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L1) Jones polynomial and Kaufman bracket

[Knots and links in $\mathbb{R}^3$. Linking number of a knot. Kaufman bracket, writhe of a knot and the Jones polynomial. Invariants of framed links.]

The main reference of this handout is the book "Quantum Invariants, A Study of Knot, 3-Manifolds, and Their Sets" by Ohtsuki. All definitions and theorems statet here are taken from section 1.1 and 1.2.

1 Knots and links

Definition 1.1. A knot is the image of a smooth (or piecewise smooth) embedding of the one sphere S^1 onto $\mathbb{R}^3$.

Definition 1.2. A link of l components is the image of a smooth (or piecewise smooth) embedding of the disjoint union of l circles into $\mathbb{R}^3$.

Definition 1.3 (**Isotopoic knots and links**)**.** Two knots (or two links) K and K' are called *isotopic* if K is obtained from K' by a continuous deformation such that there is no self-intersection at any time during the deformation.

Definition 1.4 (**Diagrams of knots and links**)**.** A knot diagram is a smooth immersion $S^1 \to \mathbb{R}^2$ with at most finitely many transversal double points such that the two paths at each double point are assigned to be the over path and the under path respectively. We call a double point of such an immersion a *crossing* of the knot diagram. When a knot diagram D is obtained as the image of a knot by a projection $\mathbb{R}^3 \to \mathbb{R}^2$, we call D a diagram of the knot. A link diagram is defined similarly.

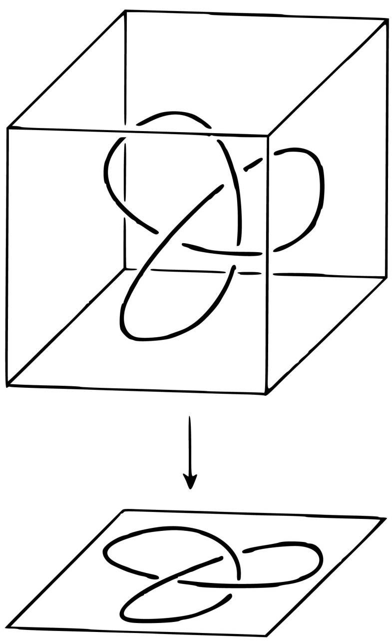


Figure 1.1 Projection of the trefoil knot

Definition 1.5 (**Isotopoic knot and link diagrams**)**.** Two knot diagrams (or two link diagrams) D and D' are called *isotopic* if D is obtained from D' by a continuous deformation such that there is no self-intersection at any time during the deformation.

Definition 1.6 (**The Reidemeister moves**)**.** A Reidemeister move operates on a small region of a diagram and is one of this types:

R1: Twist and untwist in either direction.

R2: Move one loop completely over another.

R3: Move a loop completely over or under a crossing.

Figure 1.1 The Reidemeister moves

Theorem 1.7. Let K and K' be two links and D and D' diagrams of them. Then K is isotopic to K' if and only if D is related to D' by a sequence of isotopies of $\mathbb{R}^2$ and the Reidemeister moves R1, R2 and R3.

Proof. If D and D' are related by a sequence of the moves, then K and K' are isotopic, since the moves are trivial in $\mathbb{R}^3$. We omit the other direktion. □

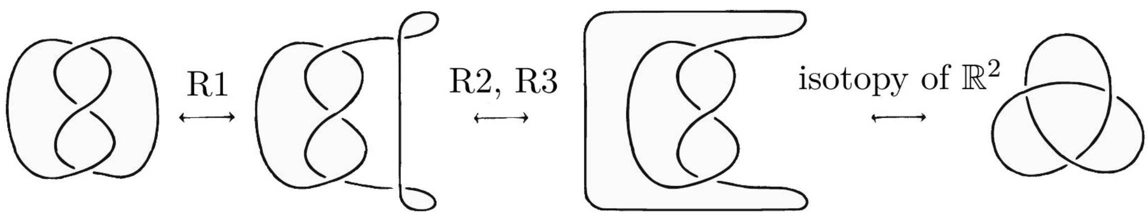


Figure 1.2 Isotopie of the Treefoilknot

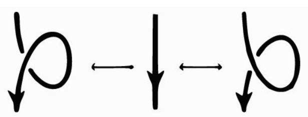
Theorem 1.1 has the following symbolic representation,

$$\{\text{knots}\}\,/\,\text{isotopy of }\mathbb{R}^3 = \{\text{knot diagrams}\}\,/\,\text{R1, R2, R3 and isotopy of }\mathbb{R}^2$$

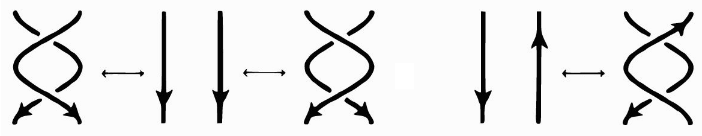
Definition 1.8. An *oriented link* is the image of an embedding of the disjoint union of oriented circles into $\mathbb{R}^3$. An *oriented diagram* is defined similarly as an immersion of oriented circles in $\mathbb{R}^2$.

