BV formalism - in perturbative algebraic QFT and using factorisation algebras

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[1] Based on joint work with Owen Gwilliam.

Outline of the talk

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- Comparison between the two was discussed in: Gwilliam, KR **CMP 2020** [1711.06674]; Benini, Perin, Schenkel [1903.03396] **CMP 2020**.

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- We work over **Nuc**, the category of *nuclear* topological vector spaces (alternative: *convenient* vector spaces).

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- We use $\mathfrak{v} : \mathbf{PAlg}^*(\mathbf{Nuc}) \to \mathbf{Nuc}$ and $\mathfrak{v} : \mathbf{Alg}^*(\mathbf{Nuc}) \to \mathbf{Nuc}$ to denote forgetful functors to vector spaces. and $\mathfrak{c} : \mathbf{PAlg}^*(\mathbf{Nuc}) \to \mathbf{CAlg}^*(\mathbf{Nuc})$ denotes the forgetful functor to commutative algebras.

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- If $\mathbf{C}$ is an additive category, we write $\mathbf{Ch}(\mathbf{C})$ to denote the category of cochain complexes and cochain maps in $\mathbf{C}$.

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- The latter means that for any causal curve $\gamma : [a, b] \to \mathcal{N}$, if $\gamma(a), \gamma(b) \in \chi(\mathcal{M})$, then for all $t \in]a, b[$, we have $\gamma(t) \in \chi(\mathcal{M})$. ($\chi$ cannot create new causal links.)

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- The symmetric monoidal structure is the disjoint union $\sqcup$.
- A morphism in $\mathbf{Loc}_n^{\otimes}$ sends disjoint components to spacelike-separated regions.

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- The pAQFT formalism provides a functor $\mathfrak{A} : \mathbf{Caus}(\mathcal{M}) \to \mathbf{Alg}^*$, which assigns a unital $*$-algebra to each "causally convex" open set (so that $\mathbf{Caus}(\mathcal{M})$ is a special subcategory of $\mathbf{Open}(\mathcal{M})$ depending on the global hyperbolic structure of $\mathcal{M}$). The algebra $\mathfrak{A}(U)$ is, in practice, a deformation quantization of the Poisson algebra $(\mathcal{O}(Sol(U)), \lfloor ., . \rfloor)$, where $\lfloor ., . \rfloor$ is the Peierls bracket.

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- Renormalization can either be done on the level of the differential (CG) or the product (pAQFT).

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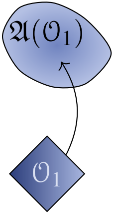

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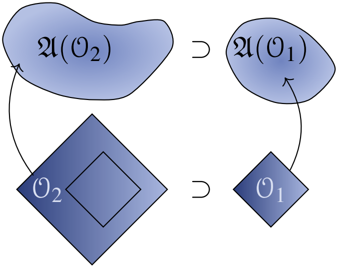
- The physical notion of subsystems is realized by the condition of isotony, i.e.: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$. We obtain a net of algebras.



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- **Time-slice axiom:** If $\mathcal{N}$ is a neighborhood of a Cauchy-surface in $\mathcal{O}$, then $\mathfrak{A}(\mathcal{N})$ is isomorphic to $\mathfrak{A}(\mathcal{O})$.
- This is a QFT version of the initial value problem (or local constancy in the time direction).

Generalizations

- Replace $\mathbb{M}$ with an arbitrary Lorentzian gloabally hyperbolic (has a Cauchy surface) manifold (M, g): locally covariant QFT on curved spacetimes ([Brunetti-Fredenhagen-Verch 03, Hollands-Wald 01, Fewster-Verch 12].

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- Advantage of the algebraic approach: it allows to separate the dynamics from the specification of the state (note that for generic M there is no preferred vacuum state).
- We can also follow the spirit of AQFT in perturbation theory,
- pAQFT is a mathematically rigorous framework that can be used to make precise calculations done in perturbative QFT.

Overview of the pAQFT approach

- Free theory obtained by the formal deformation quantization of the Poisson (Peierls) bracket: $\star$-product ([Dütsch-Fredenhagen 00, Brunetti-Fredenhagen 00, Brunetti-Dütsch-Fredenhagen 09, ...]).

