S1: Surgery presentations of 3-manifolds

References:

[BK] Bakalov, Kirillov, Lectures on Tensor Categories and Modular Functors (AMS 2001)
[PS] Prasolov, Sossinsky, Knots, links, braids and 3-manifolds (AMS 1997)

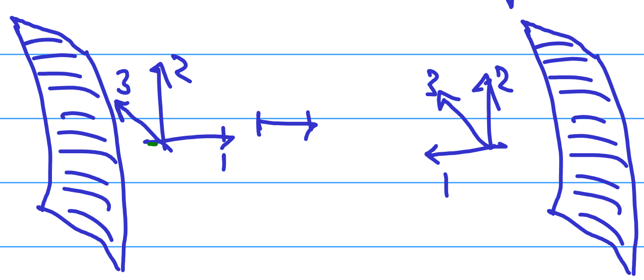
1) Warm up

Take $S^3 \simeq \mathbb{R}^3 \cup \{\infty\}$

Inversion at S^2: $\quad p \xmapsto{I} -\dfrac{p}{|p|^2} \qquad |I(p)| = \dfrac{|p|}{|p|^2} = \dfrac{1}{|p|}$

$\infty \longleftrightarrow 0$

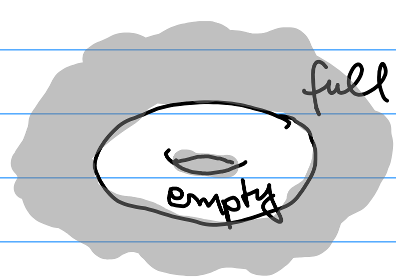
suppose no minus: $|p|=1 \Rightarrow I(p)=p$, so near S^2 looks like a reflection:



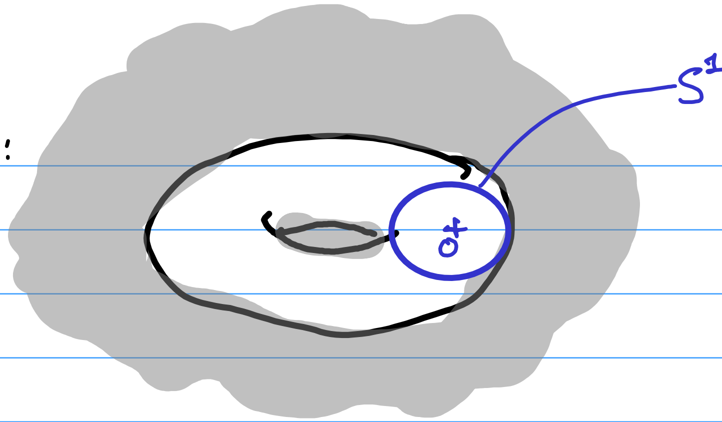
Q: apply to solid 3-ball $\quad |p| \leq 1$

A solid torus:
in $\mathbb{R}^3 \cup \{\infty\}$

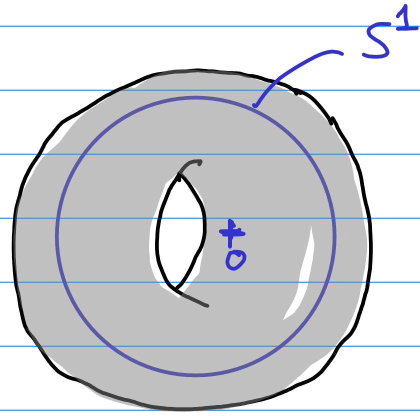
Complement of a solid torus:



Apply inversion :



get something like

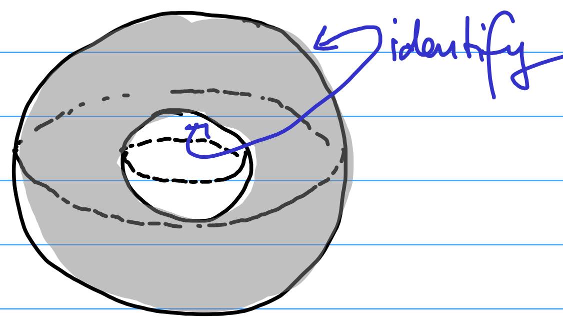


Write A for solid torus, so that $\partial A = T^2$, the 2-torus.

