Examples of invariants of ribbon graphs

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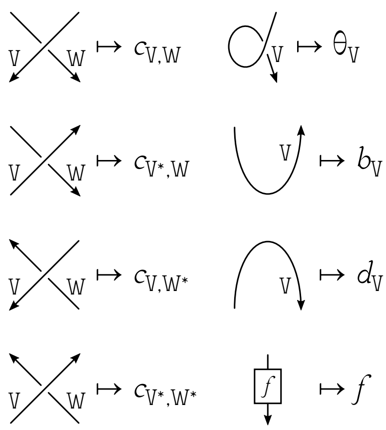
Contents

Preliminaries

See [Tur10, §§I.2.5, XI.2–3—pp. 39–40, 496–503].

Theorem 1. *Given a strict ribbon category* $(\mathcal{V}, c, \theta, (*, b, d))$, *there exists a unique covariant tensor-product-preserving functor* $F\colon \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$ *such that*



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The mirror image diagrams map to the mirror ribbon category $\overline{\mathscr{V}}$, in which

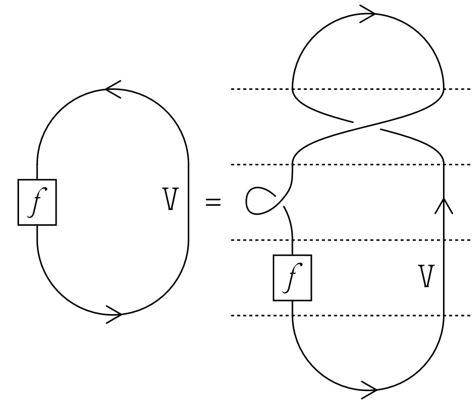
$$\bar{c}_{V,W} = (c_{W,V})^{-1} \quad \bar{\theta}_V = (\theta_V)^{-1}$$

Given a ribbon Hopf algebra (H, R, v), the category of finite-dimensional left H-modules, ${}_H\mathrm{Mod}$, is a ribbon category. In particular, $c = \tau R$ and $\theta = v$.

1 The Hopf link invariant

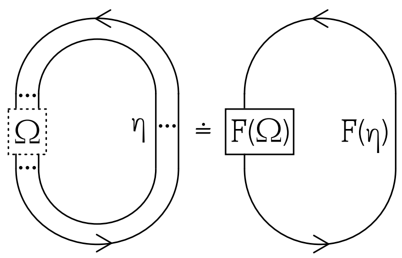
See [Tur10, §I.2.7—pp. 42–45].

Lemma 2.

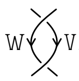


$$\doteq d_V c_{V,V^*}(\theta_V \otimes \mathrm{id}_V)(f \otimes \mathrm{id}_V) b_V = \operatorname{tr} f$$

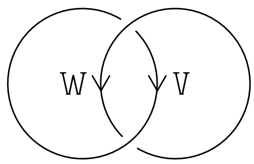
Now consider an endomorphism Ω of an object η of $\mathrm{Rib}_{\mathscr{V}}$, i.e. a ribbon graph from η to itself. We find that



$\doteq \operatorname{tr} F(\Omega)$

By taking $\Omega =$  , we obtain the **Hopf link invariant**

$\operatorname{tr}(c_{W,V} c_{V,W}) \doteq$



2 Group algebras

See e.g. [Tur10, §XI.1.2.1—p. 494].

Let G be a finite group. Consider the group algebra $K[G]$. We can define a coproduct, counit, and antipode by

$$\Delta\colon g \mapsto g \otimes g \quad \varepsilon\colon g \mapsto 1 \quad S\colon g \mapsto g^{-1}$$

The group algebra $K[G]$ is cocommutative by definition, so the natural ribbon structure is topologically trivial. In particular, the natural choice is $c = \tau$, so that $c^2 = 1$, and $\theta = \mathrm{id}$. The ribbons can pass through one another, and can untwist, so for a framed link L with components $L_1, \ldots, L_n$, respectively coloured $V_1, \ldots, V_n$,

$$F(L) = \prod_{i=1}^{n} F(L) = \prod_{i=1}^{n} \mathrm{trid}_{V_i} = \prod_{i=1}^{n} \dim V_i$$

3 Function algebras

See [Tur10, §§XI.1.2.2, 3.4.2, I.2.9.5—pp. 494, 502–3, 48].

