Khovanov homology and invariants of 4-manifolds

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MPIM / Uni Bonn

Joint work with Scott Morrison and Kevin Walker

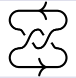
Hamburg, 2nd February 2021

Plan

1. Khovanov's categorification of the Jones polynomial
2. Link homologies
3. Break
4. Towards 4-manifold invariants

The Jones polynomial

Algorithm (The Jones polynomial)

- Input: a diagram of an oriented knot or link, for example: 
- Step 1: count $n_+ = \#($$)$ and $n_- = \#($$)$
- Step 2: rewrite  $\mapsto$) ($- q$  and ○ $\mapsto q + q^{-1}$
- Output: the coefficient of the empty diagram times $(-1)^{n_-} q^{n_+ - 2n_-}$

Theorem

Output only depends on the knot or link, not on the diagram.

Open Problem

Is the unknot ○ the only knot with Jones polynomial $q + q^{-1}$?

Generalization

The $\mathfrak{gl}_N$ quantum link polynomials $P_N \colon \{\text{framed oriented links}\} \to \mathbb{Z}[q^{\pm 1}]$.

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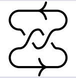
- Input: a diagram of an oriented knot or link, for example: 
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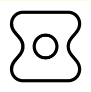
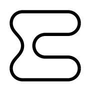
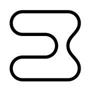
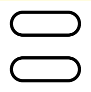


$-q$ 

$-q$ 

$(-q)^2$ 

$$= \left[(q+q^{-1})^2 \quad -2q(q+q^{-1}) \ +q^2(q+q^{-1})^2 \right]$$

Output: $1 + q^2 + q^4 + q^6$

$q^{\pm 1}]$.

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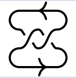
- Input: a diagram of an oriented knot or link, for example: 
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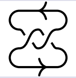
- Input: a diagram of an oriented knot or link, for example: 
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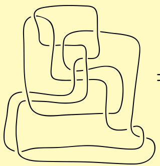

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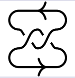
- Input: a diagram of an oriented knot or link, for example: 
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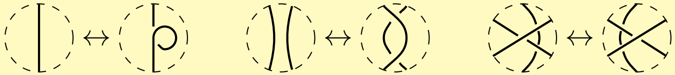
General

The $\mathfrak{gl}_N$].

Two link diagrams represent the same link if and only if they are related by a finite sequence of Reidemeister moves.



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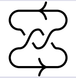
- Input: a diagram of an oriented knot or link, for example: 
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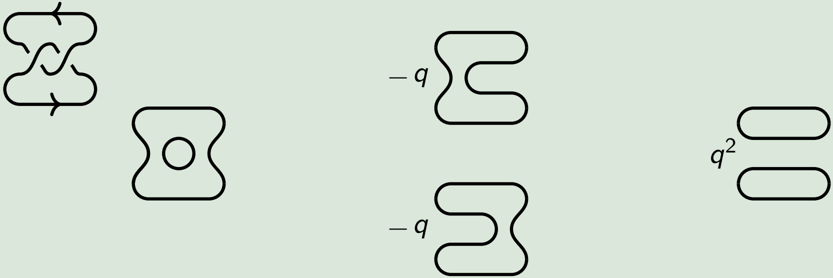
Khovanov's categorification of the Jones polynomial

Theorem (Khovanov 1999)

The Jones polynomial is the Euler characteristic of a link homology theory.

Example (The Khovanov homology of the Hopf link)



Consider the edges in the cube of resolutions

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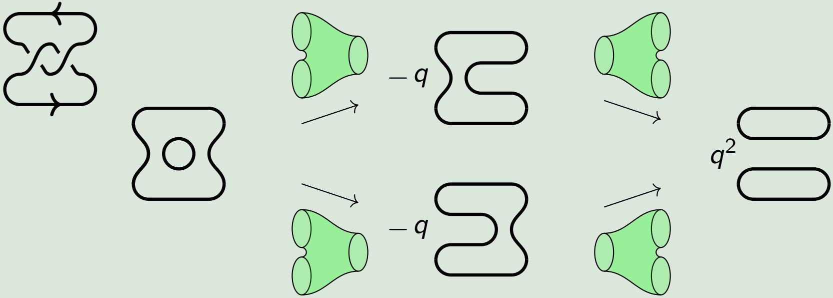