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- Interaction (with a cutoff that guarantees compact support) introduced in the causal approach to renormalization due to Epstein and Glaser ([Epstein-Glaser 73]),
- Generalization to gauge theories using homological algebra ([Hollands 07, Fredenhagen-KR 11]).

Locally covariant classical field theory I

Definition

A **locally covariant classical field theory model** of dimension n is a functor $\mathfrak{P} : \mathbf{Loc}_n \to \mathbf{PAlg}^*(\mathbf{Nuc})^{inj}$ such that the **Einstein causality** holds: given two isometric embeddings $\chi_1 : \mathcal{M}_1 \to \mathcal{M}$ and $\chi_2 : \mathcal{M}_2 \to \mathcal{M}$ whose images $\chi_1(\mathcal{M}_1)$ and $\chi_2(\mathcal{M}_2)$ are spacelike-separated, the subalgebras

$$\mathfrak{P}\chi_1(\mathfrak{P}(\mathcal{M}_1)) \subset \mathfrak{P}(\mathcal{M}) \supset \mathfrak{P}\chi_2(\mathfrak{P}(\mathcal{M}_2))$$

Poisson-commute, i.e., we have

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for any $a_1 \in \mathfrak{P}(\mathcal{M}_1)$ and $a_2 \in \mathfrak{P}(\mathcal{M}_2)$.

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Locally covariant quantum field theory III

On-shell theories

A model $\mathfrak{P}/\mathfrak{A}$ is called **on-shell** if it satisfies in addition the **time-slice axiom**: If $\chi : \mathcal{M} \to \mathcal{N}$ contains a neighborhood of a Cauchy surface $\Sigma \subset \mathcal{N}$, then the map $\mathfrak{P}\chi : \mathfrak{P}(\mathcal{M}) \to \mathfrak{P}(\mathcal{N})$ / $\mathfrak{A}\chi : \mathfrak{A}(\mathcal{M}) \to \mathfrak{A}(\mathcal{N})$ is an isomorphism.

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- is a functor $\mathfrak{P} : \mathbf{Caus}(\mathcal{M}) \to \mathbf{PAlg}^*(\mathbf{Ch}(\mathbf{Nuc}))$, so that each $\mathfrak{P}(\mathcal{O})$ is a locally convex dg Poisson $*$-algebra satisfying **Einstein causality**: spacelike-separated observables Poisson-commute at the level of cohomology.

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- it satisfies the **time-slice axiom** if for any $\mathcal{N} \in \mathbf{Caus}(\mathcal{M})$ a neighborhood of a Cauchy surface in the region $\mathcal{O} \in \mathbf{Caus}(\mathcal{M})$, then the map $\mathfrak{P}(\mathcal{N}) \to \mathfrak{P}(\mathcal{O})$ is a quasi-isomorphism.

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- for each finite collection of pairwise disjoint opens $U_1, \ldots, U_n$, with $n > 0$, and an open V containing every U_i, a morphism

$$\mathcal{F}(\{U_i\}; V) : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V),$$

Prefactorization algebras II

…and satisfying the following conditions:

- composition is associative, so that the triangle

$$\begin{array}{ccc} \bigotimes_i \bigotimes_j \mathcal{F}(T_{ij}) & \longrightarrow & \bigotimes_i \mathcal{F}(U_i) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(V) & \end{array}$$

commutes for any collection $\{U_i\}$, as above, contained in V and for any collections $\{T_{ij}\}_j$ where for each i, the opens $\{T_{ij}\}_j$ are pairwise disjoint and each contained in U_i,

Prefactorization algebras III

- the morphisms $\mathcal{F}(\{U_i\};V)$ are equivariant under permutation of labels, so that the triangle

$$\begin{array}{ccc} \mathcal{F}(U_1)\otimes\cdots\otimes\mathcal{F}(U_n) & \xrightarrow{\simeq} & \mathcal{F}(U_{\sigma(1)})\otimes\cdots\otimes\mathcal{F}(U_{\sigma(n)}) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(V) & \end{array}$$

  commutes for any $\sigma \in S_n$.

