**Hopf algebras, tensor categories and three–manifold invariants**

# Topological field theory as a functor

**The universal construction**

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[Reference: [Co] Costantino, Notes on Topological Quantum Field Theories, Winter Braids Lecture Notes (2015), 1–45]
This handout will closely follow the reference given above.

The goal of this handout will be to construct a functor from a cobordism category into vector spaces out of a given diffeomorphism invariant of manifolds by the so–called universal construction. Some of the functors gained in this way are symmetric monoidal and hence topological field theories (TFTs). However, we will see that not every TFT arises from such a universal construction.

# 1 The universal construction

## 1.1 Quantization functors

Consider a cobordism category Bord, i.e. a category together with an empty object $\varnothing$ and the notions of disjoint union, orientation reversal and boundary.

Definition.
A functor $V\colon \mathrm{Bord} \to \mathrm{Vec}_{\mathbb{K}}$ satisfying $V(\varnothing) \cong \mathbb{K}$ is called *quantization functor*.

Remark.
It depends on the author, what the exact definition of a quantization functor is. In this handout we go along with the definition of [Co].

Remark.
Obviously, any monoidal functor from Bord to $\mathrm{Vec}_{\mathbb{K}}$ is also a quantization functor. In particular, TFTs are quantization functors.

Definition.
A quantization functor $V\colon \mathrm{Bord} \to \mathbb{K}$ is called *cobordism generated*, if for all objects $\Sigma$ the associated vector space $V(\Sigma)$ is generated by the elements $V(M)(1)$ with $M \in \mathrm{Hom}(\varnothing, \Sigma)$, i.e.

$$V(\Sigma) = \mathrm{span}\,\{V(\mathrm{Hom}(\varnothing, \Sigma))(1)\}\,.$$

## 1.2 Invariants

Definition.
Consider a map $\langle - \rangle$ from closed oriented smooth manifolds of some fixed dimension $n$ to a field $\mathbb{K}$, which associates to every manifold of said type a scalar. We call such a map an *invariant (of $n$–dimensional manifolds)*, if it is constant on diffeomorphism classes, i.e. diffeomorphic manifolds are mapped to the same scalar.

Definition.
We say an invariant $\langle - \rangle$ is *multiplicative*, if we have

- $\langle M_1 \sqcup M_2 \rangle = \langle M_1 \rangle \langle M_2 \rangle$ for all closed $n$–dimensional oriented smooth manifolds $M_1, M_2$ and
- $\langle \varnothing \rangle = 1$.

## 1.3 The universal construction theorem

Theorem.
*Let* $\mathrm{Bord_n}$ *be a cobordism category and* $\langle - \rangle : \mathrm{Hom}(\varnothing, \varnothing) \to \mathbb{K}$ *a multiplicative diffeomorphism invariant of $n$–dimensional manifolds, where* $\mathrm{Hom}(\varnothing, \varnothing)$ *is referred to as a* Hom*–space of the category* $\mathrm{Bord_n}$.

*Then there exists a unique cobordism generated quantization functor* $V\colon \mathrm{Bord_n} \to \mathrm{Vec}_{\mathbb{K}}$ *whose restriction to* $\mathrm{Hom}(\varnothing, \varnothing)$ *is the given invariant* $\langle - \rangle$*.*

Proof.
Denote by $F(\Sigma) = \mathrm{span}\,\{\mathrm{Hom}(\varnothing, \Sigma)\}$ the set freely generated by all cobordisms from $\varnothing$ to $\Sigma$. Analogously, define $F'(\Sigma) = \mathrm{span}\,\{\mathrm{Hom}(\Sigma, \varnothing)\}$. Next, we want to define a pairing $\langle -, - \rangle_\Sigma : F'(\Sigma) \otimes F(\Sigma) \to \mathbb{K}$. Since we are given a multiplicative invariant, we can define on basis elements $M_1 \in F(\Sigma), M_2 \in F'(\Sigma)$ the pairing as the invariant applied to the composition $M_2 \circ M_1$, i.e.

$$\langle M_2, M_1 \rangle_\Sigma := \langle M_2 \circ M_1 \rangle = \langle M_2 \sqcup_\Sigma M_1 \rangle .$$