A commutative diagram of 1-manifolds and bordisms

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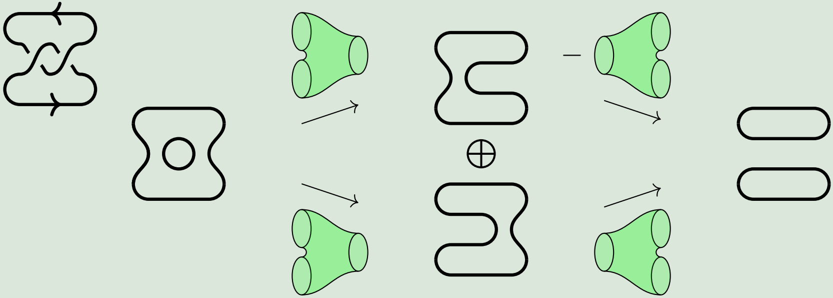

A chain complex of 1-manifolds and bordisms

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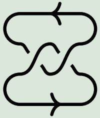

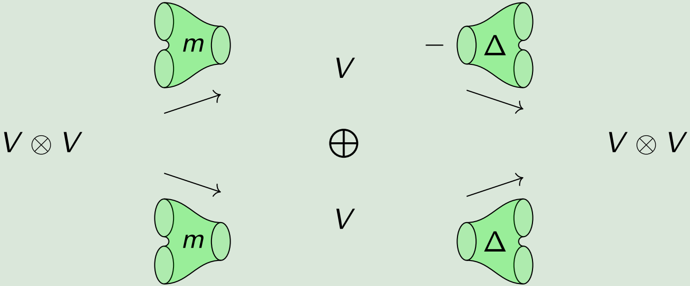

Apply TQFT corresponding to $V = H^*(\mathbb{C}P^1) \cong \mathbb{Z}[x]/\langle x^2 \rangle$

Take $H_* \implies$ Khovanov homology of the Hopf link.

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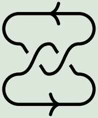

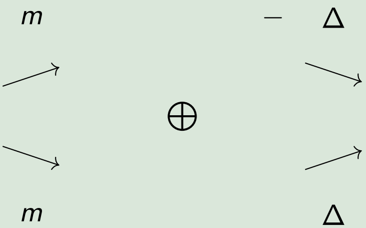

Let's compute homology...

Take $H_* \implies$ Khovanov homology of the Hopf link.

Khovanov homology and its cousins

Theorem (Khovanov 1999)

These bigraded abelian groups do not depend on the choice of link diagram (up to isomorphism) but only on the underlying link.

A family of link homologies:

$$\mathrm{Kh}(\bigcirc) \cong H^*(\mathbb{CP}^1)$$

2004: Khovanov–Rozansky homologies categorify the $\mathfrak{gl}_N$ knot polynomials.

$$\mathrm{KhR}^N(\bigcirc) \cong H^*(\mathbb{CP}^{N-1})$$

2008: Wu and Yonezawa's colored Khovanov–Rozansky homologies categorify $\wedge^k V$-colored $\mathfrak{gl}_N$ knot polynomials.

$$\mathrm{KhR}^N(\bigcirc^k) \cong H^*(\mathrm{Gr}(k, N))$$

Link homologies appear in many parts of mathematics

1. matrix factorization categories, Khovanov–Rozansky, et.al..
2. Lie theory, via category $\mathcal{O}$, Mazorchuk–Stroppel, Sussan.
3. combinatorial TQFT, foam categories, Bar-Natan, Khovanov, et.al.
4. algebraic geometry, $D^b Coh$, affine Grassmannian, Cautis–Kamnitzer, Hilbert schemes Oblomkov–Rasmussen–Shende
5. higher representation theory Khovanov–Lauda, Rouquier, Webster
6. singular Soergel bimodules, Rouquier, Khovanov, Williamson
7. symplectic geometry, via Floer homology, Seidel–Smith, Manolescu, Abouzaid, Dowlin
8. algebraic combinatorics, shuffle conjectures, Carlsson–Mellit
9. string theory, Gromov-Witten theory, Gukov–Schwarz–Vafa, Aganagic–Ekholm–Ng–Vafa, Witten
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> Link homologies appear in these contexts due to actions of categorified quantum groups.
> $\Longrightarrow$ *higher representation theory*