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Definition

A **Weiss cover** $\{U_i\}_{\{i \in I\}}$ of an open subset $U \subset M$ is a collection of opens $U_i \subset U$ such that for any finite set of points $S = \{x_1, \ldots, x_n\} \subset U$, there is some $i \in I$ such that $S \subset U_i$.

Factorization algebras I

- A factorization algebra is a prefactorization algebra for which the value on bigger opens is determined by the values on smaller opens: local-to-global property.
- A key difference to a sheaf is that we need to be able to reconstruct the "multiplication maps" from the local data, and so we need to modify our notion of cover accordingly.

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Example

Let M be a smooth n-dimensional manifold. The collection of open sets in M diffeomorphic to a disjoint union of finitely many copies of the open n-disc is a Weiss cover for M.

Factorization algebras II

Definition

A **factorization algebra** $\mathcal{F}$ is a prefactorization algebra on M such that the underlying precosheaf is a cosheaf with respect to the Weiss topology. That is, for any open U and any Weiss cover $\{U_i\}_{i\in I}$ of U, the diagram

$$\coprod_{i,j} \mathcal{F}(U_i \cap U_j) \rightrightarrows \coprod_{i} \mathcal{F}(U_i) \longrightarrow \mathcal{F}(U)$$

is a coequalizer.

Models

A **classical field theory model** is a 1-shifted Poisson (*aka* P_0) algebra $\mathcal{P}$ in factorization algebras $\mathbf{FA}(M, \mathbf{Ch}(\mathbf{Nuc}))$. That is, to each open $U \subset M$, the cochain complex $\mathcal{P}(U)$ is equipped with a commutative product $\cdot$ and a degree 1 Poisson bracket $\{-, -\}$; moreover, each structure map is a map of shifted Poisson algebras.

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Comparison of classical models

- There is a natural quasi-isomorphism

$$\iota^{cl} : \mathfrak{c} \circ \mathcal{P}|_{\mathbf{Caus}(\mathcal{M})} \Rightarrow \mathfrak{c} \circ \mathfrak{P}$$

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- Dynamics: we use a modification of the Lagrangian formalism (fully covariant).

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- Let $\mathfrak{F}$ denote the space of functionals that are polynomial and regular, i.e. $F^{(n)}(\varphi)$ is as smooth section (in general it would be distributional).

Dynamics

- Dynamics is introduced by a generalized Lagrangian S, a localization preserving map $S : \mathcal{D} \to \mathfrak{F}_{\mathrm{loc}}$, where $\mathcal{D}(M) = \mathcal{C}_0^\infty(M, \mathbb{R})$. Examples:

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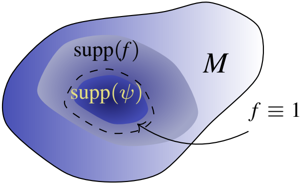

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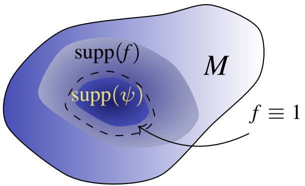

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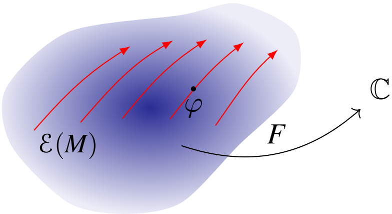

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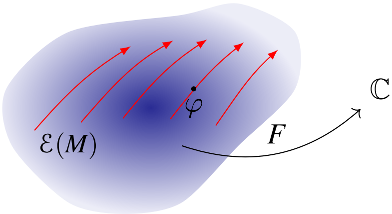

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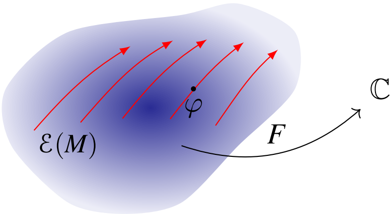

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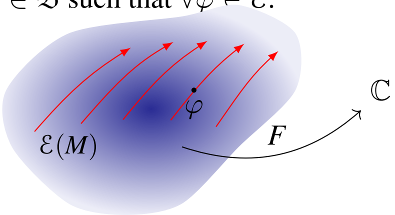
- A symmetry of S is a direction in $\mathcal{E}$ in which the action is constant, i.e. it is a vector field $X \in \mathfrak{V}$ such that $\forall \varphi \in \mathcal{E}$: $0 = \langle dS(\varphi), X(\varphi)\rangle$.



