The Dennis trace map and stable K-theory

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1. The Dennis trace map

We saw last time, that one can define the K-theory space of a ring with unit R as

$$K(R) := K_0(R) \times BGL(R)^+.$$

Here, $K_0(R)$ is there in order to obtain the correct π_0, so for higher n, $\pi_n K(R) = K_n(R)$ isn't affected by it. As $K_0(R)$ often can be determined by hand, it is important to understand the homotopy groups of $BGL(R)^+$, or at least to find approximations for them.

For any based space X and any $n > 0$ there is a canonical map

$$\pi_n(X; x) \to H_n(X).$$

For a class $[\alpha] \in \pi_n(X; x)$ with representative $\alpha\colon \mathbb{S}^n \to X$ we use a generator $\mu_n \in H_n(\mathbb{S}; \mathbb{Z})$ and the naturality of singular homology to obtain an element

$$H_n(\alpha)[\mu_n] \in H_n(X).$$

This map

$$\pi_n(X; x) \to H_n(X),\ [\alpha] \mapsto H_n(\alpha)[\mu_n]$$

is called the Hurewicz map and we denote it by h_n. You've probably seen the case $n = 1$ where we obtain

$$h_1\colon \pi_1(X; x) \to H_1(X) = \pi_1(X; x)_{ab}$$

and where h_1 is the abelianization map.

We work with path-connected spaces, so we'll drop the basepoint from the notation and if we take singular homology with integral coefficients, we'll also drop the $\mathbb{Z}$.

We learned last time that the inclusion $X \subset X^+$ induces an isomorphism on homology groups. We can start the trace map for $n \geq 1$ as the composite

$$K_n(R) = \pi_n(BGL(R)^+) \xrightarrow{h_n} H_n(BGL(R)^+) \xleftarrow{\cong} H_n(BGL(R)).$$

Singular homology of classifying spaces can be identified with group homology. If G is any discrete group, then

$$H_n(BG) \cong H_n(G).$$

This can be seen by comparing a simplicial model of BG whose set of n-simplices is G^n with the algebraic bar construction that we used to calculate $H_*(G)$.

Thus we obtain

$$H_n(BGL(R)) \cong H_n(GL(R)).$$

The next building block for the trace map is the so-called fusion map: In the group ring $\mathbb{Z}[GL_p(R)]$ you consider formal linear combinations $\sum_{i=1}^{\ell} t_i A_i$ with $t_i \in \mathbb{Z}$ and $A_i \in GL_p(R)$. For the fusion map we evaluate this formal linear combination in $M_p(R)$ using multiples and sums of matrices. This induces a map of rings

$$f\colon \mathbb{Z}[GL_p(R)] \to M_p(R).$$

However, f is *not* compatible with the stabilization maps $GL_p(R) \to GL_{p+1}(R)$ where we map A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and the corresponding ones for $M_p(R) \to M_{p+1}(R)$ where we send a $B \in M_p(R)$ to $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$.

We consider the composition

$$\xymatrix{ \mathbb{Z}[GL_p(R)]^{\otimes n} \ar[r]^t \ar@/^3pc/[rrr]^{g_p} & \mathbb{Z}[GL_p(R)]^{\otimes n+1} \ar[r]^f & M_p(R)^{\otimes n+1} \ar[r]^{tr} & R^{\otimes n+1} }$$

where t is the map that is induced by $(A_1, \ldots, A_n) \mapsto (A_1 \cdot \ldots \cdot A_n)^{-1}, A_1, \ldots, A_n)$ and where tr is the generalized trace map that you know from Morita invariance of Hochschild homology. This composite *does* commute with the stabilization maps because the extra 1s in the lower right corner just give something degenerate.

Everybody should check that g_p is actually a chain map.

So in total we obtain the Dennis trace map $K_n(R) \to \mathsf{HH}_n(R)$ as the composite

$$\xymatrix{ K_n(R) \ar[r]^{\pi_n(pr)} & \pi_n(BGL(R)^+) \ar[d]^{h_n} & & \\ & H_n(BGL(R)^+) & H_n(BGL(R)) \ar[l]_{\cong} \ar[d]^{\cong} & \\ & & H_n(GL(R);\mathbb{Z}) \ar[r]^g & \mathsf{HH}_n(R). }$$

Here, $pr\colon K(R) = K_0(R) \times BGL(R)^+ \to BGL(R)^+$ projects away $K_0(R)$.