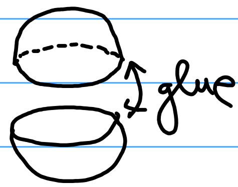
<u>Thus</u> can write $S^3 \simeq A \sqcup_f A$ for some diffeo $f : T^2 \xrightarrow{\sim} T^2$

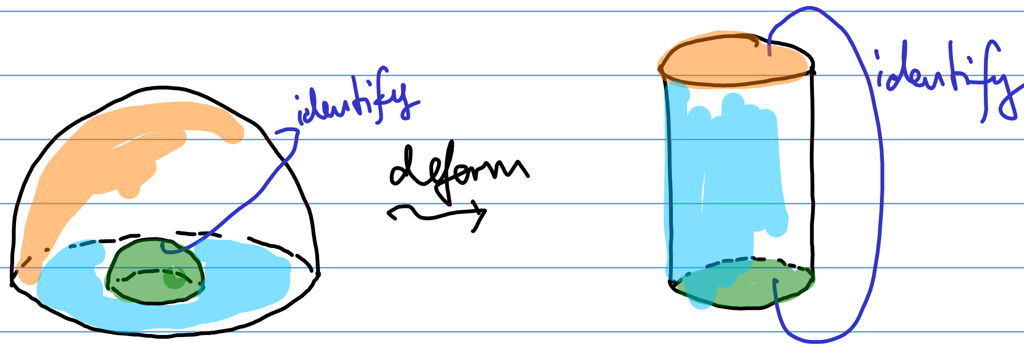
Another example :

$S^2 \times S^1$



As $S^2 = D^2 \sqcup_{S^1} D^2$ have $S^2 \times S^1 = \underbrace{D^2\times S^1}_{A} \sqcup_{\underbrace{S^1\times S^1}_{T_2}} \underbrace{D^2\times S^1}_{A}$



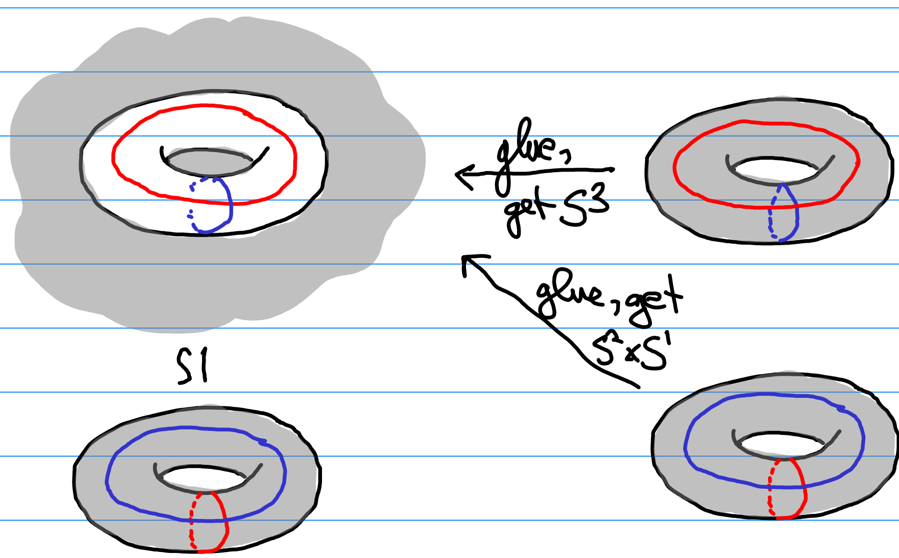


Thus can obtain 3-mf $S^2\times S^1$ from S^3 via

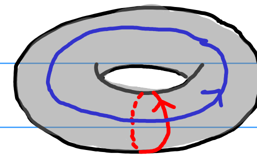
1) remove solid torus $S^3 \rightsquigarrow S^3 \setminus A$

