cdot\partial_\theta + \tfrac{h_\phi}{2}(\sin\theta)^{-1} \\ & \qquad - \big(\tfrac{h_\phi^2}{1+2h_\phi} - 2h\big)\sin\theta\Big\}\ \langle h|\phi(1,\theta)|h\rangle = 0 \end{aligned} \tag{A.111}$$

$$h_i \neq h_j:\quad \big\{3\sin\theta\cdot\partial_\theta - (1{-}2h_\phi)\cos\theta\big\}\ \langle i|\phi(1,\theta)|j\rangle = 0 \tag{A.112}$$

Equation (A.112) can of course easily be integrated, with the result, for some constant C,

$$h_i \neq h_j:\quad \langle i|\phi(1,\theta)|j\rangle = C\cdot(2\sin\theta)^{\frac{1+2h_\phi}{3}}\ . \tag{A.113}$$

Note that if $h_\phi{=}h_{12}$ we have $(1+2h_\phi)/3 = h_{13} - 2h_{12}$ and similarly for $h_\phi{=}h_{21}$. Using that for $a > -1$ we have $\int_0^\pi(\sin\theta)^a{=}\sqrt{\pi}\Gamma(\frac{1}{2}+\frac{a}{2})/\Gamma(1+\frac{a}{2})$ the expressions S_n and C_n can be given explicitly as

$$S_n = C\cdot 2^{\frac{1+2h_\phi}{3}}\sqrt{\pi}\frac{\Gamma(\frac{2h_\phi+3n+4}{6})}{\Gamma(\frac{2h_\phi+3n+7}{6})}\ , \qquad C_n = 0\ . \tag{A.114}$$

For (A.111), after some algebra (done with Mathematica), we find the recursion relations

$$\begin{aligned} S_n &= \frac{(2h_\phi{-}n{+}1)(2h_\phi{+}3n{-}2)}{4h_\phi(h_\phi{+}4h{+}n{-}1)+8h-n(3n{-}2)}\cdot S_{n-2}\ , \\ C_n &= \frac{(2h_\phi{-}n{+}1)(2h_\phi{+}3n{-}2)}{4h_\phi(h_\phi{+}4h{+}n)+8h-1-n(3n{+}4)}\cdot C_{n-2}\ . \end{aligned} \tag{A.115}$$

Since (A.111) originates from a level 2 null state it can also be integrated. With $h{=}\frac{1}{4t}\big(d^2{-}(1{-}t)^2\big)$ and $t{=}p/q$ in $M(p,q)$ (recall that $h_{12}{=}\frac{3}{4}t-\frac{1}{2}$ and $h_{13}{=}2t{-}1$) we find the two solutions to be

$$\begin{aligned} f^{11}(\theta) &= (2\sin\theta)^{1-\frac{3}{2}t}(\cos\theta)^{t-d-1}\cdot{}_2F_1\big(\tfrac{1+d-t}{2},\tfrac{2+d-t}{2}\,;\,\tfrac{3-2t}{2}\,;\,-(\tan\theta)^2\big) \\ f^{13}(\theta) &= (2\sin\theta)^{\frac{1}{2}t}(\cos\theta)^{-t-d}\cdot{}_2F_1\big(\tfrac{d+t}{2},\tfrac{d+t+1}{2}\,;\,\tfrac{2t+1}{2}\,;\,-(\tan\theta)^2\big)\ . \end{aligned} \tag{A.116}$$

These solutions are only valid for $\theta\in]0,\pi/2[$, i.e. as yet they give only half of the block. Quite often one has the situation where the left boundary is fixed to be $\mathbb{1}$. In this case the field ϕ can only couple to the identity on the left boundary. Hence for $\theta\in]\pi/2,\pi[$ we have $f^{11}(\pi-\theta)$. From there we can use analytic continuation of blocks to get

$$f(\theta) = \begin{cases} f^{11}(\pi-\theta) & ;\pi/2 \leq \theta < \pi \\ A_{11}\, f^{11}(\theta) + A_{13}\, f^{13}(\theta) & ;0 < \theta < \pi/2 \end{cases}$$
$$A_q = \sum_p \mathsf{F}_{1p} \begin{bmatrix} h & \phi \\ h & \phi \end{bmatrix} e^{2\pi i(h+h_\phi-\frac{1}{4}h_q-h_p)} \mathsf{F}_{pq} \begin{bmatrix} \phi & \phi \\ h & h \end{bmatrix} \tag{A.117}$$

Since this is the unique block with asymptotics $(2\varepsilon)^{-2h_\phi}$ for $\theta=\pi-\varepsilon$, it is proportional to the correlator

$$\langle\, \psi_h^{(1a)} |\phi_{12}(1,\theta)|\, \psi_h^{(a1)} \rangle = \tilde{C}\cdot f(\theta)\ . \tag{A.118}$$

The real gain of the recursive method comes from the cases where the exact function for the correlator is not available for numerical integration. In this case one has to resort to more complicated (and slower) methods to finding the conformal blocks, like numerically solving the differential equations. Using the recursion it is enough to compute the first few S_n and C_n to high accuracy by integrating a sufficiently good numerical solution.