Quotients and homology

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- For the scalar field, this is where the construction finishes.

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$$X(F)(\varphi) = \int dx\, X(\varphi)(x) \frac{\delta F}{\delta\varphi(x)}(\varphi)\,.$$

- We can think of derivatives $\frac{\delta}{\delta\varphi(x)}$ as "generators" of $\mathfrak{V}$.
- In literature those objects are called *antifields* and are denoted by $\varphi^\ddagger(x)$, i.e.: $\varphi^\ddagger(x) \doteq \frac{\delta}{\delta\varphi(x)}$. The grading of Koszul complex is called antifield number $\#$af.

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- As before, we can write $sX = \{X, S^{\text{ext}}(f)\}$, where S^{ext} is the extended action, which contains ghosts, antifields and often non-minimal sector needed for implementing the gauge fixing.
- The BV differential s has to be nilpotent, i.e.: $s^2 = 0$, which leads to the classical master equation (CME):

$$\{S^{\text{ext}}(f), S^{\text{ext}}(f)\} = 0\,,$$

modulo terms that vanish in the limit of constant f.

Linearization

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- Decompose $S_0 = S_{00} + \theta_0$ where S_{00} is the term with no antifields.
- Assume that $dS_{00}(\varphi) = P\varphi$, where P is a normally hyperbolic operator, so that the unique retarded and advanced Green functions $\Delta^{\mathrm{R/A}}$ exist (Green hyperbolic operator).

Poisson structure

- The Poisson bracket of the free theory is

$$\lfloor F, G \rfloor \doteq \left\langle F^{(1)}, \Delta G^{(1)} \right\rangle ,$$

where $\Delta = \Delta^{\mathrm{R}} - \Delta^{\mathrm{A}}$ is the Pauli-Jordan function for S_{00}.

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- Define $s_0 = \{., S_0\}$.
- We set $\mathfrak{P}(\mathcal{O}) = (\mathfrak{PV}(\mathcal{O}), \lfloor ., . \rfloor, \cdot, s_0)$, where $\cdot$ is the wedge product on polyvector fields and pointwise product for functionals.

Deformation quantization

- Define the $\star$-product (deformation of the pointwise product):

$$(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), W^{\otimes n} G^{(n)}(\varphi) \right\rangle \ ,$$

where W is the 2-point function of a Hadamard state and it differs from $\frac{i}{2}\Delta$ by a symmetric bidistribution: $W = \frac{i}{2}\Delta + H$.

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- We set $\mathfrak{A}(\mathcal{O}) = (\mathfrak{PV}(\mathcal{O})[[\hbar]], \star, *, s_0)$, where $*$ is the complex conjugation.

Time-ordered products I

Given the classical semistrict dg theory $\mathfrak{P}$ and its quantization $\mathfrak{A}$, the **time-ordered product** is realized as a triple $(\mathfrak{A}_T, \xi, \mathfrak{T})$ consisting of:

- a functor $\mathfrak{A}_T\colon \mathbf{Caus}(M) \to \mathbf{CAlg}^*(\mathbf{Ch}(\mathbf{Nuc}_\hbar))$ which gives the time-ordered product as a commutative product,

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- a natural isomorphism $\xi : \mathfrak{v} \circ \mathfrak{A}_T \Rightarrow \mathfrak{v} \circ \mathfrak{A}$, which identifies $\mathfrak{A}_T$ and $\mathfrak{A}$ as vector spaces,
- and a natural isomorphism of commutative algebras $\mathfrak{T} : \mathfrak{c} \circ \mathfrak{P}[[\hbar]] \Rightarrow \mathfrak{A}_T$

Time-ordered products II

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- such that for any pair of inclusions $\psi_i : \mathcal{O}_i \to \mathcal{O}$ in $\mathbf{Caus}(\mathcal{M})$, if $\psi_1(\mathcal{O}_1) \prec \psi_2(\mathcal{O}_2)$, then

$$\xi_{\mathcal{O}} \circ m_{\mathcal{T}} \circ (\mathfrak{A}_T\psi_2 \otimes \mathfrak{A}_T\psi_1) = m_\star \circ (\mathfrak{A}\psi_2 \circ \xi_{\mathcal{O}_2} \otimes \mathfrak{A}\psi_1 \circ \xi_{\mathcal{O}_1}) \,,$$

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- where $m_{\mathcal{T}}/m_\star$ is the multiplication with respect to the time-ordered/star product and the relation "$\prec$" means "not later than," i.e., there exists a Cauchy surface in $\mathcal{O}$ that separates $\psi_1(\mathcal{O}_1)$ and $\psi_2(\mathcal{O}_2)$.