2) glue A back with new bnd identification

$$\partial(S^3\setminus A) = T^2 \xrightarrow[f]{\sim} T^2 = \partial A$$

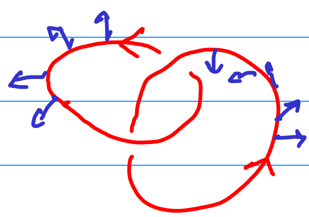


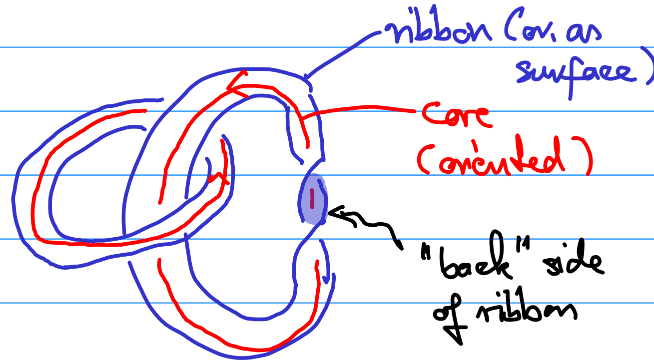
Idea: Give the two cycles on the bnd of $S^3 \setminus A$ where standard solid torus A is glued:



2) Surgery presentation

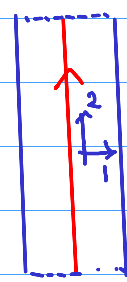
Framed oriented link $\triangleq$ ribbon link



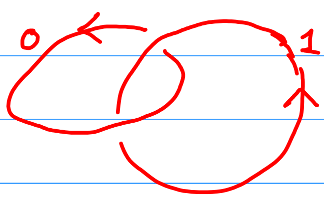


locally:

framing + or. of core give or. of ribbon.

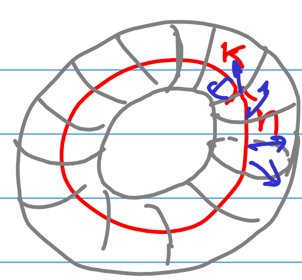


Oriented link + integer per component



number of ribbon twists

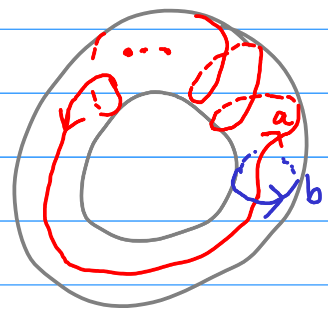
Given framed oriented link; for each component K:



