

Exercises in Algebraic Topology (master)

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Summer term 2025

Exercise sheet no 4

due: 29th of April 2025, 13:45h in H3

Again, due to the Easter break, the sheet is shorter than usual.

1 (3×3 -Lemma) (3 + 2 points)

Assume that the following diagram of abelian groups commutes and that all three columns are short exact sequences:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 \longrightarrow 0 \\
 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\
 0 & \longrightarrow & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 \longrightarrow 0 \\
 & & \downarrow g_1 & & \downarrow g_2 & & \downarrow g_3 \\
 0 & \longrightarrow & C_1 & \xrightarrow{\gamma_1} & C_2 & \xrightarrow{\gamma_2} & C_3 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

a) Prove that the third row is short exact if the first two rows are short exact and dually, that the first row is short exact if the last two rows are short exact.

b) What happens if the top and bottom rows are exact? Can you deduce that the middle row is exact or do you need an additional condition?

2 (Snake Lemma) (3 points)

Prove the famous Snake Lemma: If

$$\begin{array}{ccccccc}
 & & A' & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & A'' \longrightarrow 0 \\
 & & \downarrow f' & & \downarrow f & & \downarrow f'' \\
 0 & \longrightarrow & B' & \xrightarrow{\alpha'} & B & \xrightarrow{\beta'} & B''
 \end{array}$$

is a commutative diagram with exact rows, then there is an exact sequence

$$\ker(f') \longrightarrow \ker(f) \longrightarrow \ker(f'') \xrightarrow{\delta} \operatorname{coker}(f') \longrightarrow \operatorname{coker}(f) \longrightarrow \operatorname{coker}(f'').$$

Define δ explicitly in this case. Why “snake”?

(For an alternative: <http://www.youtube.com/watch?v=etbcKWEKnvg>)

3 (Exactness and homomorphisms) (2 + 2 points)

Let M be an abelian group and let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence of abelian groups.

(a) What can you say about the exactness of the sequence

$$0 \longrightarrow \operatorname{Hom}(M, A) \xrightarrow{\alpha_*} \operatorname{Hom}(M, B) \xrightarrow{\beta_*} \operatorname{Hom}(M, C) \longrightarrow 0?$$

(b) What happens if M is free abelian?