1.1. **Trace factors through $\mathsf{HML}_*(R)$.** Recall that

$$\mathsf{HML}_*(R; M) \cong \mathsf{Tor}_*^{\mathcal{F}(R)}(I^*, M \otimes_R (-))$$

so in particular,

$$\mathsf{HML}_*(R) \cong \mathsf{Tor}_*^{\mathcal{F}(R)}(I^*, I).$$

We also saw that this can be identified with $H_*(F(R); \mathrm{Hom})$ where Hom denotes the bifunctor $(M, N) \mapsto \mathrm{Hom}(M, N)$.

We construct a map from $H_*(GL(R); \mathbb{Z})$ to $H_*(F(R); \mathrm{Hom})$. Composed with the canonical map $\mathsf{HML}_*(R) \to \mathsf{HH}_*(R)$ this gives the Dennis trace map.

First we consider the map of rings $\mathbb{Z} \to M_n(R)$ that sends 1 to the identity matrix. This gives a map

$$H_*(GL_n(R); \mathbb{Z}) \to H_*(GL_n(R); M_n(R))$$

where on the right-hand side $M_n(R)$ carries the conjugation action of $GL_n(R)$. We compose this with

$$H_*(GL_n(R); M_n(R)) \to H_*(F(R); \mathrm{Hom}).$$

Here, we identify $GL_n(R)$ with the category that has one object $*$ and has the elements of $GL_n(R)$ as morphisms. The homology of this category is the group homology of $GL_n(R)$.

Then $M_n(R)$ is a bifunctor on this category: You send $(*, *)$ to the abelian group $(M_n(R), +)$ and $(A, B) \in GL_n(R)^2$ to the morphism $A(-)B^{-1}$.

There is a natural transformation $GL_n(R) \to F(R)$, sending $*$ to R^n and $M_n(R)$ corresponds to $\mathrm{Hom}(R^n, R^n)$.

I omit the proof that this is compatible with stabilization. This needs an explicit chain homotopy (see e. g. [L, p. 405]).

2. Stable K-theory

The canonical inclusion $BGL(R) \subset BGL(R)^+$ will be far from being a fibration. However, we can consider its *homotopy fiber*:

Let $f: X \to Y$ be an arbitrary continuous map with X and Y path connected. We can replace f by a fibration as follows. Consider the space P_f of pairs (x, ω) with $x \in X$ and $\omega \in Y^I$ such that $\omega(0) = f(x)$. So P_f is the pullback (aka fiber product)

$$\begin{array}{ccc} P_f = X \times_Y Y^I & \longrightarrow & Y^I \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

where $\mathrm{ev}_0(\omega) = \omega(0)$. The space Y^I carries the compact open topology. (For a fixed $x \in X$, you could think of all the ω's satisfying $\omega(0) = x$ as a replacement of a neighbourhood: The paths start in x and end somewhere in Y.)

We can view X as a subspace of P_f by sending an $x \in X$ to $(x, c_{f(x)})$ where c_x is the constant path at x. This inclusion is actually a homotopy equivalence because you can contract an arbitrary path back to where it started. This is compatible with the maps f and p.

Then the map $p: P_f \to Y$, $p(x, \omega) = \omega(1)$ is a fibration. For a point $y_0 \in Y$ we consider the fiber of p at y_0 and call this the *homotopy fiber of* f, $\mathrm{hfib}(f)$. If we spell out what that means then we get

$$\mathrm{hfib}(f) = \{(x, \omega) \in X \times Y^I, \omega(0) = x, \omega(1) = y_0\}.$$

This homotopy fiber is only defined up to homotopy equivalence, because of the *choice* of y_0. It sits in the diagram

$$\begin{array}{cccc} \mathrm{hfib}(f) & & & \\ & \searrow & & \\ & P_f = X \times_Y Y^I & \longrightarrow & Y^I \\ & {\scriptstyle \sim}\downarrow & \overset{p}{\searrow} & \downarrow{\scriptstyle \mathrm{ev}_0} \\ & X & \xrightarrow{\ f\ } & Y \end{array}$$

As $P_f \to Y$ is a fibration and as $X \simeq P_f$, we get a long exact sequence on homotopy groups

$$\ldots \longrightarrow \pi_n(\mathrm{hfib}(f)) \longrightarrow \pi_n(X) \xrightarrow{\pi_n(f)} \pi_n(Y) \longrightarrow \pi_{n-1}(\mathrm{hfib}(f)) \longrightarrow \ldots$$