Definition 1.9 (**The oriented Reidemeister moves**)**.** The oriented Reidemeister moves on a oriented link diagram are defined als follows.

$\overrightarrow{\text{R1}}$:



$\overrightarrow{\mathrm{R2}}$:



$\overrightarrow{\mathrm{R3}}$:

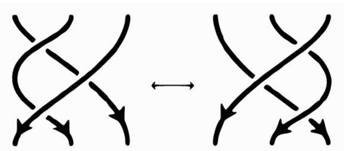
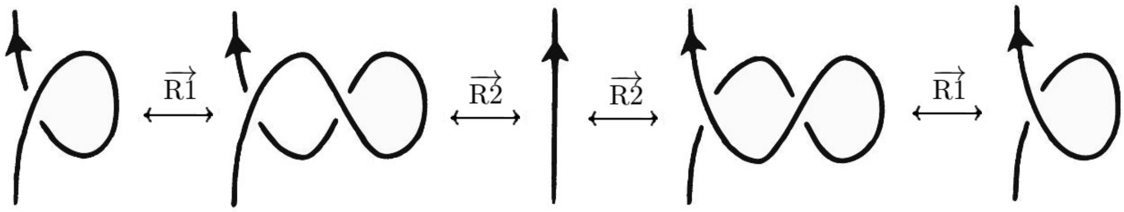


Figure 1.2 The oriented Reidemeister moves

Theorem 1.10. Let K and K' be two oriented links and D and D' oriented diagrams of them. Then K is isotopic to K' if and only if D is related to D' by a sequence of isotopies of $\mathbb{R}^2$ and the modified Reidemeister moves $\overrightarrow{\mathrm{R1}}$, $\overrightarrow{\mathrm{R2}}$ and $\overrightarrow{\mathrm{R3}}$.

Proof. It is sufficient to verify that each of the Reidemeister moves with any orientation can be obtained as a sequence of the $\overrightarrow{\mathrm{R1}}$, $\overrightarrow{\mathrm{R2}}$ and $\overrightarrow{\mathrm{R3}}$ moves. Since there are many cases we only schow the R1 move. For one orientation the R1 move is $\overrightarrow{\mathrm{R1}}$, for the other we use the following sequence.



□

2 Linking number

In an oriented diagram we call a "left pass under right"-crossing positive and a "left pass over right"-crossing negative.

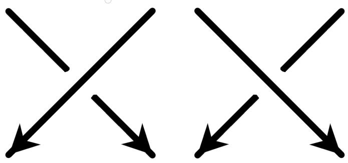


Figure 1.3 A positive and a negative crossing

Definition 2.1. The *linking number* of two components L_1 and L_2 of an oriented link is defined by

$$\begin{aligned}\mathrm{lk}(L_1, L_2) = \frac{1}{2}\ &((\text{the number of positive crossings of two strands of } D_1 \text{ and } D_2) \\ &- (\text{the number of negative crossings of two strands of } D_1 \text{ and } D_2)),\end{aligned}$$

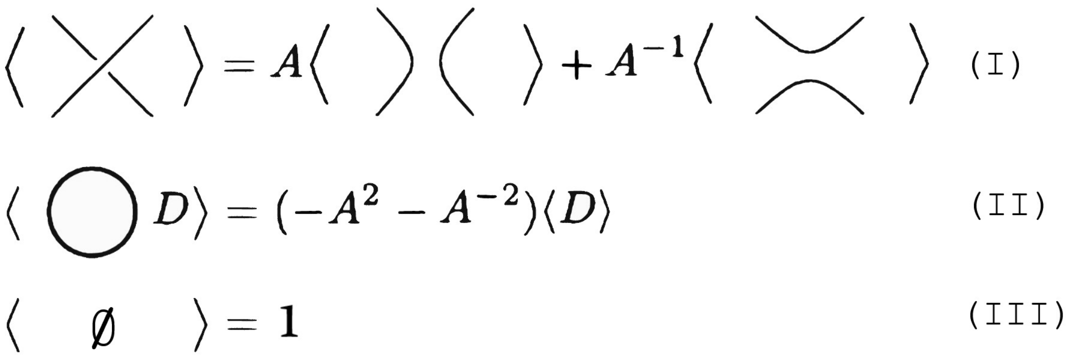
where $D_1 \cup D_2$ is a diagram of $L_1 \cup L_2$.