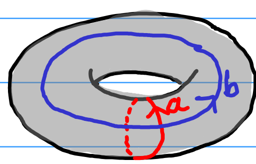
1) pick tubular nbhood of K

2) push K to bnd of tube by framing

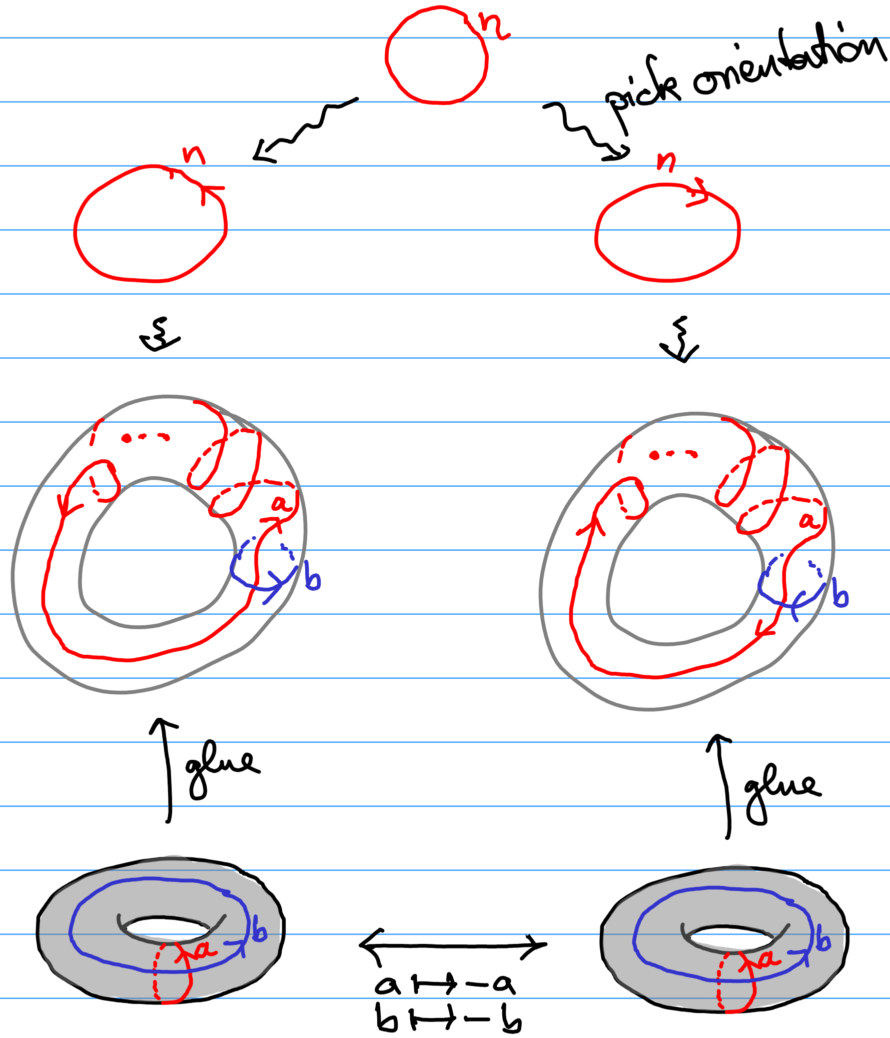
3) 2nd cycle small loop around K



4) Cut out nbhood of K and glue in A



In fact: <u>unoriented</u> framed link enough, because:



Got diffeom. 3-mf via id outside of nbhood of K and

$$a \mapsto -a\,,\; b \mapsto -b$$

on bnd, which <u>extends</u> inside A.

Def Let L be a framed (unoriented) link in S_3.
Write

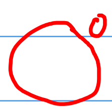
$$M_L$$

for the 3mf obtained from carrying out steps 1–4 for each component of L.

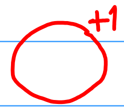
We say M_L is obtained from *surgery along the link L*.

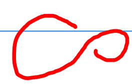
Thm (Lickorish–Wallace ~ 1960)

Every closed compact connected oriented 3mf is isomorphic to M_L for some framed link L in S^3.

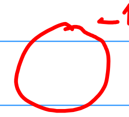
E.g. $L =$ 

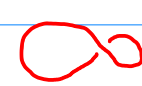
$\rightsquigarrow M_L \simeq S^2 \times S^1$