Let G be a finite abelian group. Consider the algebra of K-valued functions on G, with Dirac-delta generators $\{\delta_g\}_{g\in G}$. We can define a coproduct, counit, and antipode by

$$\Delta\colon \delta_g \mapsto \sum_{h\in G} \delta_h \otimes \delta_{h^{-1}g} \quad \varepsilon\colon g \mapsto \delta_g(1_G) \quad S\colon \delta_g \mapsto \delta_{g^{-1}}$$

It is is easy to verify that this is cocommutative, but it turns out that a nontrivial braiding is possible. Suppose that G is endowed with a pairing $b\colon G \times G \to K^*$ and a homomorphism $\phi\colon G \to K^*$ s.t $\forall g \in G, \phi(g^2) = 1$. Then take

$$R = \sum_{g,h\in G} b(g,h)\delta_g \otimes \delta_h \quad \nu = \sum_{g\in G} \phi(g) b(g,g)\delta_g$$

For a framed link L with components $L_1, \ldots, L_m$, respectively coloured $V_1, \ldots, V_m$,

$$F(L) = \prod_{1\le i<j\le m} (b(g_j, g_i) b(g_i, g_j))^{l_{ij}} \times \prod_{1\le i\le m} b(g_i, g_i)^{l_i} \phi(g_i)^{l_i+1} \dim V_i$$

where l_{ij} is the linking number of L_i and L_j, and l_i is the number of twists in L_i (the framing number). Because of the commutativity of the algebra, the formula follows by definition of l_{ij}, l_i.

4 $U_q(\mathfrak{sl}(2))$ and the Jones polynomial

See [Oht02, §4.4—pp. 85–93].

Definition 3. $U_q(\mathfrak{sl}(2))$ is a Hopf algebra generated by E, F, K, K^{-1}, with the following relations:

$$KE = q^2 EK, \qquad KK^{-1} = K^{-1}K = 1,$$

$$KF = q^{-2}FK, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

	Δ	ε	S
E	$E \otimes K + 1 \otimes E$	0	$-EK^{-1}$
F	$F \otimes 1 + K^{-1} \otimes F$	0	$-KF$
$K^{\pm 1}$	$K^{\pm 1}$	1	$K^{\mp 1}$

Note that we may formally regard $K^{\pm 1}$ as $q^{\pm H}$.

(For our purposes, we will consider q generic and not consider the root-of-unity case.)

$U_q(\mathfrak{sl}(2))$ is furthermore a ribbon Hopf algebra, with R-matrix

$$R = q^{\frac{1}{2}(H \otimes H)} \exp_q((q - q^{-1})E \otimes F)$$

and ribbon element

$$\nu = q^{-\frac{1}{2}H^2} \sum_{n=0}^{\infty} \frac{1}{[n]_q!} q^{\frac{3}{2}n(n+1)} (q^{-1} - q)^n F^n K^{-n-1} E^n$$

Here, $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, $\quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q$, and

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} q^{\frac{1}{2}n(n-1)} x^n$$

(Note that, technically, R and θ are not in $U_q(\mathfrak{sl}(2))$, unless we take a completion.)

A ribbon invariant

We can obtain a topological invariant $F^V(\Omega)$ of a ribbon graph Ω by choosing some $V \in {}_{U_q(\mathfrak{sl}(2))}\mathrm{Mod}$ and using it to colour all the ribbons (annuli and bands) of Ω. Consider the two-dimensional irreducible representation of $U_q(\mathfrak{sl}(2))$:

$$\rho_{\mathbf{C}^2}(E) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \rho_{\mathbf{C}^2}(F) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \rho_{\mathbf{C}^2}(H) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Taking $V = \mathbf{C}^2$, we can proceed to calculate $c_{V,V}$.