Time-ordered products III

- The time-ordering operator $\mathcal{T}$ is defined as:

$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle F^{(2n)}(\varphi), (\tfrac{\hbar}{2}\Delta^{\mathrm{F}})^{\otimes n} \right\rangle \equiv e^{\frac{i\hbar}{2}\partial_{\Delta^F}} F \,,$$

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- Define the time-ordered product $\cdot_{\mathcal{T}}$ on $\mathfrak{PV}[[\hbar]]$ by:

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$

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- Renormalization problem: extend $\cdot_{\mathcal{T}}$ to V local and non-linear.

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  the twist of the free quantum BV operator by the (non-local!) map that intertwines the free and the interacting theory.
- The 0th cohomology of $\hat{s}$ characterizes quantum gauge invariant observables.

Quantum BV operator II

- $\hat{s}_0$ on regular functionals can also be written as:

$$\hat{s}_0 = \{., S_0\} - i\hbar\triangle\,,$$

where $\triangle$ is the BV Laplacian, which on regular functionals is

$$\triangle X = (-1)^{(1+|X|)} \int dx \frac{\delta^2 X}{\delta\varphi^\ddagger(x)\delta\varphi(x)}\,.$$

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- In our framework this is a mathematically rigorous result, no path integral needed (in contrast to other approaches).

Towards renormalization

To extend QME and $\hat{s}$ to local observables, we need to replace $\cdot_{\mathcal{T}}$ with the renormalized time-ordered product.

Theorem (K. Fredenhagen, K.R. 2011)

The renormalized time-ordered product $\cdot_{\mathcal{T}_r}$ is an associative product on $\mathcal{T}_r(\mathfrak{PV})$ given by

$$F \cdot_{\mathcal{T}_r} G \doteq \mathcal{T}_r(\mathcal{T}_r^{-1}F \cdot \mathcal{T}_r^{-1}G)\,,$$

where $\mathcal{T}_r : \mathfrak{PV}[[\hbar]] \to \mathcal{T}_r(\mathfrak{PV})[[\hbar]]$ is defined as

$$\mathcal{T}_r = (\oplus_n \mathcal{T}_r^n) \circ \beta\,,$$

where $\beta : \mathcal{T}_r : \mathfrak{PV} \to S^\bullet \mathfrak{PV}_{\mathrm{loc}}^{(0)}$ is the inverse of multiplication m and the subscript (0) indicates functionals that vanish at $\varphi = 0$.

Renormalized QME and the quantum BV operator

- Since $\cdot_{\mathcal{T}_{\rm r}}$ is an associative, commutative product, we can use it in place of $\cdot_{\mathcal{T}}$ and define the renormalized QME and the quantum BV operator as:

$$\{e_{\mathcal{T}_{\rm r}}^{iV/\hbar}, S_0\} = 0$$

$$\hat{s}(X) \doteq e_{\mathcal{T}_{\rm r}}^{-iV/\hbar} \cdot_{\mathcal{T}_{\rm r}} \left(\{e_{\mathcal{T}_{\rm r}}^{iV/\hbar} \cdot_{\mathcal{T}_{\rm r}} X, S_0\}\right) ,$$

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- These formulas get even simpler if we use the anomalous Master Ward Identity ([Brenecke-Dütsch 08, Hollands 07]).

Renormalized QME and the quantum BV operator

- Using the MWI we obtain following formulas:

$$0 = \frac{1}{2}\{V + S_0, V + S_0\}_{\mathcal{T}_r} - \triangle_V \,,$$
$$\hat{s}X = \{X, V + S_0\} - \triangle_V(X) \,,$$

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- In the renormalized theory, $\triangle_V$ is well-defined on local vector fields, in contrast to $\triangle$.