If we apply this to the inclusion map $i\colon BGL(R) \subset BGL(R)^+$ and if we denote the homotopy fiber of i by $\Psi(R)$ we get a long exact sequence on homotopy groups:

$$\pi_2(BGL(R)^+) \longrightarrow \pi_1(\Psi(R)) \longrightarrow \pi_1 BGL(R) = GL(R) \xrightarrow{\pi_1(i)} \pi_1 BGL(R)^+ = GL(R)/E(R) \longrightarrow 1$$

As $\pi_2 BGL(R)^+ = K_2(R)$ and as we identified this with the kernel of the canonical map $St(R) \to E(R)$, we obtain that $\pi_1(\Psi(R))$ is the Steinberg group of R, $St(R)$.

Definition Let R be a ring with unit and let N be an R-bimodule. The *stable K-theory of R with coefficients in N* is defined as

$$K_n^{st}(R; N) := H_n(\Psi(R); M(N)).$$

On the right hand side we have the nth singular homology group of the space $\Psi(R)$ with local coefficients in $M(N) = \bigcup M_n(N)$. So we have to know what local coefficients are and how $St(R)$ acts on $M(N)$.

2.1. **Local coefficients.** We assume that a space X is path-connected and has a universal covering $\tilde{X}$. You know that $\pi_1(X; x)$ acts on $\tilde{X}$ and hence it acts on the singular chains of $\tilde{X}$, $S_*(\tilde{X})$, by sending a generator $\alpha\colon \Delta^n \to \tilde{X}$ to $\gamma.\alpha$ for $\gamma \in \pi_1(X, x)$. We can view $S_n(\tilde{X})$ therefore as a module over $\mathbb{Z}[\pi_1(X; x)]$. We can shift this to a right module structure by acting with inverses.

Let $\mathscr{L}$ be an abelian group and let $f\colon \pi_1(X; x) \to \mathrm{Aut}(\mathscr{L})$ be a homomorphism. Then $\mathscr{L}$ is a left $\mathbb{Z}[\pi_1(X; x)]$-module.

The *nth singular chain group of X with local coefficients in $\mathscr{L}$* is then

$$S_n(X; \mathscr{L}) := S_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X;x)]} \mathscr{L}.$$

The boundary map d on $S_*(\tilde{X})$ induces a boundary map $d \otimes_{\mathbb{Z}[\pi_1(X;x)]} \mathrm{id}$ on $S_n(X; \mathscr{L})$, but beware that the tensor product is taken over the group ring, so there is some twisting going on. The homology of this complex is then the homology of X with local coefficients in $\mathscr{L}$.

If X is a CW complex, you can do the same with cellular chains.

If the action of the fundamental group on $\mathscr{L}$ is trivial, then you just get the ordinary (singular, cellular) homology of X with coefficients in $\mathscr{L}$. But for instance if you consider $\mathbb{Z}$ with the non-trivial $\mathbb{Z}/2\mathbb{Z}$-action, then the homology of $\mathbb{R}P^2$ with these local coefficients differs from ordinary singular homology.

Another important example is group homology. If M is a G-module, then the group homology of G with coefficients in M, $H_*(G; M)$, is isomorphic to the singular homology of the classifying space BG with coefficients in the local system M. If you want to know more about this, then [DK] is an excellent source.

2.2. **The action of $St(R)$ on $M(N)$.** First of all, $GL_n(R)$ acts by conjugation on $M_n(N)$ and this action is compatible with the stabilization maps $GL_n(R) \hookrightarrow GL_{n+1}(R)$ and $M_n(N) \hookrightarrow M_{n+1}(N)$, so we get an action of $GL(R)$ on $M(N)$. The Steinberg group of R then acts on $M(N)$ via the $GL(R)$-action.

So now

$$K_n^{st}(R; M) := H_n(\Psi(R); M(N))$$

syntactically makes sense. Why is this interesting?

Theorem [Dundas-McCarthy 1994 [DM]]: For any ring with unit R and any R-bimodule N there is a natural isomorphism

$$K_*^{st}(R;N) \cong \mathsf{HML}_*(R;N).$$

Why the heck should that be true? Tom Goodwillie introduced the concept of Taylor towers for functors (like Taylor series for analytic functions). There is a different description of stable K-theory as a first derivative of algebraic K-theory in a suitable sense. Goodwillie conjectured that topological Hochschild homology should agree with stable K-theory because it also behaves like a first derivative. For background and way more details on this see [DM, DGM].

References

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