Extending this definition linearly yields a pairing on $F'(\Sigma) \otimes F(\Sigma)$.
We define a functor $V\colon \mathrm{Bord_n} \to \mathrm{Vec}_{\mathbb{K}}$ and we start by fixing $V$ on objects via

$$V(\Sigma) := F(\Sigma)/\mathrm{Ann}(F'(\Sigma)),$$

where $\mathrm{Ann}(F'(\Sigma)) = \{x \in F(\Sigma) \mid \langle y, x \rangle_\Sigma = 0, \forall y \in F'(\Sigma)\}$.
Similarly, we define a functor $V'\colon \mathrm{Bord_n^{op}} \to \mathrm{Vec}_{\mathbb{K}}$. On objects we set

$$V'(\Sigma) := F'(\Sigma)/\mathrm{Ann}(F(\Sigma)),$$

where $\mathrm{Ann}(F(\Sigma)) = \{y \in F'(\Sigma) \mid \langle y, x \rangle_\Sigma = 0 \quad \forall x \in F(\Sigma)\}$.
Remark that the pairing $\langle -, - \rangle_\Sigma : F'(\Sigma) \otimes F(\Sigma) \to \mathbb{K}$ descends to a pairing $\langle -, - \rangle_\Sigma : V'(\Sigma) \otimes V(\Sigma) \to \mathbb{K}$, which is non–degenerate by construction.
We have to define the functors $V$ and $V'$ on morphisms. Let therefore $N \in \mathrm{Hom}(\Sigma_1, \Sigma_2)$ be a morphism in $\mathrm{Bord_n}$. For a basis element $M \in \mathrm{Hom}(\varnothing, \Sigma_1)$ of $V(\Sigma_1)$ we define

$$V(N)[M] := [N \circ M] = [N \sqcup_{\Sigma_1} M].$$

This defines a functor, since for any morphism $N' \in \mathrm{Hom}(\Sigma_2, \Sigma_3)$ we have

$$V(N' \circ N)[M] = [(N' \circ N) \circ M] = [N' \circ (N \circ M)] = V(N')[N \circ M] = (V(N') \circ V(N))[M].$$

Similarly, for a morphism $M \in \mathrm{Hom}(\Sigma_2, \varnothing)$ we define

$$V'(N)[M] := [M \circ N] = [M \sqcup_{\Sigma_2} N].$$

This defines a contravariant functor $\mathrm{Bord_n} \to \mathrm{Vec}_{\mathbb{K}}$, i.e. a functor $\mathrm{Bord_n^{op}} \to \mathrm{Vec}_{\mathbb{K}}$, since for any morphism $N' \in \mathrm{Hom}(\Sigma_0, \Sigma_1)$ we have

$$V'(N \circ N')[M] = [M \circ (N \circ N')] = [(M \circ N) \circ N'] = V'(N')[M \circ N] = (V'(N') \circ V'(N))[M].$$

Now that we have the two functors $V$ and $V'$, we observe that for all morphisms $N \in \mathrm{Hom}(\Sigma_1, \Sigma_2)$ and basis elements $M_1$ of $V(\Sigma_1) = \mathrm{span}\,\{\mathrm{Hom}(\varnothing, \Sigma_1)\}$ and $M_2$ of $V(\Sigma_2) = \mathrm{span}\,\{\mathrm{Hom}(\Sigma_2, \varnothing)\}$ the equality

$$\langle V'(N)(M_2), M_1 \rangle_\Sigma = \langle M_2 \circ N \circ M_1 \rangle = \langle M_2, V(N)(M_1) \rangle_\Sigma$$

holds and conclude that by linearity and non–degeneracy of $\langle -, - \rangle_\Sigma$ either of the functors $V$ and $V'$ is uniquely determined by the other one.
To see that $V$ is indeed a quantization functor, we need that the given invariant $\langle - \rangle$ is multiplicative. We write $y, x \in F(\varnothing) = F'(\varnothing)$ in basis with $k_i, k'_j \in \mathbb{K}$ as $y = \sum_i k_i y_i$ and $x = \sum_j k'_j x_j$ and compute