Bibliography

[YBk] Ph. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory*, Springer 1998.

[AGS89a] L. Alvarez-Gaume, G. Sierra and C. Gomez, *Topics In Conformal Field Theory*, In Brink, L. (ed.) et al.: Physics and mathematics of strings 16–184 (1989).

[AGS89b] L. Alvarez-Gaumé, C. Gomez, G. Sierra, *Quantum group interpretation of some conformal field theories*, Phys. Lett. B220 (1989) 142–152.

[ALu91] I. Affleck, A.W.W. Ludwig, *Universal Noninteger "Ground-State Degeneracy" in Critical Quantum Systems*, Phys. Rev. Lett. 67 (1991) 161–164.

[ARS99] A. Y. Alekseev, A. Recknagel, V. Schomerus, *Non-commutative world-volume geometries: Branes on SU(2) and fuzzy spheres*, JHEP 9909:023 (1999), `hep-th/9908040`.

[BLZ96] V. V. Bazhanov, S. L. Lukyanov, A. B. Zamolodchikov, *Quantum field theories in finite volume: Excited state energies*, Nucl. Phys. B489 (1997) 487–531, `hep-th/9607099`.

[Bow91] P. Bowcock, *Quasi-primary fields and associativity of chiral algebras*, Nucl. Phys. B356 (1991) 367–385.

[BPPZ98] R.E. Behrend, P.A. Pearce, V.B. Petkova, J.-B. Zuber, *On the Classification of Bulk and Boundary Conformal Field Theories*, Phys. Lett. B444 (1998) 163–166, `hep-th/9809097`.

[BPPZ99] R.E. Behrend, P.A. Pearce, V.B. Petkova, J.-B. Zuber, *Boundary Conditions in Rational Conformal Field Theories*, Nucl. Phys. B579 (2000) 707–773, `hep-th/9908036`.

[BPZ84] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. B241 (1984) 333–380.

[BPZ98] R.E. Behrend, P.A. Pearce, J.-B. Zuber, *Integrable boundaries, conformal boundary conditions and A-D-E fusion rules*, J. Phys. A31 (1998) L763–L770, hep-th/9807142.

[BSc92] P. Bouwknegt, K. Schoutens, *W-symmetry in conformal field theory*, Phys. Rept. 223 (1993) 183–276, hep-th/9210010.

[Car84] J.L. Cardy, *Conformal invariance and surface critical behavior*, Nucl. Phys. B240 (1984) 514–532.

[Car86] J.L. Cardy, *Operator Content Of Two-Dimensional Conformally Invariant Theories*, Nucl. Phys. B270 (1986) 186–204.

[Car89a] J.L. Cardy, *Conformal Invariance And Statistical Mechanics*, Published in Les Houches Summer School 1988, 169–246.

[Car89b] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, Nucl. Phys. B324 (1989) 581–596.

[Car98] J.L. Cardy, *Critical Exponents near a Random Fractal Boundary*, cond-mat/9812416.

[CIZ87] A. Cappelli, C. Itzykson, J.-B. Zuber, *Modular invariant partition functions in two dimensions*, Nucl. Phys. B280 (1987) 445–465.

[CLe91] J.L. Cardy, D.C. Lewellen, *Bulk and boundary operators in conformal field theory*, Phys. Lett. B259 (1991) 274–278.

[Cor96] E. Corrigan, *Integrable field theory with boundary conditions*, hep-th/9612138.

[CPe88] J.L. Cardy, I. Peschel, *Finite size dependence of the free energy in two-dimensional critical systems*, Nucl. Phys. B300 (1988) 377–399.

[DFa84] Vl.S. Dotsenko, V.A. Fateev, *Conformal algebra and multipoint correlation functions in 2d statistical models*, Nucl. Phys. B240 (1984) 312–348.

[DFa85] Vl.S. Dotsenko, V.A. Fateev, *Four-point correlation functions and the operator algebra in 2d conformal invariant theories with central charge $c\leq 1$*, Nucl. Phys. B251 (1985) 691–734.