$L =$ 

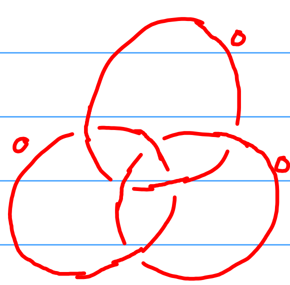
$=$  in blackb. framing

$\rightsquigarrow M_L \simeq S^3$

$L =$ 

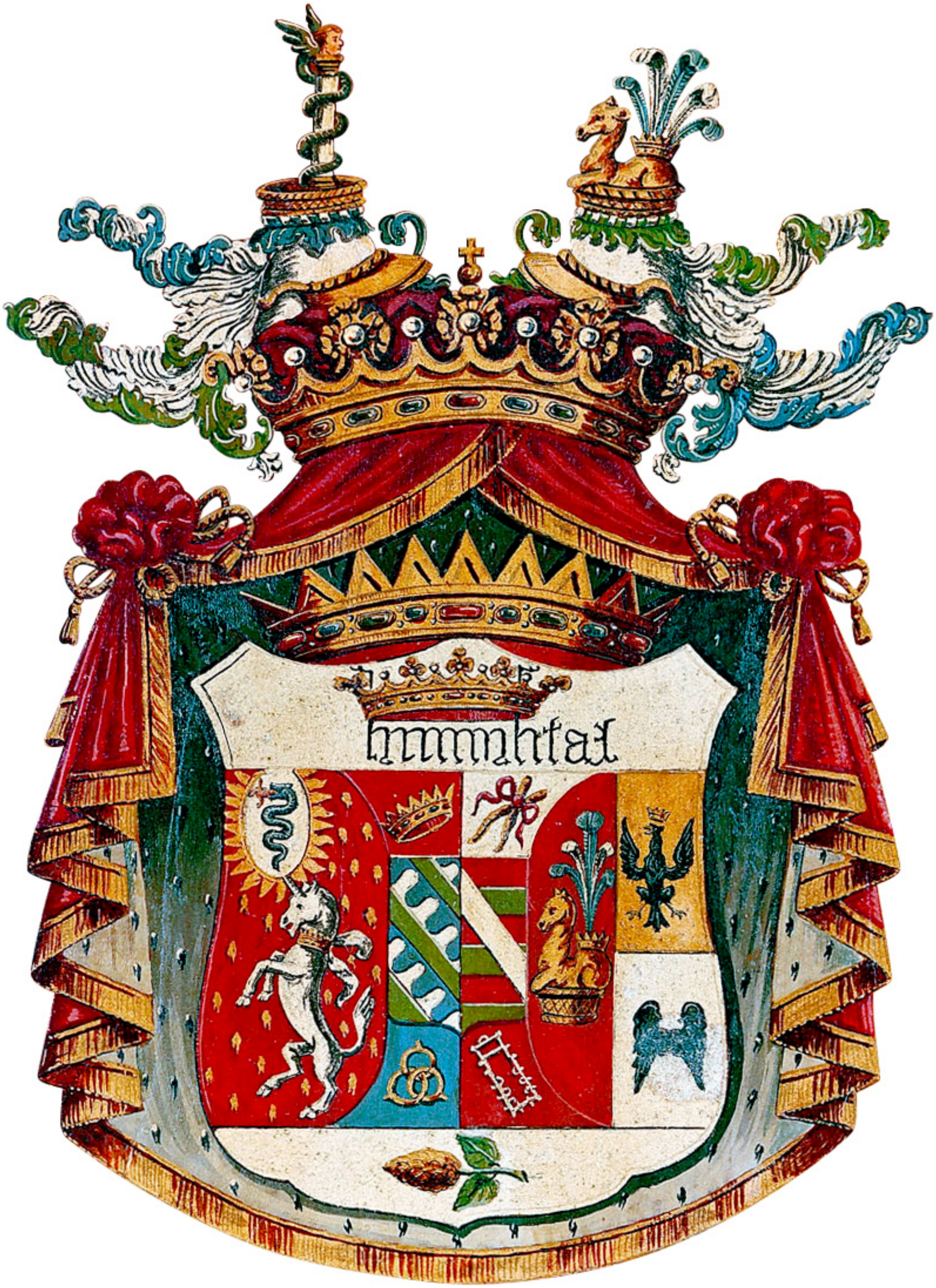
$=$  b.b.f

$\rightsquigarrow M_L = S^3$

$L =$ 

$\rightsquigarrow M_L = S^1 \times S^1 \times S^1$

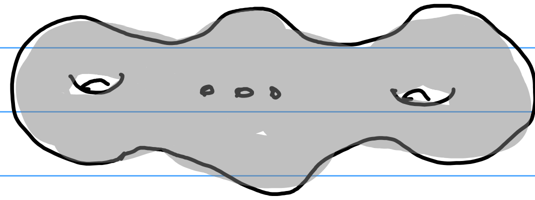
Borromean rings



From www.sempionenews.it/cronaca/un-nuovo-volto-per-i-castelli-di-cannero

Main ideas in proof: M : closed or. comp. conn. 3mf

a) Heegard splitting

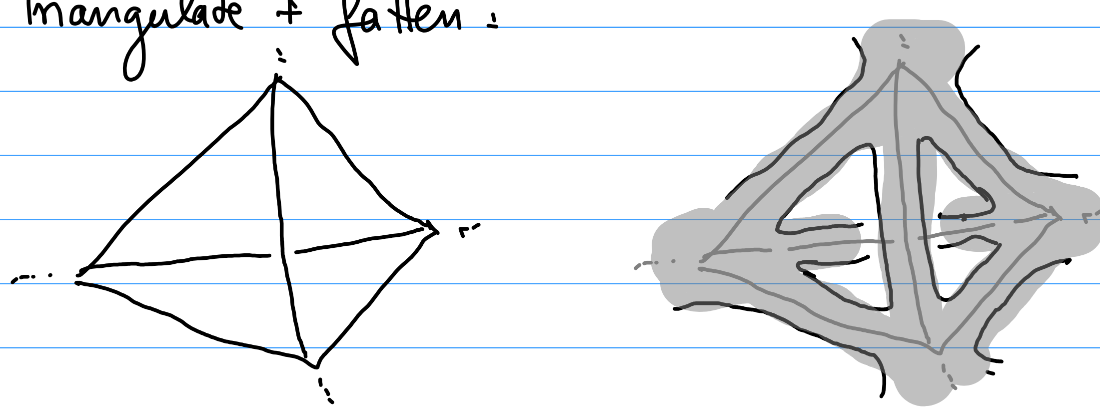
$H_g =$  genus g handlebody in $\mathbb{R}^3$