$$\begin{aligned} V(\varnothing) &= F(\varnothing)/\mathrm{Ann}(F'(\varnothing)) \\ &= \mathrm{span}\,\{\mathrm{Hom}(\varnothing, \varnothing)\} \,/\, \{x \in F(\varnothing) \mid \langle y, x \rangle_\varnothing = 0, \forall y \in F'(\varnothing)\} \\ &= \mathrm{span}\,\{\mathrm{Hom}(\varnothing, \varnothing)\} \,/ \{x \in F(\varnothing) \mid \sum_{i,j} k_i k'_j \langle y_i, x_j \rangle_\varnothing = 0, \forall y \in F'(\varnothing), \forall i, j\} \\ &= \mathrm{span}\,\{\mathrm{Hom}(\varnothing, \varnothing)\} \,/ \{x \in F(\varnothing) \mid \sum_{i,j} k_i k'_j \langle y_i \sqcup x_j \rangle = 0, \forall y \in F'(\varnothing), \forall i, j\} \\ &= \mathrm{span}\,\{\mathrm{Hom}(\varnothing, \varnothing)\} \,/ \{x \in F(\varnothing) \mid \sum_{i,j} k_i k'_j \langle y_i \rangle \langle x_j \rangle = 0, \forall y \in F'(\varnothing), \forall i, j\} \\ &= \mathrm{span}\,\{\mathrm{Hom}(\varnothing, \varnothing)\} \,/ \{x \in F(\varnothing) \mid \langle x_j \rangle = 0, \forall j\}. \end{aligned}$$

For a moment, denote by $\emptyset$ the empty manifold regarded as an element of $\mathrm{Hom}(\varnothing, \varnothing)$. Since $\langle - \rangle$ is multiplicative, we have $\langle \emptyset \rangle = 1 \in \mathbb{K}$. We can linearly extend $\langle - \rangle$ to $F(\varnothing) = \mathrm{span}\,\{\mathrm{Hom}(\varnothing, \varnothing)\}$ and obtain, that the extension $\langle - \rangle_{ext}$ is surjective onto $\mathbb{K}$, since $\langle k\emptyset \rangle_{ext} = k\langle \emptyset \rangle = k \cdot 1 = k$ for all $k \in \mathbb{K}$. Further, the kernel of the $\langle - \rangle_{ext}$ is $\{x \in F(\varnothing) \mid \langle x_j \rangle = 0, \forall j\}$. Hence, by the isomorphism theorem, $V(\varnothing) = F(\varnothing)/\ker\, \langle - \rangle_{ext} \cong \mathrm{im}\, \langle - \rangle_{ext} = \mathbb{K}$.
It is left to show that $V$ is cobordism generated. We have just seen, that the isomorphism $V(\varnothing) \cong \mathbb{K}$ is given by the invariant. Hence, the scalar $1 \in \mathbb{K}$ corresponds to the class $[\varnothing] \in V(\varnothing)$ of the empty manifold. Now by construction the functor $V$ is cobordism generated, since for all $\Sigma$ we have that

$$\begin{aligned} V(\Sigma) &= F(\Sigma)/\sim \\ &= \mathrm{span}\{\mathrm{Hom}(\varnothing, \Sigma)\}/\sim . \end{aligned}$$

But for every $N \in \mathrm{Hom}(\varnothing, \Sigma)$ we have by definition $V(N)(1) = V(N)[\varnothing] = [N \sqcup \varnothing] = [N]$, hence $V(\Sigma)$ is generated by $V(\mathrm{Hom}(\varnothing, \Sigma))$.

□

# 2 Consequences

Remark.
The functor $V$ obtained by the universal construction is not necessarily monoidal. In general we have for $V(\Sigma) = F(\Sigma)/\sim$, $V(\Sigma') = F(\Sigma')/\sim$ that $V(\Sigma \sqcup \Sigma') = F(\Sigma \sqcup \Sigma')/\sim$, where $F(\Sigma \sqcup \Sigma')$ contains in particular connected manifolds connecting $\Sigma$ and $\Sigma'$. These are a priori not contained in $V(\Sigma) \otimes V(\Sigma') = (F(\Sigma)/\sim) \sqcup (F(\Sigma')/\sim)$ and it is a surprising result, that the universal construction can produce TFTs at all. This can only happen, when all of said manifolds connecting $\Sigma$ and $\Sigma'$ are divided out of $F(\Sigma \sqcup \Sigma')$ by $\sim$.