Khovanov–Rozansky homology as a functor

$$
\begin{array}{ccc}
\left\{ \begin{array}{c} \text{links diagrams} \\ \text{movies of diagrams/m. moves} \end{array} \right\} & \xrightarrow{\ \mathrm{KhR}_N\ } & K^b(\mathrm{gr}^{\mathbb{Z}}\mathrm{Vect}) \\
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Defining KhR_N requires:

- the data of a chain complex for each link diagram (KhR04)
- the data of a chain map for every elementary movie (KhR04)
- the property of satisfying movie moves (Blanchet10 for $N = 2$)

Theorem (Ehrig–Tubbenhauer–W. via Robert–Wagner 2017)

KhR_N is a functor making the above diagram commute.

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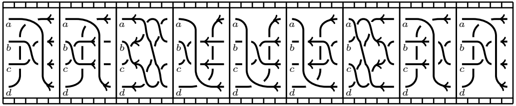
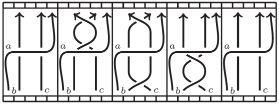
{ c

Defir

E.g. these chain maps should be homotopic to the identity:



MM10



MM6

Theo

KhR$_N$ is a functor making the above diagram commute.

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Tools and applications

Theorem (Kronheimer–Mrowka 2010)

$$\mathrm{Kh}(K) \cong \mathrm{Kh}(\bigcirc) \implies K = \bigcirc$$

Theorem (Rose–W. 2015)

For each knot K and $N = \sum N_j \in \mathbb{N}$ there exists a deformation spectral sequence:

$$\mathrm{KhR}_N(K^k) \rightsquigarrow \bigoplus_{\sum k_j = k} \bigotimes_j \mathrm{KhR}_{N_j}(K^{k_j})$$

Helps with proving the functoriality of KhR_N E.–T.–W. 2017.

Theorem (Rasmussen 2004)

Such spectral sequences give slice genus bounds, concordance invariants.

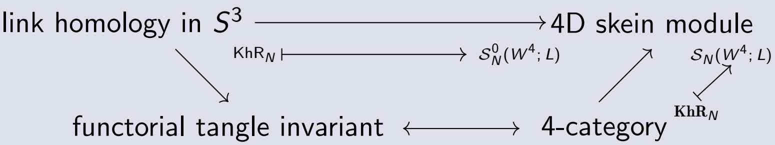
Toy application: Existence of an exotic $\mathbb{R}^4$ via the Rasmussen invariant
Serious application: the Conway knot is not slice Piccirillo 2018

Break

Summary

- Intro: Khovanov's categorification of the Jones polynomial
- Review: Khovanov–Rozansky $\mathfrak{gl}_N$ link homology KhR_N
- Theorem: KhR_N is functorial in B^3

What's next



Starting in dimension 3...

Link invariants

The $\mathfrak{gl}_N$ link polynomial $P_N\colon \{\text{framed, oriented links}\} \to \mathbb{Z}[q^{\pm 1}]$:

$$P_N(⤲) - P_N(⤱) = (q - q^{-1})P_N(\uparrow\uparrow)$$

$$P_N(\text{kink}) = q^N P_N(\uparrow), \quad P_N(L_1 \sqcup L_2) = P_N(L_1)P_N(L_2)$$

Higher categories

Ribbon category $\mathcal{C}_N := \mathrm{Rep}(U_q(\mathfrak{gl}_N))$, tangle invariants

Manifold invariants

The $\mathfrak{gl}_N$ skein module for compact, oriented M^3, $P \subset \partial M^3$ colored points:

$$\mathrm{Sk}_N(M^3; P) := \frac{\mathbb{Z}[q^{\pm 1}]\langle \mathcal{C}_N\text{-colored ribbon graphs in } (M^3, P)\rangle}{\langle \text{isotopy, local relations from } \mathcal{C}_N \text{ in } B^3 \hookrightarrow M^3\rangle}$$

Part of a 0123ε-dimensional TFT. (Fully extended 4D TFT for modular $\mathcal{C}$.)