[DGa99] G.W. Delius and G.M. Gandenberger, *Particle reflection amplitudes in a(n)(1) Toda field theories*, Nucl. Phys. B554 (1999) 325–364, hep-th/9904002.

[Dor98] P. Dorey, *Exact S Matrices*, hep-th/9810026.

[DPTW97] P. Dorey, A. Pocklington, R. Tateo, G. Watts, *TBA and TCSA with boundaries and excited states*, Nucl. Phys. B525 (1998) 641–663, hep-th/9712197.

[DRTW99] P. Dorey, I. Runkel, R. Tateo and G. Watts, *g-function flow in perturbed boundary conformal field theories*, Nucl. Phys. B578 (2000) 85–122, hep-th/9909216.

[DTa96] P. Dorey, R. Tateo, *Excited states by analytic continuation of TBA equations*, Nucl. Phys. B482 (1996) 639–659, hep-th/9607167.

[DTa97] P. Dorey, R. Tateo, *Excited states in some simple perturbed conformal field theories*, Nucl. Phys. B515 (1998) 575–623, hep-th/9706140.

[EFHHNV92] W. Eholzer, M. Flohr, A. Honecker, R. Hubel, W. Nahm, R. Varnhagen, *Representations of W-algebras with two generators and new rational models*, Nucl. Phys. B383 (1992) 249–290.

[FFFS99] G. Felder, J. Frohlich, J. Fuchs, C. Schweigert, *Correlation functions and boundary conditions in RCFT and three-dimensional topology*, hep-th/9912239.

[FFK89] G. Felder, J. Fröhlich, G. Keller, *Braid matrices and structure constants for minimal conformal models*, Comm. Math. Phys. 124 (1989) 647–664.

[FFr93] B. Feigin and E. Frenkel, *Integrals of motion and quantum groups*, Proc. CIME Summer School on Integrable Systems and Quantum Groups (1993), hep-th/9310022.

[FFZ00] V. Fateev, A. Zamolodchikov, Al. Zamolodchikov, *Boundary Liouville Field Theory I. Boundary State and Boundary Two-point Function*, hep-th/0001012.

[FGP90] P. Furlan, A.Ch. Ganchev, V.B. Petkova, *Fusion matrices and c<1 (quasi) local conformal field theories*, Int. J. Mod. Phys. A5 (1990) 2721–2735.

[FKl89] J. Fuchs, A. Klemm, *The computation of the operator algebra in nondiagonal conformal field theories*, Annals Phys. 194 (1989) 303.

[FPS93] D. Fioravanti, G. Pradisi, A. Sagnotti, *Sewing constraints and nonorientable open strings*, Phys. Lett. B321 (1994) 349–354, hep-th/9311183.

[FSc97a] J. Fuchs, C. Schweigert, *A Classifying algebra for boundary conditions*, Phys. Lett. B414 (1997) 251–259, hep-th/9708141.

[FSc97b] J. Fuchs, C. Schweigert, *Branes: From free fields to general backgrounds*, Nucl. Phys. B530 (1998) 99–136, hep-th/9712257.

[FSc99a] J. Fuchs, C. Schweigert, *Symmetry breaking boundaries. 1. General theory*, Nucl. Phys. B558 (1999) 419–483, hep-th/9902132.

[FSc99b] J. Fuchs, C. Schweigert, *Symmetry breaking boundaries. 2. More structures: Examples*, Nucl. Phys. B568 (2000) 543–593, hep-th/9908025.

[FSc00] J. Fuchs, C. Schweigert, *Solitonic sectors, alpha-induction and symmetry breaking boundaries*, hep-th/0006181.

[Gab94] M. R. Gaberdiel, *A General transformation formula for conformal fields*, Phys. Lett. B325 (1994) 366–370, hep-th/9401166.

[Gab99] M. R. Gaberdiel, *An introduction to conformal field theory*, Rept. Prog. Phys. 63 (2000) 607–667, hep-th/9910156.

[Gin88] P. Ginsparg, *Applied Conformal Field Theory*, Published in Les Houches Summer School 1988, 1–168.

[GRWip] K. Graham, I. Runkel, G. Watts, *Minimal model boundary flows and c=1 CFT*, in preparation.

[Gur93] V. Gurarie, *Logarithmic operators in conformal field theory* Nucl. Phys. B410 (1993) 535–549, hep-th/9303160.

[Ish89] N. Ishibashi, *The Boundary And Crosscap States In Conformal Field Theories*, Mod. Phys. Lett. A4 (1989) 251.