Remark.
Since we have a diffeomorphism invariant of manifolds as input into the universal construction, it makes sense to require the obtained functor to be a quantization functor. In this way we can look at the values, which the obtained functor assigns to closed manifolds regarded as morphisms from $\varnothing$ to $\varnothing$ and since $V$ is a quantization functor, we will again get a scalar in $\mathbb{K}$. Further, the universal construction is in such a way that the functor $V$ extends the given invariant, i.e. the invariant obtained from $V$ by looking at the values $V(M)(1)$ for $M \in \mathrm{Hom}(\varnothing, \varnothing)$ is precisely the invariant we started with.
Concrete: If we have a manifold $M \in \mathrm{Hom}(\varnothing, \varnothing)$, then $V(M)\colon V(\varnothing) \to V(\varnothing)$ is the map given by $[N] \mapsto [N \sqcup M]$, in particular $V(M)[\varnothing] = [M]$. Since the isomorphism $V(\varnothing) \cong \mathbb{K}$ is given by the invariant, we get that $V(M)\colon \mathbb{K} \to \mathbb{K}$ is given by $k \mapsto k\langle M \rangle$.

## 2.1 Application in dimension 1

In this section we will see, that there are TFTs which cannot be obtained by the universal construction given in section 1.

Example.
Let $Z$ be a one–dimensional TFT, i.e. a symmetric monoidal functor $\mathbb{Z}\colon \mathrm{Bord}_1 \to \mathbb{K}$. As we have seen in a previous talk, the objects in $\mathrm{Bord}_1$ are finite disjoint unions of positive $(\bullet_+)$ and negative $(\bullet_-)$ oriented points. Further, we know that $Z(\bullet_+) = V$ and $Z(\bullet_-) = V^*$ for some finite–dimensional $\mathbb{K}$–vector space $V$.
Assume $Z$ arose from a universal construction and is hence cobordism generated. We then have in particular

$$V = Z(\bullet_+) = \mathrm{span}\,\{\mathrm{Hom}(\varnothing, \bullet_+)\}\,.$$

But every one–dimensional closed oriented manifold with boundary has to have an even number of boundary points, hence $\mathrm{Hom}(\varnothing, \bullet_+) = \varnothing$. Thus, $V = \mathrm{span}\,\{\varnothing\} = \{0\}$. Analogously, we get $Z(\bullet_-) = V^* = \{0\}$.
Further, $Z$ is as a TFT a symmetric monoidal functor and hence

$$Z(\bullet_+ \sqcup \bullet_-) = Z(\bullet_+ \otimes \bullet_-) \cong Z(\bullet_+) \otimes Z(\bullet_-) = \{0\} \otimes \{0\} \cong \{0\}.$$

In fact, monoidality already implies $Z(\Sigma) = \{0\}$ for all objects $\Sigma$ in $\mathrm{Bord}_1$ (except $Z(\varnothing) = \mathbb{K}$).
Hence, $Z$ is on all non–trivial objects trivial and so it is on all morphism sets other than $\mathrm{Hom}(\varnothing, \varnothing)$. But morphisms in $\mathrm{Hom}(\varnothing, \varnothing)$ are just disjoint unions of circles. Since $Z$ is monoidal, it is enough to show that $Z$ is trivial on the morphism $S^1 \in \mathrm{Hom}(\varnothing, \varnothing)$ to show that $Z$ is trivial on all morphisms and objects other than $\varnothing$. Again, $Z$ is monoidal and we can compute $Z(S^1) = Z(\mathrm{ev} \circ \mathrm{coev}) = Z(\mathrm{ev}) \circ Z(\mathrm{coev})$, which is a composition of trivial maps, where $\mathrm{coev}\colon \mathbb{K} \to V \otimes V^*$ denotes the coevaluation map and $\mathrm{ev}\colon V \otimes V^* \to \mathbb{K}$ denotes the evaluation map.
Thus we have proven that any cobordism generated 1–dimensional TFT is trivial and we conclude, that non–trivial 1–dimenional TFTs cannot be obtained by the universal construction.

Remark.
One can ask the question, how the functor obtained from the universal construction in dimension 1 looks like.
We see that if $\Sigma$ is a 0–dimensional manifold consisting of an odd number of points, then $\mathrm{Hom}(\varnothing, \Sigma) = \varnothing$ and hence, $V(\Sigma) = \mathrm{span}\{\varnothing\} = \{0\}$.
For an object with an even number of points, the construction becomes more subtle and the result will in general be non–trivial.

## 2.2 Application to Reshetikhin–Turaev invariants

Remark.
One can show that the Reshetikhin–Turaev construction is cobordism generated and indeed obtained by the universal construction theorem applied to the Reshetikhin–Turaev invariants.