$\partial H_g = \Sigma_g$ genus g surface

Thm: $\exists\, g \geq 0$, $f : \Sigma_g \to \Sigma_g$ diffeo s.th.

$$M \cong H_g \sqcup_f H_g$$

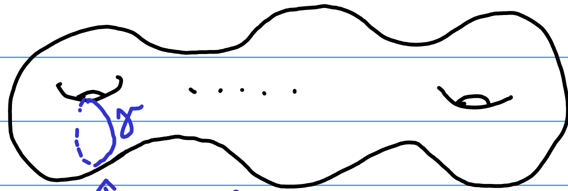
Pf: Triangulate + fatten:



$f : \Sigma_g \longrightarrow \Sigma_g$

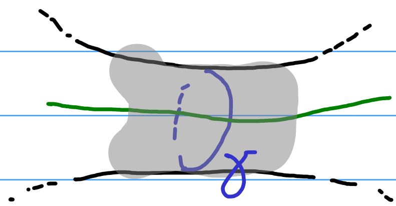
b) Thm (Dehn – Lickorish)

f is isotopic to a finite sequence of Dehn twists



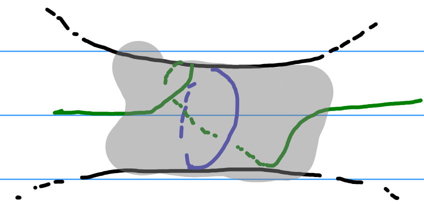
simple closed curve

Σ_g



$\xrightarrow{D_\gamma}$

is identity outside gray region



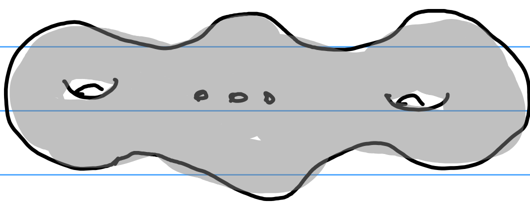
c) Dehn twists via surgery

Let $f_0 : \Sigma_g \to \Sigma_g$ be s.th. $S_3 = H_g \sqcup_{f_0} H_g$

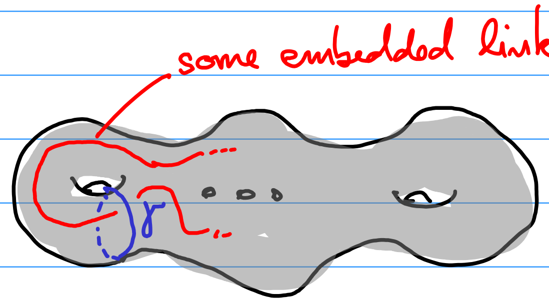
Then can write

$$H_g \sqcup_f H_g \cong H_g \sqcup_{f_0 \circ D_{\gamma_1} \cdots D_{\gamma_n}} H_g$$

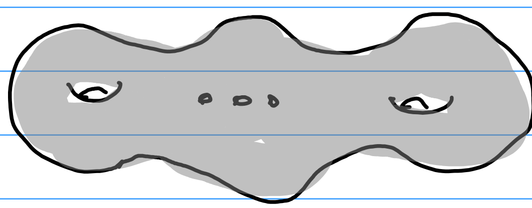
Have



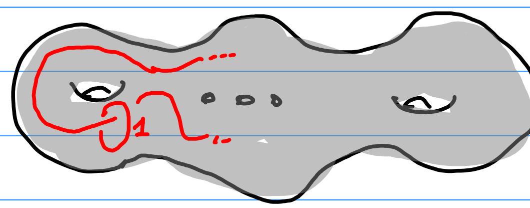
$\sqcup_{g \circ D_\gamma}$



$\cong$



$\sqcup_g$



after applying surgery on