[KWa91] H. Kausch, G. Watts, *A Study of W algebras using Jacobi identities*, Nucl. Phys. B354 (1991) 740–768.

[Lew92] D.C. Lewellen, *Sewing constraints for conformal field theories on surfaces with boundaries*, Nucl. Phys. B372 (1992) 654–682.

[LMSS95] A. LeClair, G. Mussardo, H. Saleur and S. Skorik, *Boundary energy and boundary states in integrable quantum field theories*, Nucl. Phys. B453 (1995) 581–618, hep-th/9503227.

[MSb89a] G. Moore, N. Seiberg, *Lectures On RCFT*, Presented at Trieste Spring School 1989, Physics, Geometry, and Topology, Plenum Press, New York, 1990.

[MSb89b] G. Moore, N. Seiberg, *Classical And Quantum Conformal Field Theory*, Commun. Math. Phys. 123 (1989) 177.

[MSb89c] G. Moore, N. Seiberg, *Naturality in Conformal Field Theory*, Nucl. Phys. B313 (1989) 16–40.

[Pas87] V. Pasquier, *Operator content of the ADE lattice models*, J. Phys. A20 (1987) 5707–5733.

[Pet89] V.B. Petkova, *Structure constants of the (A,D) minimal c<1 conformal models*, Phys. Lett. B225 (1989) 357–362

[PSS95a] G. Pradisi, A. Sagnotti, Ya. S. Stanev, *Planar duality in SU(2) WZW models* Phys. Lett. B354 (1995) 279–286, hep-th/9503207.

[PSS95b] G. Pradisi, A. Sagnotti, Ya. S. Stanev, *The open descendants of nondiagonal SU(2) WZW models* Phys. Lett. B356 (1995) 230–238, hep-th/9506014.

[PSS96] G. Pradisi, A. Sagnotti, Ya. S. Stanev, *Completeness conditions for boundary operators in 2-D conformal field theory*, Phys. Lett. B381 (1996) 97–104, hep-th/9603097.

[PTe99] B. Ponsot, J. Teschner, *Liouville bootstrap via harmonic analysis on a noncompact quantum group*, hep-th/9911110.

[PZu94] V.B. Petkova, J.-B. Zuber, *On structure constants of sl(2) theories*, Nucl. Phys. B438 (1995) 347–372, hep-th/9410209.

[RRS00] A. Recknagel, D. Roggenkamp, V. Schomerus, *On relevant boundary perturbations of unitary minimal models*, hep-th/0003110.

[RSc97] A. Recknagel, V. Schomerus, *D-branes in Gepner models*, Nucl. Phys. B531 (1998) 185–225 hep-th/9712186.

[Rue99] P. Ruelle, *Symmetric boundary conditions in boundary critical phenomena*, J. Phys. A32 (1999) 8831–8850, hep-th/9904100.

[Run98] I. Runkel, *Boundary structure constants for the A-series Virasoro minimal models*, Nucl. Phys. B549 (1999) 563–578, hep-th/9811178

[Run99] I. Runkel, *Boundary structure constants for the D-series Virasoro minimal models*, Nucl. Phys. B579 (2000) 561–589, hep-th/9908046.

[Son88b] H. Sonoda, *Sewing Conformal Field Theories. 2*, Nucl. Phys. B311 (1988) 417.

[YZa90] V.P. Yurov, A.B. Zamolodchikov, *Truncated conformal space approach to scaling Lee-Yang model*, Int. J. Mod. Phys. A5 (1990) 3221–3246.

[Zam84] A.B. Zamolodchikov, *Conformal Symmetry In Two-Dimensions: An Explicit Recurrence Formula For The Conformal Partial Wave Amplitude*, Commun. Math. Phys. 96 (1984) 419–422.

[Zam86] A. B. Zamolodchikov, *"Irreversibility" Of The Flux Of The Renormalization Group In A 2-D Field Theory*, JETP Lett. 43 (1986) 730–732, Pisma Zh. Eksp. Teor. Fiz. 43 (1986) 565–567.

[Zam90] Al. B. Zamolodchikov, *Thermodynamic Bethe Ansatz In Relativistic Models. Scaling Three State Potts And Lee-Yang Models*, Nucl. Phys. B342 (1990) 695–720.

[Zam95] Al. B. Zamolodchikov, *Mass scale in the sine-Gordon model and its reductions*, Int. J. Mod. Phys. A10 (1995) 1125–1150.

[prN] R. Nepomechie, private communication.

[prP] V.B. Petkova, private communication.

[prW] G. Watts, private communication.