...upgrading to dimension 4

Link invariants

The $\mathfrak{gl}_N$ Khovanov–Rozansky link homology

$$\mathrm{KhR}_N\colon \{\text{links/link cobordisms}\} \to K^b(\mathrm{gr}^{\mathbb{Z}}\mathrm{Vect}), \quad \chi_q \circ \mathrm{KhR}_N = P_N$$

More recently, in Morrison–Walker–W. 2019:

Higher categories

A 'ribbon 2-category' resp. disk-like 4-category categorifying $\mathrm{Rep}(U_q(\mathfrak{gl}_N))$.

Manifold invariants

A 'skein module' $\mathcal{S}_N(W^4; L)$ for compact, oriented, smooth W^4, $L \subset \partial W^4$.

$$\mathcal{S}_N(B^4; L) \cong \mathrm{KhR}_N(L).$$

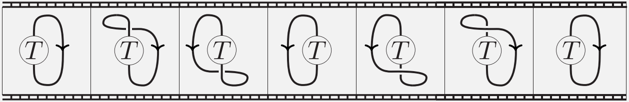
Part of a 01234ε-dimensional TFT?

Morally: a categorification of Crane–Yetter theory over $\mathbb{Z}[q, q^{-1}]$.

Functoriality in S^3

For $\mathcal{S}_N(B^4; L) \cong \mathrm{KhR}_N(L)$ we need KhR_N for links in $S^3 = B^3 \cup \{\infty\}$.

- links in S^3 generically avoid ∞
  $\implies$ same chain complexes
- link cobordisms in $S^3 \times I$ generically avoid $\infty \times I$
  $\implies$ same chain maps
- link cobordism isotopies in $S^3 \times I^2$ might intersect $\infty \times I^2$ transversely
  $\implies$ a new movie move to check, non-local if viewed from B^3



Theorem (M.-W.-W. 2019)

KhR_N is invariant under the sweeparound move, thus functorial in S^3.

Ribbon 2-category via KhR_N for tangles

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{tangle diagrams}\\ \text{movies of diagrams/m. moves}\end{array}\right\} & \xrightarrow{\llbracket - \rrbracket_N} & H^*\mathrm{Ch}^b(N\mathrm{Foam}) \\
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\end{array}$$

Theorem (M.-W.-W. 2019)

There is a braided monoidal (dg) 2-category $\mathbf{KhR}_N$ with

- Objects: tangle boundary sequences
- 1-morphisms: Morse data for tangle diagrams
- 2-morphisms: $\mathbf{KhR}_N(T_1, T_2) := H^*\mathrm{Ch}^b(N\mathrm{Foam})(\llbracket T_1 \rrbracket_N, \llbracket T_2 \rrbracket_N)$.

Think of $\mathbf{KhR}_N$ as categorification of $\mathrm{Rep}(U_q(\mathfrak{gl}_N))$.

Towards derived 4D skein modules & 5D TFT

Questions

Is $\mathbf{KhR}_N$ 4-dualizable and $SO(4)$-fixed in a suitable 5-category of braided monoidal dg 2-categories (with suitable extra structure on the generating object)?
$\rightsquigarrow$ a local 01234ε-D oriented TFT via stratified factorization homology?

Proposed direct construction for the 4ε part:

Theorem (M.-W.-W. 2019)

KhR_N controls a disk-like 4-category, determines $\mathcal{S}_N(W^4; L)$ via the blob complex (Morrison–Walker).

Here we simplify even further: just degree zero blob homology.

What a quantum topologist would do...

In analogy to

$$\mathrm{Sk}_N(M^3; P) := \frac{\mathbb{Z}[q^{\pm 1}]\langle \text{framed, oriented tangles in } (M^3, P)\rangle}{\langle \ker RT_N \text{ in } B^3 \hookrightarrow M^3\rangle}$$

we would like to define $\mathcal{S}_N^0(W^4; L)$ as:

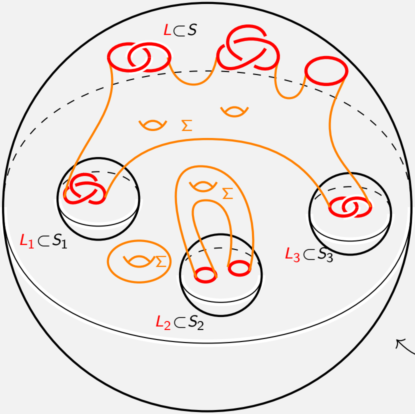
$$\frac{\Bbbk\langle \text{framed, oriented surfaces in } (W^4, L)\rangle}{\langle \ker [\![-]\!]_N \text{ in } B^4 \hookrightarrow W^4\rangle}$$

Problem: Want $\mathcal{S}_N^0(B^4; L) \cong \mathrm{KhR}_N(L)$, but this is not always spanned by images of cobordisms maps.

