

Exercises in Algebraic Topology (master)

Prof. Dr. Birgit Richter

Summer term 2024

Exercise sheet no 6

due: 10th of May 2024, 11:45h in H2

1 (Euler characteristic) (2 + 2 + 2 + 2 points)

Let X be a finite CW complex. The *Euler characteristic* of X , $\chi(X)$, is then defined as

$$\chi(X) := \sum_{n \geq 0} (-1)^n \text{rk}(H_n(X; \mathbb{Z})).$$

Here, rk denotes the rank of a finitely generated abelian group, *i.e.*, the number of its free summands. Then $\chi(X)$ is well-defined.

a) Let $c_n(X)$ denote the number of n -cells of X . Prove that

$$\chi(X) = \sum_{n \geq 0} (-1)^n c_n(X).$$

b) What is $\chi(X)$ for a oriented compact closed surface of genus g , F_g for $g \geq 0$?

c) What can you say about $\chi(X \cup Y)$ if X and Y are finite CW complexes and if $X \cap Y$ is a subcomplex of X and Y ?

d) For finite CW complexes X and Y , show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

e) Assume that X is a finite CW complex, $n \geq 1$ and that $p: \tilde{X} \rightarrow X$ is an n -sheeted covering. Prove that $\chi(\tilde{X}) = n\chi(X)$.

2 (Moore spaces) (2 points)

Let G be an arbitrary finitely generated abelian group.

Construct a CW complex $M(G, n)$ whose reduced homology is concentrated in degree n with $\tilde{H}_n(M(G, n)) \cong G$. Such a space is called a *Moore space of type* (G, n) .

3 (Non-orientable surfaces) (2 points)

For $g \geq 2$ consider a regular $2g$ -gon $P_{2g} \subset \mathbb{R}^2$ with vertices z_1, \dots, z_{2g} . We identify edges according to

$$(1-t)z_{2j-1} + tz_{2j} \sim tz_{2j+1} + (1-t)z_{2j}$$

(here the indices are to be read mod $2g$) and call the quotient $N_g = P_{2g}/\sim$ the *closed non-orientable surface of genus* g . You know N_2 .

Calculate the homology of N_{2g} using the cellular chain complex.