Using $\rho_V(E^2) = \rho_V(F^2) = 0$,

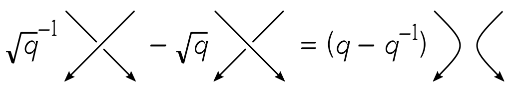
$$
\begin{aligned}
(\rho_V \otimes \rho_V) \exp_q((q - q^{-1})E \otimes F) \\
&= \rho_V(1) \otimes \rho_V(1) + (q - q^{-1})\rho_V(E) \otimes \rho_V(F) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
(\rho_V \otimes \rho_V) q^{\frac{1}{2}(H \otimes H)} &= q^{\frac{1}{2}(\rho_V(H) \otimes \rho_V(H))} \\
&= q^{\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \\
&= \begin{bmatrix} \sqrt{q} & 0 & 0 & 0 \\ 0 & \sqrt{q}^{-1} & 0 & 0 \\ 0 & 0 & \sqrt{q}^{-1} & 0 \\ 0 & 0 & 0 & \sqrt{q} \end{bmatrix}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
c_{V,V} &= \tau(\rho_V \otimes \rho_V)\mathcal{R} \\
&= \sqrt{q}^{-1} \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{bmatrix}
\end{aligned}
$$

It is straightforward to verify that the following skein relations are satisfied:



$$\sqrt{q}^{-1} c_{V,V} - \sqrt{q} c_{V,V}^{-1} = (q - q^{-1}) \mathrm{id}_{V \otimes V}$$

A link invariant

To get a link invariant from the ribbon invariant, we must deal with the extra information contained in a ribbon graph, i.e. the framing. In ${}_{U_q(\mathfrak{sl}(2))}\mathrm{Mod}$, the twist is

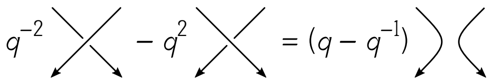
$$
\begin{aligned}
\theta_{V,V} = \rho_V(\nu) &= \rho_V(q^{-\frac{1}{2}H^2})\rho_V(K^{-1} + q^2(q^{-1} - q)FK^{-2}E) \\
&= \begin{bmatrix} \sqrt{q}^{-1} & 0 \\ 0 & \sqrt{q}^{-1} \end{bmatrix} \begin{bmatrix} q^{-1} & 0 \\ 0 & q^{-1} \end{bmatrix} = \sqrt{q}^{-3} I
\end{aligned}
$$

which happens to be a scalar. For a link diagram Ω, the writhe $w(\Omega)$ is defined as the number of positive crossings minus negative crossings. The combination

$$\theta_{V,V}^{w(\Omega)} F^V(\Omega)$$

then, gives us an invariant of the underlying link L.

Now the skein relations



$$q^{-2} c_{V,V} - q^2 c_{V,V}^{-1} = (q - q^{-1})\mathrm{id}_{V\otimes V}$$

are satisfied, which means that, up to a normalization and reparameterization, we have obtained the Jones polynomial.

5 Modular tensor categories

See [Tur10, §II.1—pp. 72–78], [Tak01, §4—pp. 638–640].

Definition 4. An Ab-category is a category $\mathcal{V}$ in which there is an addition on morphisms, i.e. $\forall V, W \in \mathcal{V}$, $\mathrm{Hom}(V, W)$ is an additive abelian group.

If $\mathcal{V}$ is monoidal, $K = \mathrm{End}(\mathbf{1}) = \mathrm{Hom}(\mathbf{1}, \mathbf{1})$ is a commutative ring, called the **ground ring**. Now $\mathrm{Hom}(V, W)$ is a left K-module with scalar multiplication $kf = k \otimes f$.