$\implies$ consider decorated framed, oriented surfaces.

Lasagna algebra

Khovanov–Rozansky homology is an algebra for the lasagna operad



$$\rightarrow \quad \begin{pmatrix} \mathrm{KhR}_N(S^3, L) \\ \uparrow \mathrm{KhR}(D) \\ \bigotimes_i \mathrm{KhR}_N(S_i, L_i) \end{pmatrix}$$

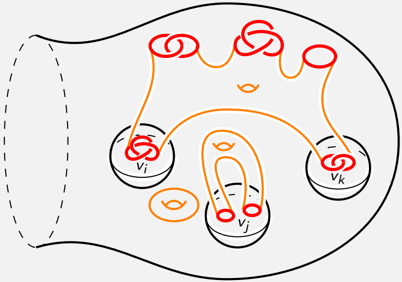
A lasagna diagram D with

L_i, L : input/output links

Σ : f., o. surface in $(B^4 \setminus \sqcup_i B_i^4; L \sqcup_i L_i)$

Khovanov–Rozansky skeins

A lasagna filling of W^4 with a link $L \subset \partial W^4$ is the data of:



B_i^4 : finitely many disjoint 4-balls in $W^{4\circ}$

L_i : input links in ∂B_i^4

Σ : f., o. surface in $(W^4 \setminus \sqcup_i B_i^4; L \sqcup_i L_i)$

$v_i \in \mathrm{KhR}_N(\partial B_i^4, L_i)$

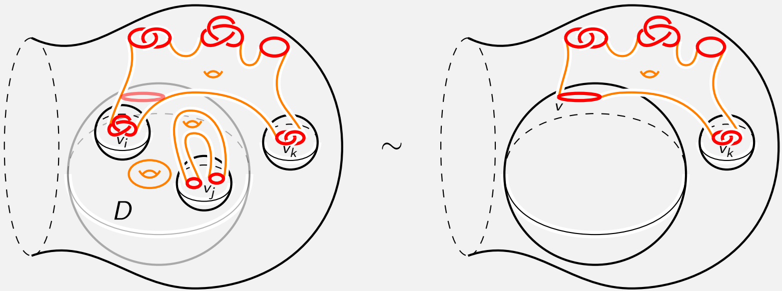
Definition of $\mathcal{S}_N^0(W^4; L)$

Definition

We define the $\mathbb{Z} \times \mathbb{Z} \times H_2(W^4, L)$-graded vector space

$$\mathcal{S}_N^0(W^4; L) := \Bbbk\langle \text{lasagna fillings of } (W^4, L)\rangle / \sim$$

where the equivalence relation $\sim$ is generated by



with $v = \mathrm{KhR}(D)(v_i \otimes \cdots \otimes v_j)$.

To finish, some examples

Example (B^4)

$\mathcal{S}_N(B^4; L) \cong \mathcal{S}_N^0(B^4; L) \cong \mathrm{KhR}(L)$ from the definition.

Example ($B^3 \times S^1$)

$\mathcal{S}_2(B^3 \times S^1; L)$ is related to the Hochschild homology of Khovanov's arc algebra and to Rozansky's homology theory for links L in $S^2 \times S^1$.

Theorem (Manolescu–Neithalath 2020)

If W^4 is a 2-handle body with a single 0-handle, $L \subset S^3$ the attaching link of the 2-handles, then

$$\mathcal{S}_N^0(W^4; \emptyset) \cong \underline{\mathrm{KhR}}_N(L)$$

where $\underline{\mathrm{KhR}}_N(L)$ depends on KhR_N of cables of L.

E.g. $\dim_q \left(\mathcal{S}_N^0(S^2 \times D^2; \emptyset, 0)\right) = \prod_{k=1}^{N-1} \frac{1}{1-q^{2k}}$, results for $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$.