Theorem 2.2. The linking number $\mathrm{lk}(L_1, L_2)$ is an isotopy invariant of an oriented link $L_1 \cup L_2$.

Proof. The linking number is invariant under the $\overrightarrow{\mathrm{R1}}, \overrightarrow{\mathrm{R2}}$ and $\overrightarrow{\mathrm{R3}}$ moves, hence the proposition is obtained from Theorem 1.10. □

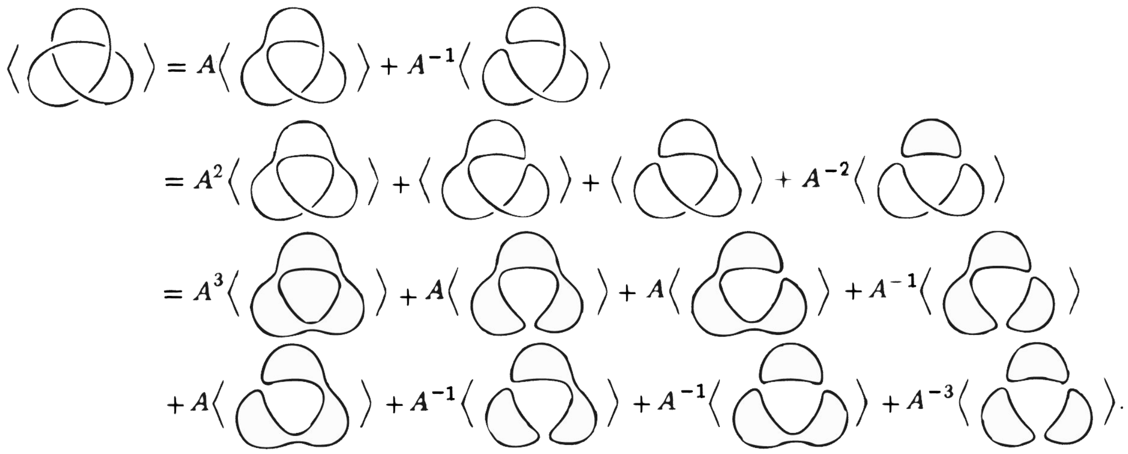
3 The Kaufmann bracket

Definition 3.1. For a link diagram D in $\mathbb{R}^2$ the Kauffman bracket $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ of D is defined as follows. We consider the following three recursive formulae



where D is a diagram without crossings.

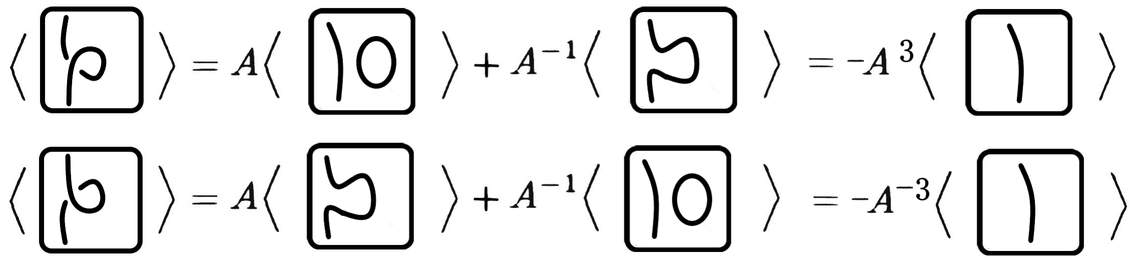
Example 3.2. Let D be the canonical diagram of the Trefoil knot. We calculat the Kaufmann bracket of the Trefoil knot $\langle D \rangle$. At first resolve all crossings with the first formula.



Since a diagram without crossings is the disjoint union of loops we obtail the value of the bracket recursively by (II) and (III).

$$\langle D \rangle = (-A^2 - A^{-2})(-A^5 - A^{-3} + A^{-7}).$$

To obtain an isotopy invariant of links from the Kauffman bracket of their diagrams, it is sufficient to show the invariance of the Kauffman bracket under the R1, R2 and R3 moves. Unfortunately, $\langle D \rangle$ does change under the R1 move on a diagram D as can be seen by the following calculation.



R2 and R3 do not change $\langle D \rangle$, we omit the calculation. To deal with R1, we modify $\langle D \rangle$ by *writhe* of D.

4 The Jones polynomial

Definition 4.1. For an oriented diagram D we define the *writhe* of D by

$$\begin{aligned} w(D) = & \text{ (the number of positive crossings of } D) \\ & -(\text{the number of negative crossings of } D). \end{aligned}$$

Theorem 4.2. Let L be an oriented link, and D an oriented diagram of L. Then,

$$(-A^3)^{-w(D)} \langle D \rangle$$

is invariant under the R1, R2 and R3 moves, where $\langle D \rangle$ is the Kauffman bracket of D with its orientation forgotten. In particular, it is an isotopy invariant of L.