Comparison (free scalar field) I

- Classical case is almost trivial on the level of algebras, since both CG and FR work with the space of regular polynomials and $\mathcal{P}(\mathcal{O}) = (\mathfrak{PV}(\mathcal{O}), \delta_{S_0}, \{.,.\}) = \mathfrak{v} \circ \mathfrak{P}(\mathcal{O})$ for $\mathcal{O} \in \mathbf{Caus}(\mathcal{M})$.

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- The quantum case is a bit subtler. The pAQFT approach assigns a dg algebra $\mathfrak{A} = (\mathfrak{PV}[[\hbar]], \delta_{S_0}, \star)$ whereas the CG approach assigns merely a cochain complex $(\mathfrak{PV}[[\hbar]], \delta_{S_0} - i\hbar\triangle)$.

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- The key is to use the time-ordering machinery.

Comparison (free scalar field) II

- The time-ordering operator $\mathcal{T}$ provides a cochain isomorphism

$$\mathcal{A} = (\mathfrak{PV}[[\hbar]], \hat{s}_0) \xrightarrow{\mathcal{T}} (\mathfrak{PV}[[\hbar]], \delta_{S_0}) = \mathcal{P}[[\hbar]].$$

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- Since underlying vector spaces are in our case the same, we have

$$\mathcal{A}\big|_{\mathbf{Caus}(\mathcal{M})} \xrightarrow{\iota^q = \mathcal{T}} (\mathfrak{PV}[[\hbar]], \delta_{S_0}) = \mathfrak{v} \circ \mathfrak{A}.$$

Different perspectives I

Quantum observables are described either by deforming the product (from $\cdot$ to $\cdot_{\mathcal{T}}$) and keeping the differential as δ_{S_0} or, equivalently, by deforming the differential (from δ_{S_0} to $\hat{s}_0 = \delta_{S_0} - i\hbar\triangle$) and keeping the product.

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- Note that δ_{S_0} is a derivation with respect to $\star$, but not with respect to $\cdot_{\mathcal{T}}$, since

$$\delta_{S_0}(X \cdot_{\mathcal{T}} Y) = (-1)^{|X|}\delta_{S_0}X \cdot_{\mathcal{T}} Y + X \cdot_{\mathcal{T}} \delta_{S_0}Y - i\hbar\{X, Y\}_{\mathcal{T}},$$

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- Equivalently to deforming the product, one can deform the differential (CG approach) from δ_S to $\hat{s}$. Again we have

$$\hat{s}_0(X \cdot Y) = (-1)^{|X|}\hat{s}_0X \cdot Y + X \cdot \hat{s}_0Y - i\hbar\{X, Y\}\,.$$

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- Modulo $\mathrm{Im} s_0$ we have:

$$[G, F]_\star = G \star \beta_+(F) - \beta_-(F) \star G = G \cdot_{\mathcal{T}} \beta_+(F) - \beta_-(F) \cdot_{\mathcal{T}} G$$

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- Recall that $\star$ arises from the deformation of the Peierls bracket $\lfloor .,. \rfloor$. We want to relate this Poisson bracket to the (-1)-shifted Poisson bracket $\{\}$.

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- The time-slice axiom implies that for F as before, there exists Ψ such that $\beta_- F - \beta_+ F = s_0 \Psi$.
- We rewrite the $\star$ commutator as

$$[G,F]_\star = G \star \beta_+(F) - \beta_-(F) \star G = G \cdot_{\mathcal{T}} \beta_+(F) - \beta_-(F) \cdot_{\mathcal{T}} G$$
$$= G \cdot_{\mathcal{T}} (\beta_- F - \beta_+ F) = G \cdot_{\mathcal{T}} s_0 \Psi \,,$$

for some Ψ.

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- Hence we can write the Peierls bracket as

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- Hence

$$\lfloor G, F \rfloor = \{G, \Psi\} \mod \hbar \, , \mathrm{Im} s_0 \, , ,$$

which can be thought of as the intrinsic definition of the Peierls bracket, given the antibracket and a theory satisfying time-slice axiom.

Thank you for your attention!