Definition 5. An object V of a monoidal Ab-category $\mathcal{V}$ is called **simple** if End(V) is a rank-1 free K-module. In other words, V is simple if scalar multiplication defines a bijection $K \to \mathrm{End}(V)$.

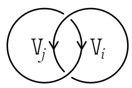
For instance,

- $\mathbf{1}$ is always simple.
- In the category Vect_K of vector spaces over a field K, the simple objects are the 1-dimensional vector spaces.

Definition 6. A monoidal Ab-category $\mathcal{V}$ with direct sum $\oplus$ is called **semisimple** if every object can be written as a direct sum of simple objects.

Definition 7. A semisimple ribbon category $\mathcal{V}$ with a complete basis of simple objects $\{V_i\}_{i\in I}$ is a **modular category** if $S = [S_{i,j}]_{i,j\in I}$ is an invertible matrix, where

$$S_{i,j} = \mathrm{tr}(c_{V_j,V_i} c_{V_i,V_j}) \doteq$$



Example 8. For example, in the group algebra case, the simple objects just correspond to elements of G, so

$$S_{i,j} = \dim V_i \dim V_j = 1,$$

(using the formula for framed links), which is clearly not bijective (unless G is the trivial group).

Example 9. In the function algebra case,

$$S_{i,j} = b(g_j, g_i)b(g_i, g_j)\phi(g_i)\phi(g_j),$$

which form an invertible matrix iff $[b(g_j, g_i)b(g_i, g_j)]_{i,j}$ is invertible.

Definition 10. The purpose of modular categories, as far as we are concerned, is to define invariants of 3-manifolds. To accomplish this goal, we will need to select two elements of $\mathscr{V}$.

1. A **rank** $\mathscr{D}$ is an element of K s.t.

$$\mathscr{D}^2 = \sum_{i\in I} (\dim V_i)^2$$

   There may be many ranks, or none, and the invariant will depend on the choice of one.

2. Since V_i is simple, θ acts in V_i as a scalar $v_i \in K$, which is furthermore invertible. We define

$$\Delta_{\mathscr{V}} = \sum_{i\in I} v_i^{-1}(\dim V_i)^2 \in K$$

6 Factorisable Hopf algebras

See [Tak01, §§2-4—pp. 636-640].

Definition 11. For a (finite-dimensional) quasi-triangular Hopf algebra (H, R), we define the **Drinfeld map** as

$$\Phi: H^* \longrightarrow H \quad (1)$$
$$f \longmapsto \mu \circ (\mathrm{id} \otimes f) \circ (R_{21}R) \quad (2)$$

If Φ is an isomorphism, H is called **factorizable**.

Theorem 12. *Let* H *be a semisimple ribbon Hopf algebra over an algebraically closed field* K*. If* H *is factorizable then* ${}_H\mathrm{Mod}$ *is modular.*

Example 13. For example, in the group algebra case,

$$\Phi: f \mapsto \mu \circ (\mathrm{id} \otimes f) \circ (1 \otimes 1) = f(1)$$

is clearly not bijective (unless G is the trivial group).

Example 14. In the function algebra case, on a basis element $f \in G$

$$\Phi: f \mapsto \mu \circ (\mathrm{id} \otimes f) \circ \sum_{g,h\in G} (b(h, g)b(g, h)\delta_g \otimes \delta_h) = \sum_{g\in G} b(f, g)b(g, f)\delta_g$$

so $R_{21}R$ acts as a matrix, and Φ is bijective iff $[b(h, g)b(g, h)]_{g,h\in G}$ is an invertible matrix.

References

[Oht02] Tomotada Ohtsuki. *Quantum Invariants. A Study of Knot, 3-Manifolds, and Their Sets*. World Scientific, 2002.

[Tak01] Mitsuhiro Takeuchi. “Modular Categories and Hopf Algebras”. In: *Journal of Algebra* 243.2 (2001), pp. 631–643.

[Tur10] Vladimir G. Turæv. *Quantum Invariants of Knots and 3-Manifolds*. 2nd ed. De Gruyter, 2010.