Proof. It is sufficient to check the invariance under the moves. □

It is an elementary exercise to show that the values of $(-A^3)^{-w(D)} \langle D \rangle$ belong to the polynomial ring $\mathbb{Z}[A^2, A^{-2}]$.

Definition 4.3 (Jones polynomial)**.** Putting $A^2 = t^{-1/2}$ we define the quotient of the invariant by $(-A^2 - A^{-2})$ as the *Jones polynomial* $V_L(t)$ of an oriented link L, i.e., we set

$$V_L(t) = \frac{(-A^3)^{-w(D)}}{(-A^2 - A^{-2})} \langle D \rangle \Bigg|_{A^2 = t^{-1/2}} \in \mathbb{Z}[t^{1/2}, t^{-1/2}].$$

Example 4.4. For the trivial knot K_0 the Jones polynomial $V_{K_0}(t) = 1$. Let L be the Trefoil knot with orientation and D its oriented diagram. Then the Jones polynomial is

$$\begin{aligned} V_L(t) &= \frac{(-A^3)^{-w(D)}}{(-A^2 - A^{-2})} \langle D \rangle \Bigg|_{A^2 = t^{-1/2}} \\ &= (-A^3)^{-3}(-A^5 - A^{-3} + A^{-7}) \Big|_{A^2 = t^{-1/2}} \\ &= t + t^3 - t^4. \end{aligned}$$

Theorem 4.5. The Jones polynomial satisfies the following relation, called the *skein relation* of the Jones polynomial,

$$t^{-1} V_{L_+} - t V_{L_-}(t) = (t^{1/2} - t^{-1/2}) V_{L_0}(t).$$

Here, L_+, L_-, and L_0 are three oriented links, which are identical except for a ball, where they have a positive, a negative or no crossing, respectively.

5 The Kaufmann brackets as an invariant of framed links

Definition 5.1. A *framed link* is the image of an embedding of a disjoint union of annuli into $\mathbb{R}^3$. The *underlying link* of a framed link is the link obtained by restricting an annulus $S^1 \times [0,1]$ to its center line $S^1 \times \{1/2\}$.

Definition 5.2. Since framed knots are orientable manifolds they can be projected onto $\mathbb{R}^2$ such that the annuli are "flat". We say such diagram is obtained by *blackboard framing*.

Theorem 5.3. Let L and L' be two framed links, and D and D' diagrams of them by blackboard framings. Then, L is isotopic to L' if and only if D is related to D' by a sequence of isotopies of $\mathbb{R}^2$ and the $\mathcal{R}1$ (See figure 1.4), R2 and R3 moves.

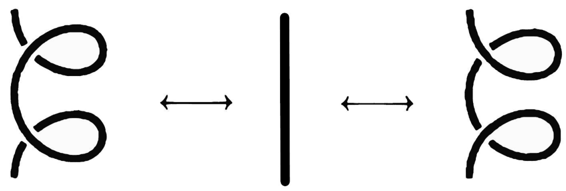


Figure 1.4 The $\mathcal{R}1$ move

Proof. It is trivial to show that, if D is related to D' by a sequence of the moves, then L is isotopic to L'.
Conversely, suppose that L is isotopic to L'. Then, by forgetting the framings D and D' are related by a sequence of isotopies of $\mathbb{R}^2$ and R1, R2 and R3 moves by Theorem 1.1. Modifying the sequence we obtain a sequence of isotopies of $\mathbb{R}^2$ and the $\mathcal{R}1$, R2 and R3 moves using the fact that D and D' have the same framing. □

Corollary 5.4. Let L and L' be two framed links, and D and D' diagrams of them by blackboard framings. Further, we regard D and D' as diagrams on $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Then, L is isotopic to L' if and only if D is related to D' by a sequence of isotopies of S^2 and R2 and R3 moves.

Proof. We obtain the $\mathcal{R}\infty$ move by pulling a loop around the Sphere followed by a sequence of R2 and R3 moves. □

Theorem 5.5. Let L be a framed link, and let D be a diagram of L by blackboard framing. Then, the Kauffman bracket $\langle D \rangle$ is an isotopy invariant of L. We denote the invariant also by $\langle L \rangle$ and call it the Kauffman bracket of a framed link L.

Proof. As shown before, $\langle D \rangle$ is invariant under the R2 and R3 moves on a diagram D. Hence, by Corollary 5.3 it is an isotopy invariant of a framed link L. □

Remark 5.6. It is known that there are nontrivial links with Jones polynomial equal to that of the corresponding unlinks by the work of Morwen Thistlethwaite. It is an open problem, if there is a none trivial knot with Jones polynomial equal to 1.